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Ant Colony Optimization Technique to Solve the Min-Max Single Depot Vehicle Routing Problem

Koushik S Venkata Narasimha, Manish Kumar, *Member, ASME*

Abstract — **This paper implements a swarm intelligence based algorithm called ant colony optimization to solve the min-max Single Depot Vehicle Routing Problem (SDVRP). A traditional SDVRP tries to minimize the total distance travelled by all the vehicles to all customer locations. The min-max SDVRP, on the other hand, tries to minimize the maximum distance travelled by any vehicle. This problem is of specific significance for time-critical applications where one wants to minimize the time taken to attend any customer. The algorithm developed is an extension of SDVRP algorithm developed by Bullnheimer et al. in 1997 based upon ant colony optimization. A computer simulation model using the MATLAB is developed. A comparative study is carried out to evaluate the proposed algorithm's performance with respect to the algorithm developed by Carlsson et al. in terms of the optimality of solution and time taken to reach the solution.**

I. Introduction

With the growing business and hence increase in the complexity of transportation of materials, minimizing the cost of logistics becomes a significant factor in reducing the overall cost [1]. In the last decade, research suggests that 10% to 15% of the traded goods correspond to the transportation costs. Therefore utilization of computerized methods for planning transportation routes will result in significant savings ranging from 5% - 20% [2]. A well known problem in this area, called Single Depot Vehicle Routing Problem, has been studied for the past five decades. The problem involves finding tours of vehicles (with constraints on maximum distance they can travel) from a depot that visits a given number of delivery points and minimizes the total distance travelled by all vehicles combined. Solving the problem is computationally extensive and is known to be Nondeterministic Polynomial (NP) Hard problem. NP Hard are those set of problems where the computational effort required for finding the solution increases exponentially with the size of the problem. Hence an approximate solution is more desirable as it is considered to be a tradeoff to the computation time. This paper presents a similar solution to the interesting version of this problem called min-max Single Depot Vehicle Routing Problem. The objective of the problem is to minimize the maximum distance travelled by a vehicle (instead of total distance travelled as in SDVRP). This problem is often of interest when minimization of time taken to visit all points is more important than the total distance travelled; a usual factor of interest in emergency management situations. In emergency management, the objective is to use all available vehicles to minimize the time taken to attend to all points needing emergency resources. Other applications of this problem are in defense and computer networking. For example, assigning tours to a group of UAVs engaged in large scale surveillance operation by solving min-max SDVRP will minimize the maximum time of travel of UAVs, and hence help achieve desired objectives in time-critical scenarios. In computer networking, depots represent servers, vehicles represent data packets, and customers represent clients, a network routing topology generated by solving the min-max problem would result in minimizing the maximum latency between a server and a client. In what follows, we provide background on different versions of this problem used in literature, discuss the min-max SDVRP problem and methods available in literature, and provide background on ant colony optimization technique used to solve similar problems. We then present the problem formulation, our approach, followed by simulation results, discussions, and conclusions.

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Koushik S Venkata Narasimha is pursuing his graduate studies at the University of Cincinnati, OH 45221 USA (Phone: 513-346-8551; e-mail: venkatkh@mail.uc.edu).
Manish Kumar is with University of Cincinnati, OH 45221 USA. He is an Assistant Professor at the Department of Mechanical Engineering. (e-mail: kumarmu@ucmail.uc.edu).

II. Background

A. TSP and VRP

Travelling Salesman Problem (TSP) is a well known combinatorial optimization problem. Here, the goal is to find a closed tour of minimal length connecting 'n' given cities or customer locations. Each city must be visited once and only once. Vehicle Routing Problem (VRP) is closely related to TSP because it consists of many TSPs with common start and end cities. In VRP, there is single depot (hence sometimes called single depot vehicle routing problem or SDVRP), and 'k' vehicles with capacity and distance restrictions. The objective is to find the minimum cost (overall travel distance) vehicle route so that: i) every customer location is visited exactly once; ii) all vehicles routes begin and end at the depot; iii) vehicle capacity and distance restrictions are not violated. VRP was first proposed by Dantzig in 1959 [3]. From then it has been studied extensively and serves as one of the benchmark problems in the field of optimization. The computational complexities involved in solving TSPs and VRPs are well known to be NP hard [4]. These problems have influenced the emergence of fields such as operations research, polyhedral theory and complexity theory.

B. Variants of VRP

The common form of VRP is the capacitated VRP (CVRP) wherein a set of vehicles of uniform capacity must visit customer places once and only once from a common depot at a minimum transit cost [5]. Multiple Depot VRP (MDVRP) is an extension of the CVRP with more number of depots. The objective here is to visit all the customer locations once and only once while minimizing the total travel distance. VRP with time windows (VRPTW) is also a similar type of problem with an additional restriction in terms of time window associated with each customer. The objective here is to minimize the number of vehicles used and sum of travel time and waiting time needed to supply all customers.

C. Min-Max VRP

There is another important variant of VRP closely related to VRPTW called Min-Max VRP. This problem was first formulated Carlsson et al. in 2007 [6]. Unlike the traditional objective of minimizing the tour lengths of the vehicles, the purpose of this problem is to minimize the maximum length of a tour in VRP. Carlsson et al. developed two different heuristics to solve min-max MDVRP. The first heuristic is a linear programming based technique. Using this algorithm, the customers are assigned to vehicles at depots and TSP routes are generated for each vehicle and its respective customers. Second heuristic namely region partition method is also developed which helps in generating fast approximate initial solutions. On the basis of a popular BHH theorem (1959) by Beardwood, Halton and Hammersley, they show that theoretically there exists a lower and upper bounds to the longest tour length, L, which denotes the optimal solution to min-max MDVRP. Using this upper bound, they developed a tour partition heuristic which generates a feasible solution for the MDVRP. Building upon this, they develop an asymptotic bound for L and they conclude that the optimal solution to min-max MDVRP with uniformly distributed points will numerically approach to a value proportional to $\sqrt{n/k}$, which is the value of optimal TSP tour of all customers split by number of vehicles, under the constraint. In this paper, we use this upper bound and lower bounds on L in the algorithm based on ant colony optimization.

D. TSP and VRP using Swam Optimization Techniques

Swarm intelligence using the foraging strategies of ants was first applied to TSP in [7]. This optimization technique is called ant-colony optimization. The basic idea underlying this ant based algorithm is to use a positive feedback mechanism, based on an analogy with the trail laying trail following behavior of some species of ants and some other social insects, and to reinforce those portions of good solutions that contribute to the quality of these solutions [8]. Since the algorithm involves simple rules of each ant with decentralized control, this technique is computationally efficient and fairly easy to implement. A major disadvantage of this method is getting stuck in local minima. This issue has been addressed using several approaches such as introduction of randomness in choosing the next city as well as via mechanisms such as 1-opt and 2-opt techniques. This algorithm was applied to solve SDVRP [9], where a hybrid ant system algorithm was presented and then improved using specific information (savings, capacity utilization). In 2004, Bell and McMullen [10] developed multiple ant colony method to solve the vehicle routing problem. This method uses separate specialized ant groups with unique pheromone depositions for each vehicle to solve the VRP. This separation is intended to differentiate paths typically used in first vehicle route from those used by subsequent vehicles. It is believed that this technique is more useful when the size of the problem and the number of vehicles required increase. Also, their paper reinstates the route improvement strategies such as 2-opt heuristic and candidate list technique which were initially proposed by Bullnheimer et al [9]. In this paper, we make use of 2-opt heuristics to improve the solution.

III. Problem Description

The paper focuses on the min-max SDVRP in which the objective is to minimize the maximal length of a vehicle's tour in a traditional SDVRP. To mathematically formulate this problem, consider *'n'* customer/delivery points, a depot V_0 and *'k'* vehicles at the depot. All the vehicles are initially located at the depot. The vehicles are required to visit all customer points and return to the depot from where it started its journey. The problem as stated earlier is to decide the tours of each vehicle so that the distance travelled by the vehicle with maximum tour length is minimized. Mathematically [6], the aim is to

$$\begin{aligned} &\text{Minimize } \lambda \\ &\text{subject to } TSP(S_i) \le \lambda, \forall i \qquad (1) \\ &\cup S_i = N \end{aligned}$$

where N is the set of all customer locations, $|N| = n$, $S_i (\subset N)$ is the subset of customers assigned to vehicle *'i'* and *TSP(S)* is the minimal Travelling Salesman Problem (TSP) tour of customers in set *S*.

IV. Approach

Ant colony optimization technique is used in this paper to solve the min-max SDVRP. Our approach is based on the ant colony optimization method used by Bullnheimer et al. [9] in solving the traditional SDVRP. A nice introduction to ant colony optimization technique can be found in [8]. The traditional SDVRP, as defined by Bullnheimer et al. [9], can be represented by a complete weighted digraph $G = (V,A,d)$ where $V = \{v_0,v_1,..,v_n\}$ is a set of vertices and $A = \{(v_i,v_j) : i \neq j\}$ is a set of arcs. The vertex v_0 denotes the depot, the other vertices of V represent cities or customers. A cost variable is associated with each arc called d_{ij} that can represent distance or travel time between city i and city j. In traditional SDVRP, the aim is to find minimum cost vehicle routes, i.e., tours for the vehicles so that total distance travelled by all vehicles is minimized. In ant colony optimization, artificial ants searching the solution space simulate real ants searching the environment; the objective function corresponds to the quality of food sources and an adaptive memory corresponds to the pheromone trails. To aid the search procedure through a set of possible solutions, the artificial ants are provided with local heuristic functions which represent greedy approach based upon local information. The artificial pheromones represent the global desirability of a solution. To appropriately guide the search process using the local and global information, ant colony algorithms typically use some parameters that include heuristic desirability, pheromone updating rule, and probabilistic transition rule. The definitions of these parameters are problem specific and are critical to solve the optimization problem in consideration. Moreover, to enhance the accuracy of the solution, 2-opt-heuristic method has been used in this paper. This is a local search heuristic that involves removing two edges, and reconnecting the path to obtain a better solution. Basically, this amounts to reordering the customer visit considering only 2 cities at a time. This is carried out to ensure that there is no possibility to shorten the tour length based on some neighborhood reordering.

For the min-max SDVRP, the definitions of these parameters are similar to the TSP problem. The heuristic desirability of visiting city j from city i (or the visibility) is represented by η_{ij} and is equal to reciprocal of d_{ij}:

$$\eta_{ij} = \frac{1}{d_{ij}} \tag{2}$$

The probabilistic transition rule is indicated by p_{ij} which represents the probability of choosing city j from city i, and is given by:

$$p_{ij} = \begin{cases} \dfrac{[\tau_{ij}]^{\alpha}[\eta_{ij}]^{\beta}}{\sum_{h\in\Omega}[\tau_{ih}]^{\alpha}[\eta_{ih}]^{\beta}} & if\ v_j \in \Omega \end{cases} \tag{3}$$

$$else \quad p_{ij} = 0$$

Here, τ_{ij} is the pheromone concentration on path from i to j, $\Omega \in V$ is a set of feasible cities and it includes cities that have not been visited. The parameters α and β represent the biases for pheromone trail and visibility respectively, and they represent parameters to weigh pheromone concentration (which represents learnt knowledge for more global solution) with respect to visibility (which represents local heuristic desirability) in the transition rule.

The pheromone update rule is given by:

$$\tau_{ij}^{new} = \rho\tau_{ij}^{old} + \sum_{k=1}^{m}\Delta\tau_{ij}^{k} + \sigma\Delta\tau_{ij}^{*} \tag{4}$$

The details of individual terms of Eq. 4 are explained in the algorithm shown in Table 1.

To solve SDVRP, the artificial ants construct vehicle routes by successively choosing the next cities using probabilistic transition rule shown in Eq. 3. If the probability 'p' value for transition to a particular j^{th} city from the i^{th} city is more, then chances of that city getting chosen as the next city is high. During every transition, the tour length is calculated and whenever the choice of next city would lead to an infeasible solution i.e., if the tour length exceeds the vehicle distance constraint (the maximum distance that a vehicle can travel) L, then the depot is chosen as the next city (so that the tour of that vehicle is finished) and a new tour is started for a new vehicle. It can be seen that the maximal length of a tour in this method will be L, and hence this serves as a critical parameter in finding out the min-max solution to the proposed problem. This parameter, as proven by Carlsson et al. [6], is bounded between two values as per their following lemma:

Lemma 1: For a general planar graph representing the SDVRP,

$$\frac{TSP(D \cup N) - TSP(D)}{k} \le L \le \frac{TSP(N)}{k} + 2 * d(D,N) \tag{5}$$

In Eq. 5, N denotes the set of customer points, D denotes set of depots (in our case, D=1), k denotes number of vehicles in a depot, $d(A,B)$ denotes the largest distance between an arbitrary pair of points in two different sets A and B i.e., $d(A,B) = \max_{x \in A, y \in B} \|x - y\|$. A proof of the above lemma is provided in reference [6].

Since L represents the distance constraint, it is the maximum distance travelled by a vehicle. Our approach is based upon finding the least value of L, called L^*, that will still lead to a valid solution of the traditional SDVRP. This solution of traditional SDVRP will also be the solution of the min-max SDVRP. In this paper, the above lemma has been used to guide the search of L^*. A high level description of the algorithm used to solve the min-max SDVRP is shown in Table I.

V. Results and Discussions

A MATLAB based program was developed to implement

TABLE I

Ant Colony Optimization Algorithm to Solve Min-Max SDVRP

/*Initialization*/

1. V = set of x,y coordinates of the cities, V_0 = x,y coordinate of the depot

2. $L\_max = \dfrac{TSP(V)}{k} + 2 * d(V_0, V)$

3. $L\_min = \dfrac{TSP(V \cup V_0)}{k}$

4. **While** ($L > L\_min$)

a. **For** every edge (i,j) do

$\tau_{ij}(0) = \tau_0$

End For

/* Main Loop*/

b. **For** $t = 1$ to t_max do

Initialize *Length_tour* and tour T^k

For $k = 1$ to m do

Build tour T^k (t) by applying $n - 1$ times the following Step

Choose the next city j with probability

$$p_{ij} = \frac{[\tau_{ij}]^{\alpha}[\eta_{ij}]^{\beta}}{\sum_{h \in \Omega}[\tau_{ih}]^{\alpha}[\eta_{ih}]^{\beta}} \quad \text{if } v_j \in \Omega$$

$$else \quad p_{ij} = 0$$

Calculate length of the tour *Length_tour*

If $Length\_tour + d0(j) > L$

Choose depot as the next city

Calculate the *Length_tour* accordingly

End If

End For

Find minimum of *Length _tour* and tour T

If an improved tour is found **then**

Update it as *T_opt* and *L_opt*

End If

Count the number of vehic, *no_of_vehic* based on number of times the ant has visited depot in the tour *T_opt* and also find tour of each vehicle, *L_opt_vehic*

Using 2-opt Heuristic, if possible Obtain a better *T_opt* and *L_opt*

For every edge (i,j) do

Update pheromone trails by applying the rule:

$$\tau_{ij}^{new} = \rho\tau_{ij}^{old} + \sum_{k=1}^{m} \Delta\tau_{ij}^{k} + \sigma\Delta\tau_{ij}^{*}$$

$$\text{Where } \Delta\tau_{ij}^{k} = \begin{cases} \dfrac{Q}{Length\_tour} & if\,(i,j) \in T \\ 0 & \text{otherwise} \end{cases}$$

$$\text{and } \Delta\tau_{ij}^{*} = \begin{cases} \dfrac{Q}{L\_opt} & if\,(i,j) \in T\_opt \\ 0 & \text{otherwise} \end{cases}$$

End For **(define Q above)**

End For

c. **If** $no\_of\_vehic \le k$, where k is the number of vehicles in the depot

$L = L - \delta L$ *and* ***go to*** 'b' until $no\_of\_vehic = k+1$

and the respective L is L^*

Else $L = L + \delta L$

End If

Ant Colony Optimization Algorithm to Solve Min-Max SDVRP Contd.

End While

5.**Print** *T_opt*, *L_opt*, L^*

6.**Stop**

m=no. of ants, n=no. of cities, $\alpha = 1$, $\beta = 5$, $\rho = 0.25$, $Q = 100$, $\tau_0 = 10^{-6}$, $\sigma = 5$

the proposed solution to the min-max SDVRP. The proposed algorithm was verified using a number of simulations. Here, we present results from three simulated scenarios: i) with 15 cities; ii) with 18 cities; iii) with 25 cities. Each of the above scenarios has been tested for 1-4 vehicles. For each of the above scenario, the locations of the cities were obtained using uniform random distribution on a 100 unit X 100 unit planar region. In the figures presented in this section the x and y axes correspond to axes of the planar region, and the circles represent the locations of cities in the region. We also present a comparison of our results with that obtained from the method proposed by Carlsson et al [6]. A MATLAB program that implements the Carlsson's Linear Programming based algorithm is freely available over the internet [11], and has been used in this paper for the comparison purposes. It may be noted that their program does not work for 2 vehicle problems, and hence the results provided here do not include their results for 2 vehicle problems.

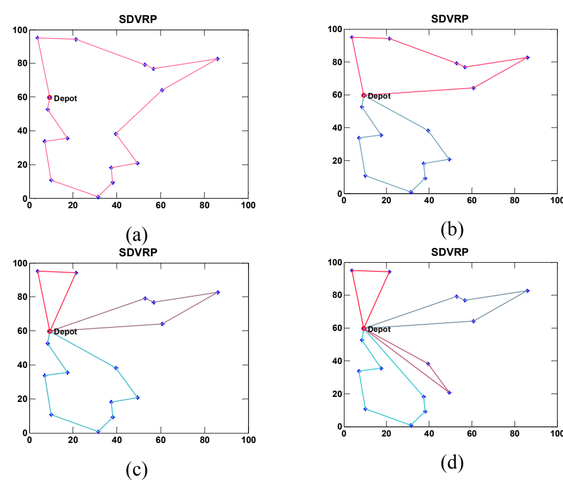


Fig. 1. Results obtained using the proposed ant colony based approach for scenario (i).

Figs. 1 show the solution obtained for scenario (i). Fig. 1a was obtained when the vehicle distance constraint (the distance it can travel) was *L_max*. This particular case is equivalent to TSP because there is only one vehicle involved with a capacity more than TSP.The same can be observed through the results obtained. The distance travelled by that vehicle is the TSP distance which is 323.59. Fig. 1b was obtained when the number of vehicles *(no_of_vehic)* was set to 2. Here the L^* was found to be

230. In this case, vehicle 1 travels 173 units of distance, while vehicle 2 travels 205 units of distance with the total distance travelled being 378. Fig. 1c was obtained when the number of vehicles was set to 3. In this case, L^* obtained was 180. The total distance travelled, however, increases to 427 from 378. Here, vehicle 1 travels 173 units, vehicle 2 travels 164 units, and vehicle 3 travels 89 units. Fig. 1d was obtained when the number of vehicles was set to 4. L^*

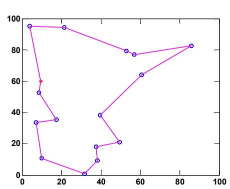
TABLE II

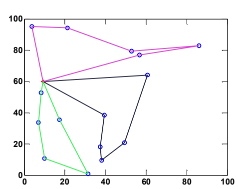
COMPARSION OF ANT COLONY BASED AND LP BASED TECHNIQUE TO SOLVE MIN-MAX SDVRP FOR A 15 CITY PROBLEM

No. of Vehicles	Distance travelled(*L_opt_vehic*) in case of Ant Colony based method	Distance travelled by vehicles in case of LP based method
1	Vehic 1 = 323.59*	Vehic 1 = 323.59*
2	Vehic 1 = 173 Vehic 2 = 205**	-
3	Vehic 1 = 173** Vehic 2 = 164 Vehic 3 = 89	Vehic 1 = 178 Vehic 2 = 201** Vehic 3 = 136
4	Vehic 1 = 154 Vehic 2 = 89 Vehic 3 = 164** Vehic 4 - 112	Vehic 1 = 143 Vehic 2 = 211** Vehic 3 = 101 Vehic 4 = 133

* denotes the TSP solution for the problem
** denotes the maximum distance travelled by a vehicle in a set of vehicle routes.

obtained in this case was 170 with vehicle 1 travelling 154 units, vehicle 2 travelling 89 units, vehicle 3 travelling 164 units, and vehicle 4 travelling 112 units. Fig 2 show the corresponding results obtained from Carlsson's Linear Programming (LP) based method. Fig. 2a shows the 1 vehicle case and it can be observed that it is same as Fig. 1a which is nothing but the TSP solution and the total distance being TSP length which is 323.59 units.


(a)


(b)

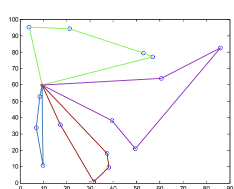

(c)

Fig. 2. Results obtained using LP based approach developed by Carlsson et al. for scenario (i).

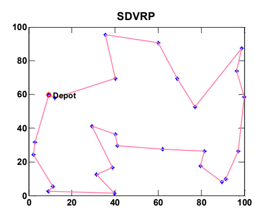
This program had some problems in obtaining solution in the case of 2 vehicles. Fig. 2b shows the 3 vehicles case where the lengths traversed by vehicles are: Vehicle 1: 178 units, vehicle 2: 201 units, and vehicle 3: 136 units.

TABLE III

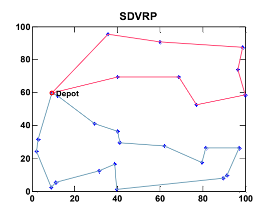
COMPARSION OF ANT COLONY BASED AND LP BASED TECHNIQUE TO SOLVE MIN-MAX SDVRP FOR A 18 CITY PROBLEM

No. of Vehicles	Distance travelled(*L_opt_vehic*) in case of Ant Colony based method	Distance travelled by vehicles in case of LP based method
1	Vehic 1 = 340.14*	Vehic 1 = 340.14*
2	Vehic 1 = 258** Vehic 2 = 189	-
3	Vehic 1 = 135 Vehic 2 = 219** Vehic 3 = 197	Vehic 1 = 184 Vehic 2 = 184 Vehic 3 = 229**
4	Vehic 1 = 135 Vehic 2 = 189 Vehic 3 = 183 Vehic 4 = 197**	Vehic 1 = 168 Vehic 2 = 181 Vehic 3 = 213** Vehic 4 = 198

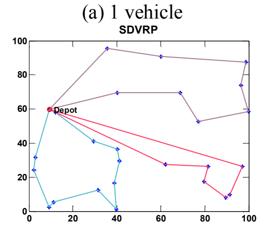
* denotes the TSP solution for the problem
** denotes the maximum distance travelled by a vehicle in a set of vehicle routes.



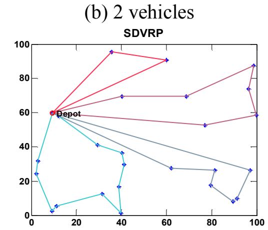
(a) 1 vehicle



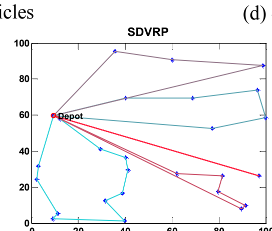
(b) 2 vehicles



(c) 3 vehicles



(d) 4 vehicles



(e) 5 vehicles

Fig. 3. Results obtained using the proposed ant colony based approach for scenario (iii).

Hence, total length is 515 and maximum length being 201 which is of vehicle 2.

Fig. 2c shows the 4 vehicles case. The lengths traversed in this case are vehicle 1: 143 units, vehicle 2: 211 units, vehicle 3: 101 units and vehicle 4:133 units. Thereby, total length is equal to 588 units and maximum length is 211 units. Table II compares the results obtained from the proposed ant colony based method with respect to the LP based method.

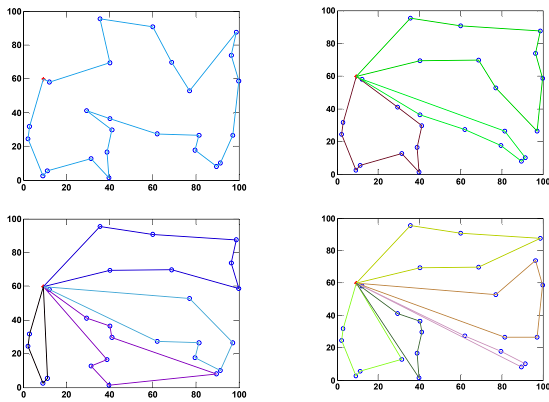


Fig. 4. Results obtained using LP based approach developed by Carlsson et al. for scenario (iii).

TABLE IV

COMPARSION OF ANT COLONY BASED AND LP BASED TECHNIQUE TO SOLVE MIN-MAX SDVRP FOR A 25 CITY PROBLEM

No. of Vehicles	Distance travelled(*L_opt_vehic*) in case of Ant Colony based method	Distance travelled by vehicles in case of LP based method
1	Vehic 1 = 471.17*	Vehic 1 = 470.89*
2	Vehic 1 = 290** Vehic 2 = 240	-
3	Vehic 1 = 173 Vehic 2 = 240** Vehic 3 = 217	Vehic 1 = 171 Vehic 2 = 282** Vehic 3 = 197
4	Vehic 1 = 173 Vehic 2 = 217** Vehic 3 = 216 Vehic 4 = 128	Vehic 1 = 231 Vehic 2 = 224** Vehic 3 = 222 Vehic 4 = 117
5	Vehic 1 = 173 Vehic 2 = 196 Vehic 3 = 201 Vehic 4 = 202** Vehic 5 = 187	Vehic 1 = 136 Vehic 2 = 239** Vehic 3 = 194 Vehic 4 = 203 Vehic 5 = 140

* denotes the TSP solution for the problem
** denotes the maximum distance travelled by a vehicle in a set of vehicle routes.

For scenarios (ii) and (iii), a similar simulation exercise has been done and the results obtained are compared with respect Carlsson's LP based method as shown in Tables III and IV. For the scenario (iii), the actual routes obtained by the proposed ant colony method and the Carlsson's LP based method are shown in Figures 3 and 4 respectively. It is quite clear from the comparison made with the Carlsson's method that the proposed ant colony based method provides more optimal results. However, this method takes more computational time to converge to the solution.

VI. CONCLUSION & FUTURE WORK

The paper presents an ant colony optimization based algorithm to solve an interesting class of problem called min-max SDVRP. Unlike a traditional SDVRP, which minimizes the total distance travelled, the min-max SDVRP minimizes the maximal distance travelled by a vehicle. This version of problem holds immense applications for time-critical problems. The proposed approach makes use of the distance constraint in traditional SDVRP to find its optimal value. This optimal distance constraint, when used in a traditional SDVRP, minimizes the maximal distance travelled by a vehicle. Verification of the proposed technique has been carried with the help of extensive simulations, and results have been compared with the results obtained using a known LP based algorithm. It has been established that the results obtained using the proposed ant colony based approach provides more optimal results although takes more calculation time as compared to the LP based algorithm when the number of cities is in the range of 15- 25. Future work includes extending the solution to multi-depot problem. Also, implementation of some of the heuristics like 2-opt or 3-opt with the candidate list method are planned in order to expedite convergence towards the solution.

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