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Stability Crossing Boundaries and Fragility Characterization of PID Controllers for SISO Systems with I/O Delays

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Abstract— **This paper focuses on the closed-loop stability analysis of single-input-single-output (SISO) systems subject to input (or output) delays in the presence of PID-controllers. More precisely, using a geometric approach, we present a simple and user-friendly method for the closed-loop stability analysis as well as for the fragility of such PID controllers. The proposed approach is illustrated on several examples encountered in the control literature.**

Index Terms— **PID, Delay, SISO, Fragility, Geometry**

I. INTRODUCTION

In this paper, inspired by the geometric ideas developed by Gu *et al.* [5] we start by developing a simple method to derive the stability regions in the gain parameters space of a PID-controller for a SISO system subject to (constant) time-delay. And next, we propose a *simple algorithm* to analyze the fragility of a given PID-controller for *any* SISO system subject to I/O delays. The method is based on the *Implicit Function Theorem* [6] and related properties, and requires three "ingredients":

(i) the construction of the *stability crossing boundaries (surfaces)* in the parameter-space defined by "P" (proportional), "I" (integral) and "D" (derivative) gains,
(ii) the explicit computation of the crossing direction (towards stability or instability) when such a surface is traversed,
(iii) finally, the explicit computation of the distance of some point to the closest stability crossing boundaries.

In the procedure above, the first step sends back to the D-decomposition method suggested by Neimark [13] in the 40s (see [9] for further comments) or to the parameter space approach (see, for instance, [1], [3] or [15] and the references therein). In the sequel, the stability crossing boundaries (surfaces for PID, curves for PI or PD controllers) represent the collection of all points for which the corresponding characteristic equation of the closed-loop system has at least one root on the imaginary axis. These boundaries define a "partition" of the parameter-space in several regions, each region having a constant number of unstable roots for all the parameters inside the region. Next, using an argument based on implicit function theorem one derives if a region has more or fewer unstable roots compared with its neighboring regions. This allows to detect the regions with no unstable roots which correspond to controller gains guaranteing the stability of the closed-loop system. This methodology has also advantages from the robustness point of view. Precisely, choosing controller gains inside a stability region and far from all the stability crossing boundaries that bound the region, the stability of the closed-loop system is ensured even for some small bounded variations of the controller gains.

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II. THE FREQUENCY MODEL

For the sake of brevity, let us consider now the class of *strictly proper* SISO open-loop systems with I/O delays given by the transfer function:

$$G(s) := \frac{P(s)}{Q(s)} e^{-s\tau} = c^T \left(sI_n - A\right)^{-1} b e^{-s\tau}, \qquad (1)$$

where $\left(A, b, c^T\right)$ is a state-space representation of the open-loop system. As mentioned in the Introduction, our aim is two-fold. First, design a PID controller

$$C(s) = k\left(1 + T_d s + \frac{1}{T_i s}\right) = k_p + k_d s + \frac{k_i}{s} \qquad (2)$$

that stabilizes the plant (1). Our second goal, is to derive an appropriate PID controller (k_p^*, k_d^*, k_i^*) and the largest positive value d such that the controller (2) stabilizes the system (1) for any k_p, k_d and k_i, as long as

$$\sqrt{\left(k_p - k_p^*\right)^2 + \left(k_d - k_d^*\right)^2 + \left(k_i - k_i^*\right)^2} < d.$$

It is clear that the closed-loop dynamics is characterized by the equation:

$$1 + G(s)C(s) = 0, \qquad (3)$$

which rewrites as:

$$f(s;\, k_p, k_d, k_i) := \frac{1}{G(s)} + \left(k_p + k_d s + \frac{k_i}{s}\right) = 0. \qquad (4)$$

Our approach follows the lines presented in [8], [11]-[12]. More precisely, we want to derive the *stability crossing boundaries* $\mathcal{T}$ which is the set of parameters $(k_p, k_d, k_i) \in \mathbb{R}_+^3$ such that (4) has imaginary solutions. As the parameters (k_p, k_d, k_i) cross the stability crossing boundaries, some characteristic roots cross the imaginary axis.We also consider $\Omega = \{\omega \in \mathbb{R} \mid \exists (k_p, k_d, k_i) \in \mathbb{R}_+^3$ such that $f(j\omega; k_p, k_d, k_i) = 0\}$ the set of frequencies where the number of unstable roots of (4) changes. The set Ω will be called *stability crossing set*.

III. Stability crossing characterizations

Considering that Ω is known, the stability crossing boundaries are simply characterized by:

Proposition 1: The stability crossing boundaries associated to (4) are described as follows:

$$\begin{cases} k_p = -\Re\left(\dfrac{Q(j\omega)}{P(j\omega)}e^{j\omega\tau}\right) \\ k_i = k_d\omega^2 + \omega\Im\left(\dfrac{Q(j\omega)}{P(j\omega)}e^{j\omega\tau}\right) \end{cases}, \quad \forall\omega\in\Omega. \tag{5}$$

Proof: From its definition, $\mathcal{T}$ is the set of parameters $(k_p, k_d, k_i) \in \mathbb{R}^3$, for which exists at least an $\omega \in \Omega$ such that $f(j\omega, k_p, k_d, k_i) = 0$. Therefore, both the real and the imaginary parts of $f(j\omega; k_p, k_d, k_i)$ have to be zero. Straightforward computation shows that:

$$\Re f(j\omega;\cdot) = k_p + \Re\, G(j\omega)^{-1},$$

which leads to the first relation stated in (5). On the other hand,

$$\Im f(j\omega;\cdot) = \Im\, G(j\omega)^{-1} + k_d\omega - k_i/\omega,$$

which allows us to deduce the second relation in (5). ■

Remark 1: For any fixed $\omega^* \in \Omega$, one obtains a section of a stability crossing surface which consists in a straight line parallel to the (k_d, k_i) plane and passing through the point $\left(-\Re\left(\dfrac{Q(j\omega)}{P(j\omega)}e^{j\omega\tau}\right), 0, \omega\Im\left(\dfrac{Q(j\omega)}{P(j\omega)}e^{j\omega\tau}\right)\right)$. The slope of this line in the (k_d, k_i) plane is always positive and is given by ω^2.

Remark 2: From the *Proposition* 1 it is clear that $k_i = 0$ represents a boundary.

Remark 3: Let the relative degree of the system (1) be $\rho = 1$. Then, the closed-loop system (1) becomes a system of *neutral-type* (see, e.g., [7], [9]) and

$$\left(k_p, \left|\frac{q_n}{p_{n-1}}\right|, ki\right) \quad \text{and} \quad \left(k_p, -\left|\frac{q_n}{p_{n-1}}\right|, ki\right)$$

belong to the stability crossing surfaces. Here, p_{n-1} and q_n represent the leading coefficients of the polynomials $P(s)$ and $Q(s)$, respectively.

A. Stability crossing sets

In the sequel, we present a practical methodology to derive the stability crossing set. For the sake of brevity, we suppose the following technical assumption is satisfied:

Assumption 1: There exist some bounds $\left(\underline{k_p^*}, \overline{k_p^*}\right)$, $\left(\underline{k_d^*}, \overline{k_d^*}\right)$ and $\left(\underline{k_i^*}, \overline{k_i^*}\right)$ of the controller gains.

These bounds can be arbitrarily fixed and, in principle, they are chosen by the designer according to the physical constraints of the model/controller. In this context, when Assumption 1 holds, the section of the stability crossing surface obtained for a fixed $\omega \in \Omega$ reduces to a segment (see Remark 1).

Proposition 2: Consider that Assumption 1 holds. Then the stability crossing set Ω is a union of bounded intervals consisting in all frequencies that simultaneously satisfy the following conditions:

$$\begin{cases} \underline{k_p^*} \le -\Re\left(\dfrac{Q(j\omega)}{P(j\omega)}e^{j\omega\tau}\right) \le \overline{k_p^*} \\ \exists\ \underline{k_d^*} \le k_d \le \overline{k_d^*} \text{ s.t. } \underline{k_i^*} \le k_d\omega^2 + \omega\Im\left(\dfrac{Q(j\omega)}{P(j\omega)}e^{j\omega\tau}\right) \le \overline{k_i^*}. \end{cases} \tag{6}$$

Proof: The characterization of the stability crossing set Ω given by (6) follows straightforward from (5) and Assumption 1. In order to prove the boundedness of the crossing set Ω, we notice that due to the assumption that the transfer $G(\cdot)$ is strictly proper, one has $\lim_{\omega\to+\infty}\left|G(j\omega)^{-1}\right| = +\infty$. In other words, this means that either $\lim_{\omega\to+\infty}\left|\Re G(j\omega)^{-1}\right| = +\infty$ or $\lim_{\omega\to+\infty}\left|\Im G(j\omega)^{-1}\right| = +\infty$, which contradicts either the first relation in (6) or the second one. ■

Remark 4: Propositions 1 and 2 lead to the following algorithm to determine both the stability crossing set Ω and the stability crossing boundaries $\mathcal{T}$:

- **Step 1:** One solves the system $\underline{k_p^*} \le -\Re\dfrac{1}{G(j\omega)} \le \overline{k_p^*}$ getting a union of intervals.
- **Step 2:** For all ω derived at the previous step one computes k_p and derive the equation of the line (k_d, k_i) given by the second equation in (5).
- **Step 3:** Finally, one keeps only those frequencies ω for which the line (k_d, k_i) derived at the previous step intersects the rectangle $[(\underline{k_d^*}, \underline{k_i^*}); (\overline{k_d^*}, \underline{k_i^*}); (\overline{k_d^*}, \overline{k_i^*}); (\underline{k_d^*}, \overline{k_i^*})]$.

Consider now, that either k_d or k_i is fixed. Let us also denote by $\mathcal{T}_h$, $h \in \{i, d\}$ the crossing curve when d or i is fixed and consider the following decomposition into real and imaginary parts:

$$R_0 + jI_0 = j\left.\frac{\partial f(s;k_p,k_h)}{\partial s}\right|_{s=j\omega}, \tag{7}$$

$$R_1 + jI_1 = -\left.\frac{\partial f(s;k_p,k_h)}{\partial k_h}\right|_{s=j\omega}, \tag{8}$$

$$R_2 + jI_2 = -\left.\frac{\partial f(s;k_p,k_h)}{\partial k_p}\right|_{s=j\omega}. \tag{9}$$

Then, since $f(s; k_p, k_h)$ is an analytic function of s, k_p and k_h, the implicit function theorem indicates that the tangent of $\mathcal{T}_h$ can be expressed as

$$\begin{pmatrix} \dfrac{dk_p}{d\omega} \\ \dfrac{dk_h}{d\omega} \end{pmatrix} = \begin{pmatrix} R_2 & R_1 \\ I_2 & I_1 \end{pmatrix}^{-1} \begin{pmatrix} R_0 \\ I_0 \end{pmatrix} = \frac{1}{R_1I_2 - R_2I_1}\begin{pmatrix} R_1I_0 - R_0I_1 \\ R_0I_2 - R_2I_0 \end{pmatrix}, \tag{10}$$

provided that

$$R_1I_2 - R_2I_1 \ne 0. \tag{11}$$

It follows that $\mathcal{T}_h$ is smooth everywhere except possibly at the points where either (11) is not satisfied, or when

$$\frac{dk_p}{d\omega} = \frac{dk_h}{d\omega} = 0. \tag{12}$$

Remark 5: If (12) is satisfied, then straightforward computations show us that $R_0 = I_0 = 0$. In other words, $s = j\omega$ is a multiple solution of (15).

B. Classification of the stability crossing boundaries

It is worth noting here that k_p, k_d and k_i continuously depend on ω. Therefore, in order to classify the stability crossing boundaries we will first classify the intervals belonging to the stability crossing set. Precisely, a deeper analysis of Proposition 2 allows us to say that ω^* is an end of an interval belonging to Ω if and only if one of the following condition is satisfied:

- **Type 1:** $-\Re\dfrac{1}{G(j\omega^*)} = k_p^*$, where k_p^* is either $k_p^* = \underline{k_p^*}$ or $k_p^* = \overline{k_p^*}$. In this case, $\omega^* \in \Omega$ and the stability crossing surface approach a segment parallel to the (k_d, k_i) plane given by $k_p = k_p^*$ and

$$k_i = k_d \cdot (\omega^*)^2 + \omega^* \Im \frac{1}{G(j\omega^*)},$$
$$\underline{k_d^*} \le k_d \le \overline{k_d^*},\ \underline{k_i^*} \le k_i \le \overline{k_i^*}$$

- **Type 2:** $-\dfrac{1}{\omega^*}\Im\dfrac{1}{G(j\omega^*)} = k_d^*$. In this case $\omega^* \in \Omega$ and the stability crossing surface ends in the point $\left(-\Re\dfrac{1}{G(j\omega^*)}, -\dfrac{1}{\omega^*}\Im\dfrac{1}{G(j\omega^*)}, 0\right)$, included in the (k_p, k_d) plane.
- **Type 3:** $\omega^*\Im\dfrac{1}{G(j\omega^*)} = k_i^*$. In this case $\omega^* \in \Omega$ and the stability crossing surface ends in the point $\left(-\Re\dfrac{1}{G(j\omega^*)}, 0, \omega^*\Im\dfrac{1}{G(j\omega^*)}\right)$, included in the (k_p, k_i) plane.

Similarly to [5], we classify the stability crossing boundaries in 8 types in function of the kind of the left and right ends of the corresponding frequency crossing interval. Precisely, we say that a crossing surface is of type ab, $a, b \in \{1, 2, 3\}$ if it corresponds to a crossing interval (ω_l, ω_r) with ω_l of type a and ω_r of type b. Let us notice that generally the intervals (ω_l, ω_r) are closed.

C. Crossing direction

As explained in [4], [17], a pair of imaginary zeros $(\bar{s}, s)$ of the characteristic equation $f(s;\ k_p, k_d, k_i) = 0$ cross the imaginary axis through the gates $-j\omega$, $j\omega$ respectively, as (k_p, k_d, k_i) moves from one side of a stability crossing surface to the other side. The direction of crossing may be calculated using implicit function theorem as described in [5], [10]. Precisely, the characteristic equation $f(s;\ k_p, k_d, k_i) = 0$ defines an implicit function s of variables k_p, k_d and k_i. The definition of $f(s;\ k_p, k_d, k_i)$ given by (4) allows us to compute the following partial derivatives:

$$\begin{aligned} \frac{\partial s}{\partial k_p} &= \frac{s^2G^2(s)}{k_iG^2(s) - k_ds^2G^2(s) + s^2G'(s)}, \\ \frac{\partial s}{\partial k_d} &= \frac{s^3G^2(s)}{k_iG^2(s) - k_ds^2G^2(s) + s^2G'(s)}, \\ \frac{\partial s}{\partial k_i} &= \frac{sG^2(s)}{k_iG^2(s) - k_ds^2G^2(s) + s^2G'(s)}. \end{aligned} \tag{13}$$

Let $(\bar{k}_p, \bar{k}_d, \bar{k}_i)$ a point belonging to a stability crossing surface and let $s = j\bar{\omega},\ \bar{\omega} > 0$ be the corresponding imaginary zero of the characteristic equation. Let $\mathbf{x} = (x_p, x_d, x_i)$ be a unit vector that is not tangent to the surface. Let us also use the following notation $\overrightarrow{k} = (k_p, k_d, k_i)$ and $\overrightarrow{k}_* = (\bar{k}_p, \bar{k}_d, \bar{k}_i)$.

Proposition 3: A pair of zeros of (4) moves from the left half complex plane (LHP) to the right half complex plane (RHP) as (k_p, k_d, k_i) moves from one side of a stability crossing surface to the other side through $(\bar{k}_p, \bar{k}_d, \bar{k}_i)$ in the direction of $\mathbf{x}$ if

$$\Re\left(\frac{\partial s}{\partial k_p}x_p + \frac{\partial s}{\partial k_d}x_d + \frac{\partial s}{\partial k_i}x_i\right)\Bigg|_{s=j\bar{\omega},\ \overrightarrow{k}=\overrightarrow{k}_*} > 0. \tag{14}$$

The crossing is from the RHP to the LHP if the inequality (14) is reversed.

Proof: The proof follows directly from the fact that the derivative of the implicit function s along the direction given by $\mathbf{x}$ in the point $(\bar{k}_p, \bar{k}_d, \bar{k}_i)$ is

$$\left.\frac{ds}{dx}\right|_{(\bar{k}_p,\bar{k}_d,\bar{k}_i)} = \left(\frac{\partial s}{\partial k_p}x_p + \frac{\partial s}{\partial k_d}x_d + \frac{\partial s}{\partial k_i}x_i\right)_{(\bar{k}_p,\bar{k}_d,\bar{k}_i)}$$

Thus the real part of the previous directional derivative is computed as the right part of (14) ■

IV. Fragility Analysis of PID Controllers

Consider now the *PID fragility problem*, that is the problem of computing the maximum controller parameters deviation without loosing the closed-loop stability. In other words, given the parameters (k_p^*, k_d^*, k_i^*) such that the roots of the closed-loop characteristic equation:

$$Q(s) + P(s)\Big(k_p^* + k_d^*s + \frac{k_i^*}{s}\Big)e^{-s\tau} = 0, \tag{15}$$

are located in $\mathbb{C}_-$ (that is the closed-loop system is asymptotically stable), find the maximum parameter deviation $d \in \mathbb{R}_+$ such that the roots of (3) stay located in $\mathbb{C}_-$ for all controllers (k_p, k_d, k_i) satisfying:

$$\sqrt{(k_p - k_p^*)^2 + (k_d - k_d^*)^2 + (k_i - k_i^*)^2} < d.$$

This problem can be more generally reformulated as: *find the maximum parameter deviation* d *such that the number of unstable roots of (3) remains unchanged.*

First, let us introduce some notation:

$$\begin{aligned} \mathcal{T} &= \bigcup_{l=1}^{N} \mathcal{T}_l, \quad \mathcal{T}_l = \big\{(k_p, k_d, k_i)\big|\omega \in \Omega_l\big\}, \\ \overrightarrow{k(\omega)} &= \left(k_p(\omega), k_d(\omega), k_i(\omega)\right)^T, \quad \overrightarrow{k^*} = \left(k_p^*, k_d^*, k_i^*\right)^T, \\ \overrightarrow{k_{ab}}(\omega) &= \left(k_a(\omega), k_b(\omega)\right)^T, \quad \overrightarrow{k_{ab}^*} = \left(k_a^*, k_b^*\right)^T, \end{aligned}$$

where $a, b \in \{p, i, d\}$. Let us also denote $d_{\mathcal{T}} = \min_{l \in \{1,\ldots,N\}} d_l$, where

$$d_l = \min_{(k_p,k_d,k_i)\in\mathcal{T}_l}\left\{\sqrt{(k_p - k_p^*)^2 + (k_d - k_d^*)^2 + (k_i - k_i^*)^2}\right\}.$$

A. PI-PD Controller Fragility

Let $k_d = k_d^* \in \mathbb{R}$ or $k_i = k_i^* \in \mathbb{R}$ be fixed, we have the following result:

Proposition 4: The maximum parameter deviation, without changing the number of unstable roots of the closed-loop equation (3) can be expressed as:

PI-Controller: Let $k_d = k_d^*$ be fixed,

$$d_{pi}^* = \min\left\{|k_i^*|, \min_{\omega\in\Omega_{f_{pi}}}\left\{\left\|\overrightarrow{k_{pi}}(\omega) - \overrightarrow{k_{pi}^*}\right\|\right\}\right\}. \tag{16}$$

PD-Controller: Let $k_i = k_i^*$ be fixed, then $d_{pd}^* =$

$$d_{pd}^* = \min\left\{k_{d\infty}, \left|k_p^* - k_p(0)\right|, \min_{\omega\in\Omega_{f_{pd}}}\left\{\left\|\overrightarrow{k_{pd}}(\omega) - \overrightarrow{k_{pd}^*}\right\|\right\}\right\}, \tag{17}$$

with,

$$k_{d\infty} = \begin{cases} \min\left\{\left|k_d^* - \left|\frac{q_n}{p_m}\right|\right|, \left|k_d^* + \left|\frac{q_n}{p_m}\right|\right|\right\} & \text{if} \quad m = n-1 \\ \emptyset & \text{if} \quad m < n-1 \end{cases}$$

and $\Omega_{f_{ab}}$, $a, b \in \{p, i, d\}$ is the set of roots of the function $f_{ab} : \mathbb{R}_+ \mapsto \mathbb{R}$,

$$f_{ab}(\omega) \quad \triangleq \quad \left\langle \left(\overrightarrow{k_{ab}}(\omega) - \overrightarrow{k_{ab}^*}\right), \frac{d\overrightarrow{k_{ab}}(\omega)}{d\omega}\right\rangle, \tag{18}$$

where " $\langle\cdot,\cdot\rangle$" means the inner product.

Proof: We consider that the pair (k_a^*, k_b^*) belongs to a region generated by the crossing curves. Since the number of unstable roots changes only when (k_a, k_b) get out of this region, our objective is to compute the distance between (k_a^*, k_b^*) and the boundary of the region. Furthermore, the boundary of such a region consist of pieces of crossing curves and possibly one segment of the k_p axis (in the case of $PI-$fragility) or a segment of the shifted axis $k_i + k_p(0)$ (in the case of $PD-$fragility). In order to compute the minimal distance between (k_a^*, k_b^*) and a crossing curve we only need to identify the points where the vector $(k_a - k_a^*, k_b - k_b^*)$ and the tangent to the curve are orthogonal. In other words we have to find the solutions of

$$f_{ab}(\omega) = 0,$$

where f_{ab} is defined by (18). Taking into account the relation (10) we may write (18) as

$$(k_a - k_a^*)(R_1 I_0 - R_0 I_1) + (k_b - k_b^*)(R_0 I_2 - R_2 I_0).$$

It is worth to mention that the stability region is defined in $(\underline{\omega}, \overline{\omega})$ and, therefore, (18) will have a finite number of roots. Let us consider $\{\omega_1, \ldots, \omega_M\}$ the set of all the roots of $f_{ab}(\omega)$ when we take into account all the pieces of crossing curves belonging to the region around (k_a^*, k_b^*). Since the distance from (k_a^*, k_b^*) to the $k_p(\omega)$ axis is given by $|k_b^*|$ (for the $PI-$fragility) and the distance from (k_a^*, k_b^*) to the shifted axis $k_i + k_p(0)$ is given by $|k_p^* - k_p(0)|$ (for the $PD-$fragility), one obtains that

$$d_{pi}^* = \min\left\{|k_i^*|, \min_{\ell=\{1,\ldots,M\}}\left\{\left\|\overrightarrow{k_{pi}}(\omega_\ell) - \overrightarrow{k_{pi}^*}\right\|\right\}\right\},$$

in the case of $PI-$fragility, or

$$d_{pd}^* = \min\left\{\left|k_p^* - k_p(0)\right|, \min_{\ell=\{1,\ldots,M\}}\left\{\left\|\overrightarrow{k_{pd}}(\omega_\ell) - \overrightarrow{k_{pd}^*}\right\|\right\}\right\},$$

in the case of $PD-$fragility, which are just another way to express (16)-(17). ■

B. DI Projection

Let $k_p = k_p^* \in \mathbb{R}$ be fixed, we have the following result:

Proposition 5: The maximum parameter deviation from (k_d^*, k_i^*), without changing the number of unstable roots of the closed-loop equation (3) can be expressed as:

$$d_{di}^* = \min\left\{|k_i^*|, \min_{\omega_\ell\in\Omega_{k_p^*}}\left\{\left|\frac{\omega_\ell^2 k_d^* - k_i^* + \omega_\ell \Im\left\{\frac{Q(j\omega_\ell)}{P(j\omega_\ell)}e^{j\omega_\ell\tau}\right\}}{\sqrt{(\omega_\ell)^4 + 1}}\right|\right\}\right\}, \tag{19}$$

where $\Omega_{k_p^*}$ is the set of roots of the function $f_{k_p^*} : \mathbb{R}\times\mathbb{R}_+ \mapsto \mathbb{R}$,

$$f_{k_p^*}\left(k_p^*, \omega\right) \triangleq k_p^* + \Re\left\{\frac{Q(j\omega)}{P(j\omega)}e^{j\omega\tau}\right\}. \tag{20}$$

Proof: The proof follows similar geometric arguments as those used to prove *Proposition* 4, and for the sake of brevity will be omitted. ■

Remark 6: Observe that (20) has an uncountable number of solutions, however in *Proposition* 5 we have considered the set including the corresponding (k_d^*, k_i^*) points.

C. PID Fragility Algorithm

In order to obtain the obtain the PID fragility we present the following algorithm:

- **Step 1:** Let $k_{pid}^* \in \mathbb{R}^3$ be fixed. Then, set $d = \min\left\{d_{pi}^*, d_{pd}^*, d_{di}^*\right\}$.
- **Step 2:** Sweep over all $\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ and compute $k_{p\theta}^* = k_p^* + d\sin\theta$.
- **Step 3:** Solve $f_{k_p^*}\left(k_{p\theta}^*, \omega\right) = 0$ and denote by Ω_θ the set of solutions.
- **Step 4:** Compute,

$$d_\theta^* = \min_{\omega_\ell\in\Omega_\theta}\left\{\left|\frac{(\omega_\ell)^2 k_d^* - k_i^* + \omega_\ell\Im\left\{\frac{Q(j\omega_\ell)}{P(j\omega_\ell)}e^{j\omega_\ell\tau}\right\}}{\sqrt{(\omega_\ell)^4 + 1}}\right|\right\}.$$

- **Step 5:** If $d_\theta^* < d\cos\theta$ then set $d = d_\theta^*/\cos\theta$ and go to step 2. Otherwise continue to step 2.
- **Step 6:** If $\theta = \frac{\pi}{2}$, the procedure is finish and d is the PID fragility for the controller $\left(k_p^*, k_d^*, k_i^*\right)$.

V. Illustrative examples

In order to motivate the previous results, we consider in the sequel some numerical examples.

A. Stability crossing boundaries classification

Example 1: Finally, lets consider the SISO plant [2],

$$G(s) = \frac{-s^4 - 7s^3 - 2s + 1}{(s+1)(s+2)(s+3)(s+4)(s^2+s+1)} e^{-\frac{1}{20}s}. \tag{21}$$

By choosing the rectangle: $0 \leq k_p \leq 5$, $-12 \leq k_i \leq 5$, $0 \leq k_d \leq 10$, we obtain the following cases: Based in these

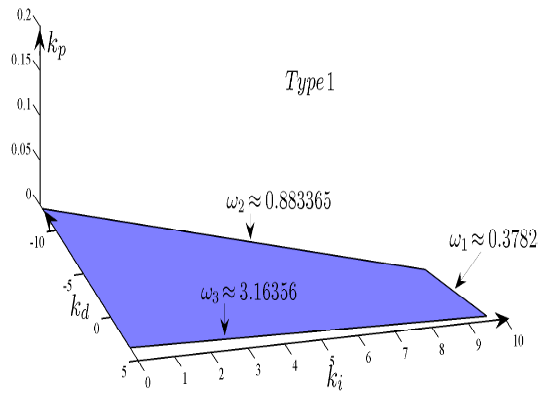

Fig. 1. Boundary classification Type 1 for the system (21).

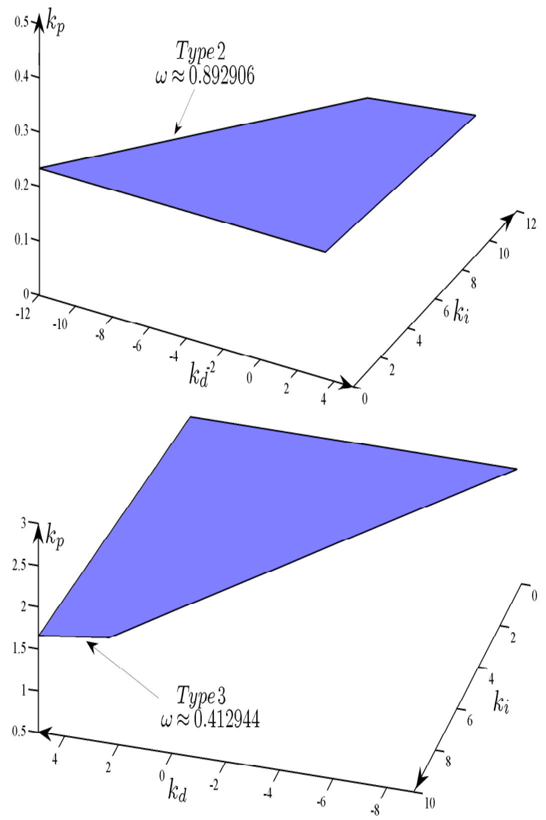

Fig. 2. Stability crossing boundaries classification for the system (21). (Upper) Type 2. (Lower) Type 3.

results, the table I classifies the cases cited above.

TABLE I

CLASSIFICATION INTERVALS TYPE FOR THE SYSTEMS (21).

Interval	**Classification**
$[0.37823, 3.16356]$	Type 11
$[0.37823, 0.89290]$	Type 12
$[0.37823, 0.41294]$	Type 13
$[0.89290, 3.16356]$	Type 21
$[0.41294, 3.16356]$	Type 31
$[0.41294, 0.89290]$	Type 32

B. PID fragility analysis

Example 2: Consider the following system [14]:

$$G(s) = \frac{s^3 - 4s^2 + s + 2}{s^5 + 8s^4 + 32s^3 + 46s^2 + 46s + 17} e^{-s}. \tag{22}$$

By choosing $k_p^* \in \left[0, \frac{9}{2}\right]$, we obtain the stability region depicted in Fig.3. Next, in order to illustrate the proposed

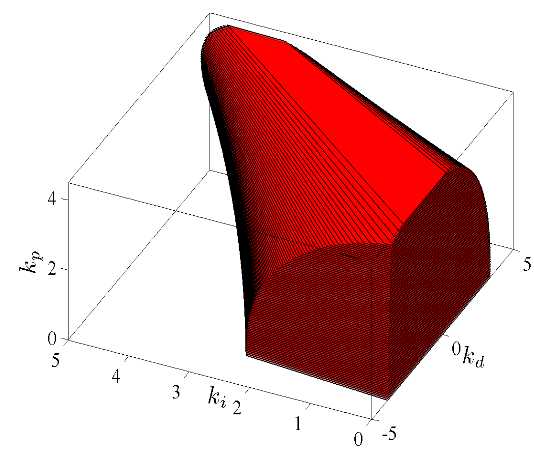

Fig. 3. The PID stability region for $k_p \in \left[0, \frac{9}{2}\right]$.

PID fragility-algorithm, consider $\left(k_p^*, k_d^*, k_i^*\right) = (2, 3, 3)$, leading to the values in Table II and depicted in Fig.4.

TABLE II

PID FRAGILITY FOR THE EXAMPLE (22).

Controller $\left(k_p^*, k_d^*, k_i^*\right)$	**Fragility** (PI, PD, DI)	**Initial** PID-Fragility	**PID-Fragility** $\min\left\{d^*, d_\theta^*\right\}$
$(2, 3, 3)$	$d_{pi}^* = 1.68051$ $d_{pd}^* = 1.33313$ $d_{di}^* = 1.27520$	$d^* = 1.27520$	$d_\theta^* = 1.26295$

Example 3 (unstable, non-minimal phase system): Consider the following plant [8],

$$G(s) = \frac{s - 2}{s^2 - \frac{1}{2}s + \frac{13}{4}} e^{-\frac{1}{2}s}. \tag{23}$$

The interest in the analysis of this system, remains in the fact that the closed-loop plant becomes a system of *Neutral-Type*. Now, applying the same procedure as before, and considering $k_p^* \in (0.32595, 1.625)$ we obtain the following stability region.

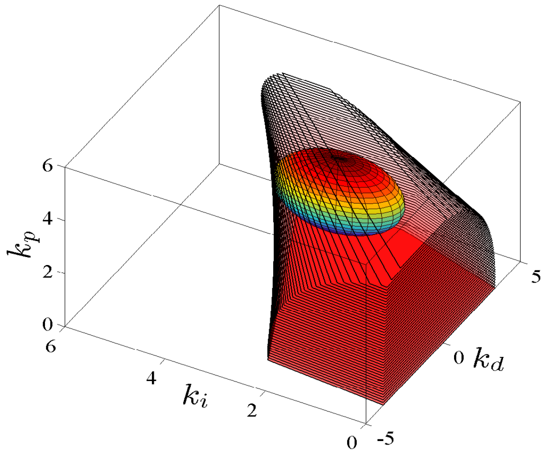

Fig. 4. PID-fragility for the controller $\left(k_p^*, k_d^*, k_i^*\right) = (2, 3, 3)$.

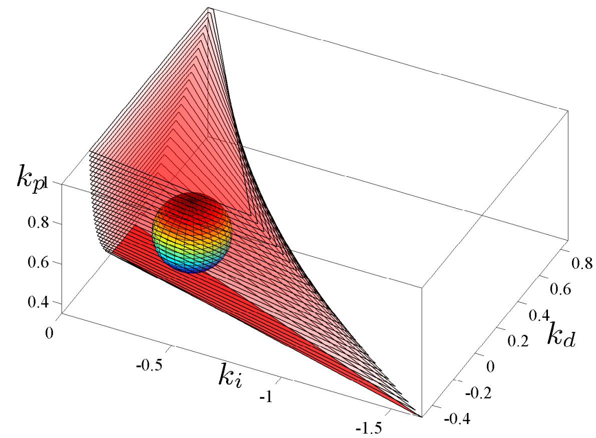

Fig. 6. PID-fragility for the controller $\left(k_p^*, k_d^*, k_i^*\right) = \left(\frac{5}{8}, -\frac{1}{10}, -\frac{2}{5}\right)$.

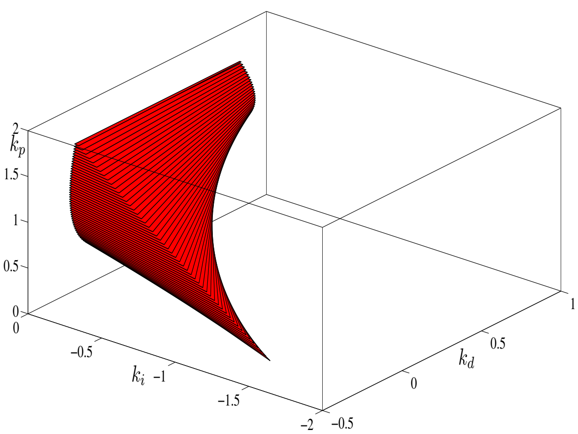

Fig. 5. The PID stability region of Neutral-Type .

For the fragility analysis, lets consider the controller $\left(k_p^*, k_d^*, k_i^*\right) = \left(\frac{5}{8}, -\frac{1}{10}, -\frac{2}{5}\right)$, leading to the results summarized in Table III. Figure 6 illustrate such a results.

TABLE III

PID FRAGILITY FOR THE EXAMPLE (23).

Controller $\left(k_p^*, k_d^*, k_i^*\right)$	**Fragility** (PI, PD, DI)	**Initial** PID-Fragility	**PID-Fragility** $\min\left\{d^*, d_\theta^*\right\}$
$\left(\frac{5}{8}, \frac{-1}{10}, \frac{-2}{5}\right)$	$d_{pi}^* = 0.29314$ $d_{pd}^* = 0.16758$ $d_{di}^* = 0.16782$	$d^* = 0.16758$	$d_\theta^* = 0.16453$

VI. CONCLUSIONS

In this paper, we focused on stabilizing a class of SISO linear systems with constant delay in the input or output by using PID controllers. First, by exploiting the system properties we have characterized the stability crossing boundaries in the parameter-set defined by the controller's parameters. Second, we have developed a simple geometrical method to construct the PID stability region, that characterize the set of all stabilizing controller parameter. Finally, a simple geometric-based algorithm is derived for computing the fragility of PID-controllers. To prove the efficiency of the proposed methods, several illustrative examples have been considered. It is important to note that such an idea can be easily extended to *proper* SISO systems with I/O delays.

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