

# $\mathcal{H}_2$ Performance Bounds for Voltage Regulation on a Spatially-Invariant DC Power Grid

Michael Rinehart, Mardavij Roozbehani, and Munther A. Dahleh

**Abstract**—Despite the importance of voltage regulation to operational reliability and safety of power grids, little is understood about how the topology of the physical layer, and the architecture of the communication and control layer impact the ability of the system to regulate the voltages. In this paper, we apply spatial-invariance analysis to develop analytical upper and lower bounds for the  $\mathcal{H}_2$  gain of a DC power grid with respect to voltage regulation. These bounds establish a relation between network architecture and performance, and reveal fundamental limits to performance that cannot be overcome using feedback control.

## I. INTRODUCTION

Voltage regulation is critical to the safety and quality of service in electricity transmission and distribution grids, and complex, hierarchical control policies are used to guarantee that the line voltages are within acceptable limits. Real-world control schemes for regulating both the voltages and frequency of the power grid are outlined in [1], [2], [3], [4], and they involve a combination of localized automatic control and system-wide control mechanisms. The sensitivity of voltage regulation (as well as frequency regulation) to load disturbances in the network also has had the broader effect of limiting the adoption of renewable energy sources, which contribute variable power to the grid [5]. While some recent advances have been made in addressing the challenges in integration of the renewable resources [6], this remains an active area of research.

Despite the importance of voltage regulation, little is known about the effect of the structure of the power grid on the ability to regulate voltage. This is in part due to the fact that the analysis of large-scale networks is computationally demanding, and, to date, despite much advances in theory, there are no effective tools that simplify the analysis of asymmetric and possibly heterogeneous networks. Furthermore, given the state of the art in decentralized control, it is not currently understood how network topology, the ratio of distribution points to suppliers, and demand uncertainty impact the ability to regulate voltages. In this paper, we examine these effects within a mathematical framework that allows the large-scale nature of the power grid to be viewed from local interactions, yielding analytic performance bounds for the closed-loop power grid. Specifically, the  $\mathcal{H}_2$  gain from demand disturbances to voltage error is related to the

amount of network interconnectedness, the ratio of distribution points to suppliers, demand variation, and transmission-line impedance, and fundamental limits for performance (independent of the feedback law) are determined using only these factors. This work has immediate application in cost-benefit analyses on the fundamental limits on integration of renewable energy sources into the current power grid. Furthermore, the research reveals the relationship between the transmission-line impedances and the range of voltages that can be achieved at the regulation points of the grid, opening the possibility for new approaches to power grid control and design.

In the last decade, numerous techniques for analysis and design of controllers for decentralized systems have emerged. Of particular relevance to the developments in this paper are the analysis techniques for spatially-invariant systems, first discussed in [7], and further developed in [8] for distributed control of systems representable by fractional transformations on spatial and temporal operators, in [9] for distributed control design for systems with group symmetry, and in [10] for developing analytic performance bounds for platoons of agents. In this paper, we take a similar approach to [10] in establishing  $\mathcal{H}_2$  performance bounds for voltage regulation. The power grid model we study is a spatially-invariant DC power grid model. Spatial invariance yields a tractable model for analysis and obtaining analytical performance bounds as it allows us to apply the multi-dimensional Discrete Fourier Transform to generate algebraic relationships between the system's performance and the network and controller parameters. We consider a direct current (DC) power grid because it simplifies analysis by removing extraneous states from the system, and because of the growing importance of DC systems in real-world power networks [11]. The approach developed in this paper is in principle extendable to voltage regulation in AC power grids at the expense of a more involved notation and technical development. This can be done for instance by using linearized models of AC power flow in rectangular form [12], [13].

The outline of this paper is as follows. In Section II, we review spatial invariance analysis and the multi-dimensional Discrete Fourier Transform. The interested reader may consult [14], [15] for comprehensive introduction to the theory. In Section III, we develop a model for a DC power grid that satisfies the spatial-invariance property, and in Section IV, we develop upper and lower performance bounds for the network that depend only on network parameters. Finally, we offer some closing remarks in Section V.

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## II. DEFINITIONS, NOTATION, AND FORMULATION

### A. Spatially-Invariant Operators and the Multidimensional Discrete Fourier Transform

To construct a tractable finite model for the power grid that is amenable to analysis, we structure the power grid as a multidimensional torus. We denote the discrete torus over the integers  $\{1, \dots, N\}$  as  $\mathbb{Z}_N$ , and the  $d$ -dimensional torus as  $\mathbb{Z}_N^d$ . An element  $k \in \mathbb{Z}_N^d$  can be expressed as a vector  $k = [k_1, k_2, \dots, k_d]^T$  with  $k_i \in \{1, \dots, N\}$ .

We define the shift operator  $T_i$  along the  $i^{\text{th}}$  coordinate over the multi-dimensional torus as follows: for any function  $f$  over  $\mathbb{Z}_N^d$ ,  $(T_i f)(k) = f(k + e_i)$ . This leads to our first definition regarding spatially-invariant operators.

*Definition 1:* An operator  $O$  is spatially-invariant if  $O(T_i f) = T_i O(f)$  for all  $i$  and for all functions  $f$  over  $\mathbb{Z}_N^d$ .

Just as a time-invariant operator can be expressed in terms of an impulse response that is convolved against a time-domain signal, a spatially-invariant operator  $O$  can be expressed as an array of functions  $\bar{O}$  defined over the torus  $\mathbb{Z}_N^d$  [10], so that

$$(Of)(k) = (\bar{O} * f)(k) = \sum_{l \in \mathbb{Z}_N^d} \bar{O}(k-l)f(l).$$

Because the discrete torus is a compact abelian group, the Discrete Fourier Transform (DFT) [10], [14] can be applied to any function  $f : \mathbb{Z}_N^d \rightarrow \mathbb{C}$ . The DFT of a function  $f$  over the multi-dimensional torus is defined as

$$\hat{f}(n) = \sum_{k \in \mathbb{Z}_N^d} f(k) e^{-j \frac{2\pi}{N} n \cdot k}, \quad \forall n \in \mathbb{Z}_N^d \quad (1)$$

where,  $j$  is the imaginary unit and  $n \cdot k = \sum_i n_i k_i$ . We denote  $\mathcal{F}$  as the DFT operator, and for any appropriate function  $f$ , we let  $\hat{f} = \mathcal{F}(f)$  be its Fourier transform<sup>1</sup>.

The inverse DFT  $\mathcal{F}^{-1}$  is given by a similar transformation:

$$f(k) = \frac{1}{N^d} \sum_{n \in \mathbb{Z}_N^d} \hat{f}(n) e^{j \frac{2\pi}{N} n \cdot k}, \quad \forall k \in \mathbb{Z}_N^d. \quad (2)$$

Just as the DFT can be used to transform convolution in time into pointwise multiplication in the frequency domain, it can be used to transform spatial convolution in space to multiplication over spatial frequencies. Specifically,

$$\mathcal{F}(\bar{O} * f) = \mathcal{F}(\bar{O})\mathcal{F}(f).$$

### B. Spatially-Invariant LTI Systems

Now, consider a linear time-invariant (LTI) system over the multi-dimensional torus:

$$\begin{aligned} \frac{d}{dt} x(k, t) &= A(k)x(t) + B(k)u(t) + E(k)w(t) \\ y(k, t) &= C(k)x(t) \end{aligned} \quad (3)$$

<sup>1</sup>Note the similarity to the standard DFT used for discrete-time periodic signals. An  $N$ -periodic discrete-time signal over  $\mathbb{Z}$  is simply a function over  $\mathbb{Z}_N$ .

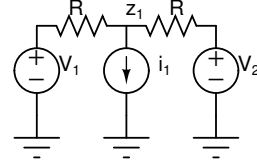


Fig. 1: A portion of a one dimensional DC power grid. The voltage generators  $V_1$  and  $V_2$  attempt to regulate the load voltage  $z_1$  subject to a random load current  $i_1$ .

where  $k \in \mathbb{Z}_N^d$ , and the state  $x(k, t)$ , the control input  $u(k, t)$ , and the noise  $w(k, t)$  at each  $k$  are real vectors. We denote  $x(t) = (x(k, t))_k$  as a stacked vector, and  $u(t)$  and  $w(t)$  are defined similarly. Note that the state  $x(k, t)$  at location  $k$  evolves as a function of the state  $x(k', t)$  at all locations  $k'$  (as well as of the inputs).

Generally, if there are many points in the torus (that is,  $N^d$  is large), analysis of (3) can be very difficult. However, we can simplify the analysis if we assume that it is spatially invariant.

*Definition 2:* System (3) is spatially-invariant if there exist spatially-invariant operators  $A, B, E$ , and  $C$  so that

$$\begin{aligned} \frac{d}{dt} x(t) &= \bar{A} * x(t) + \bar{B} * u(t) + \bar{E} * w(t) \\ y(t) &= \bar{C} * x(t). \end{aligned} \quad (4)$$

Intuitively, spatial-invariance implies that the state  $x(k)$  at  $k$  “reacts” to its neighbor  $x(k')$  exactly as  $x(k+s)$  reacts to its neighbor  $x(k'+s)$ . Because the operators are now spatially-invariant, we can take DFTs on both sides of (4), yielding  $N^d$  decoupled systems:

$$\begin{aligned} \frac{d}{dt} \hat{x}(n, t) &= \hat{A}(n)\hat{x}(n, t) + \hat{B}(n)\hat{u}(n, t) + \hat{E}(n)\hat{w}(n, t) \\ \hat{y}(n, t) &= \hat{C}(n)\hat{x}(n, t). \end{aligned} \quad (5)$$

## III. A SPATIALLY-INVARIANT MODEL FOR A DC POWER GRID

The primary goal of this work is to study the performance limits of voltage regulation in terms of the networks’ parameters. To this end, we must develop a model for a DC power grid that is amenable to analysis. The spatial invariance properties discussed previously will be a cornerstone to this analysis.

Figure 1 illustrates a portion of a 1-dimensional DC power grid. The generators are voltage sources  $V_k$ , and the load is a current source that draws a random current  $i_1$  out of the network. The goal of the network is to control the generator voltages so that the output voltage  $z_1$  is zero. For a given  $V_1, V_2$ , and  $i_1$ , the output voltage  $z_1$  is

$$z_1 = \frac{1}{2}(V_1 + V_2) - \frac{R}{2}i_1.$$

We do not assume that the generator voltages  $V_k$  can be controlled instantaneously, but rather have integrator dynam-

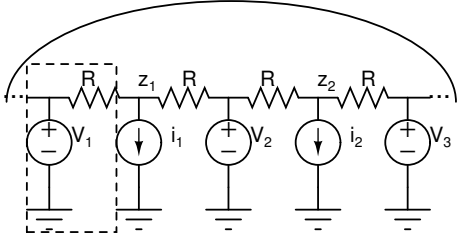


Fig. 2: A one dimensional DC power grid. The dashed box corresponds to an element of the grid. Note that because the analysis is over a circle, the circuit must close on itself.

ics of the form

$$\frac{d}{dt}V_k(t) = u_k(t)$$

where  $u_k$  is a control signal that is to be designed. Generally, constraining  $|u_k|$  would limit the ramp-time of these voltage sources, though we will not place such constraints in our analysis.

Finally, the current  $i_1$  is given by a noise-driven stable system

$$\frac{d}{dt}i_1(t) = -\alpha i_1(t) + \beta w_1(t)$$

where  $\alpha > 0$  and  $w_1$  is a noise source.

Our choice of subscripts is intentional. We are associating the left generator with the load to its right. If we expand the system to a 1-dimensional network of generators and loads (as shown in Figure 2), we can write the dynamics rather conveniently as

$$\begin{aligned} \frac{d}{dt}V_k(t) &= u_k(t) \\ \frac{d}{dt}i_k(t) &= -\alpha i_k(t) + \beta w_k(t) \\ z_k(t) &= V_k(t) + V_{k+1}(t) - Ri_k(t) \end{aligned}$$

where we removed the  $\frac{1}{2}$ -factor from the output for simplicity. One can immediately see that if we define

$$\begin{aligned} \bar{A}(k) &= \begin{cases} \begin{bmatrix} 0 & 0 \\ 0 & -\alpha \end{bmatrix}, & k = 0 \\ 0 & , \text{ otherwise} \end{cases} \\ \bar{B}(k) &= \begin{cases} \begin{bmatrix} 1 \\ 0 \end{bmatrix}, & k = 0 \\ 0 & , \text{ otherwise} \end{cases} \\ \bar{E}(k) &= \begin{cases} \begin{bmatrix} 0 \\ \beta \end{bmatrix}, & k = 0 \\ 0 & , \text{ otherwise} \end{cases} \\ \bar{C}(k) &= \begin{cases} \begin{bmatrix} 1 & -R \end{bmatrix}, & k = 0 \\ \begin{bmatrix} 1 & 0 \end{bmatrix}, & k = -1 \\ 0 & , \text{ otherwise} \end{cases} \end{aligned}$$

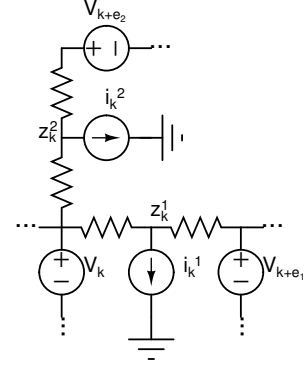


Fig. 3: A two dimensional DC power grid.

and further define  $x = [V \ i]^T$  (by stacking the elements), then the dynamics of the network are spatially-invariant in the form of (4).

Now we extend our notation to the multi-dimensional setting. We use the 2-dimensional illustration in Figure 3 to motivate the notation. Now, generator  $k$  is associated with two currents sources – the one above it, and the one to the right of it. The current to the right is in the  $e_1$  direction, so we denote it as  $i_k^1$ . The convention for  $i_k^2$  is similar. There are also two output voltages  $z_k^1$  and  $z_k^2$  associated with generator  $k$ . In  $d$ -dimensions, we extend this to  $d$  voltages and currents for each generator  $k$ . The general dynamics for this case follow from the 1-dimensional case and can be written as

$$\begin{aligned} \frac{d}{dt}V_k(t) &= u_k(t) \\ \frac{d}{dt}i_k^l(t) &= -\alpha i_k^l(t) + \beta w_k^l(t) \\ z_k^l(t) &= V_k(t) + V_{k+e_l}(t) - Ri_k^l(t). \end{aligned}$$

Once again, define  $w_k = [w_k^1 \cdots w_k^d]$ , and  $x = [V \ i]^T$ , where  $i_k = [i_k^1 \cdots i_k^d]^T$ .

Toward the goal of understanding the limits of linear feedback control on the DC power grid, we need to incorporate a parameterized feedback control law into our model. We do this by defining

$$u(t) = \bar{G} * V(t) + \bar{F} * i(t) = [\bar{G} \ \bar{F}] * x(t)$$

where  $\bar{G}$  and  $\bar{F}$  are spatially-invariant operators with  $\bar{G}(k) \in \mathfrak{R}$  and  $\bar{F}(k) \in \mathfrak{R}^{1 \times d}$ . We also make a reasonable assumption on the structure of these operators.

*Assumption 1:*  $\bar{G}(k) = \bar{G}(-k)$  and  $\bar{F}(k) = \bar{F}(-k)$ .

Symmetric operators not only simplify the analysis that is to follow, but they also make intuitive sense as the network itself is symmetric in all directions.

The definitions for the spatially-invariant operators in this case also follow immediately from the 1-dimensional case

and the definition of  $u(t)$ :

$$\begin{aligned}\bar{A}(k) &= \begin{cases} \begin{bmatrix} \bar{G}(0) & \bar{F}(0) \\ 0 & -\alpha I_d \end{bmatrix}, & k = 0 \\ \begin{bmatrix} \bar{G}(k) & \bar{F}(k) \\ 0 & 0 \end{bmatrix}, & \text{otherwise} \end{cases} \\ \bar{E}(k) &= \begin{cases} \begin{bmatrix} 0 \\ \beta I_d \end{bmatrix}, & k = 0 \\ 0, & \text{otherwise} \end{cases} \\ \bar{C}(k) &= \begin{cases} \begin{bmatrix} 1_d & -R I_d \end{bmatrix}, & k = 0 \\ e_i \begin{bmatrix} 1 & 0 \end{bmatrix}, & k = -e_i \\ 0, & \text{otherwise.} \end{cases}\end{aligned}\quad (6)$$

Note that we no longer require  $\bar{B}$  since  $u$  is a feedback control law; it is folded into  $\bar{A}$ . Finally, taking the DFT of these operators using (1), we get

$$\begin{aligned}\hat{A}(n) &= \begin{bmatrix} \hat{G}(n) & \hat{F}(n) \\ 0 & -\alpha I_d \end{bmatrix} \\ \hat{E}(n) &= \begin{bmatrix} 0 \\ \beta I_d \end{bmatrix} \\ \hat{C}(n) &= [1_d \quad -R I_d] + \sum_l e_l [1 \quad 0] e^{j \frac{2\pi}{N} n l}.\end{aligned}$$

#### IV. ANALYTIC $\mathcal{H}_2$ PERFORMANCE BOUNDS

##### A. Computing the $\mathcal{H}_2$ Gain

We refer the reader to [7] and [10] for a more detailed description of computing the  $\mathcal{H}_2$  gain of a spatially-invariant system, and we instead simply state the results directly.

Because the  $n^{\text{th}}$  system in (5) is an LTI system, its  $\mathcal{H}_2$  gain can be computed as

$$J_n = \text{Tr} \left( \hat{E}^*(n) P(n) \hat{E}(n) \right) \quad (7)$$

where  $M^*$  is our notation for the Hermitian-transpose of a matrix  $M$ , and the positive-semidefinite matrix  $P(n)$  satisfies the Lyapunov equation

$$\hat{A}^*(n) P(n) + P(n) \hat{A}(n) = -\hat{C}^*(n) \hat{C}(n). \quad (8)$$

The  $\mathcal{H}_2$  gain of the network is simply the sum of the individual system gains. However, we are interested in the average gain to each  $z_k^l$ , so, by symmetry, we are interested in computing

$$J = \frac{1}{N^d} \sum_{n \in \mathbb{Z}_N^d} J_n$$

(which is normalized by the number of building blocks). An analytic expression for  $J$  is given in the following lemma.

*Lemma 1:* The normalized  $\mathcal{H}_2$  gain of system (4) with stable dynamics given by (6) is

$$\begin{aligned}J &= \frac{\beta^2}{\alpha} \frac{1}{N^d} \left[ \frac{R^2 d N^d}{2} + \sum_{n \in \mathbb{Z}_N^d} \sum_{i=1}^d \frac{R (1 + \cos \frac{2\pi}{N} n_i)}{\hat{G}(n) - \alpha} \hat{F}_i(n) \right. \\ &\quad \left. + \sum_{n \in \mathbb{Z}_N^d} \sum_{i=1}^d \frac{d + \sum_{l=1}^d \cos \frac{2\pi}{N} n_l}{\hat{G}(n) (\hat{G}(n) - \alpha)} \hat{F}_i^2(n) \right]\end{aligned}$$

where  $\hat{F}(n) = [\hat{F}_1(n), \dots, \hat{F}_d(n)]$ . Also,  $\hat{G}(n) < 0$  is a necessary condition for closed-loop stability.

*Proof:* First, because the matrix  $\hat{A}(n)$  is a block matrix, we similarly partition  $P(n)$  as

$$P(n) = \begin{bmatrix} X(n) & Z(n) \\ Z^*(n) & Y(n) \end{bmatrix}. \quad (9)$$

Note that with this partitioning of  $P(n)$ ,  $J_n$  in (7) is simply given by  $J_n = \beta^2 \text{Tr}(Y_n)$ , which further only operates on the diagonal elements of  $Y_n$ .

We now begin to solve for  $P(n)$  in terms of  $A(n)$  using (8). Using the block partitioning for both matrices, we get for the upper-left hand equality:

$$\begin{aligned}2 \text{Re} \left\{ X(n) \hat{G}(n) \right\} &= -2d - 2 \sum_{l=1}^d \cos \frac{2\pi}{N} n_l \\ \Rightarrow X(n) &= \frac{-d - \sum_{l=1}^d \cos \frac{2\pi}{N} n_l}{\hat{G}(n)}\end{aligned}\quad (10)$$

where the implication follows from the fact that both  $\hat{G}(n)$  and  $X(n)$  are scalars,  $\hat{G}(n) \in \mathfrak{R}$  by Assumption 1, and  $P(n) = P^*(n)$ , meaning that  $\hat{X}(n)$  must be real. Also, since  $P(n) > 0$ ,  $X(n)$  must be nonnegative, meaning that  $\hat{G}(n) < 0$  is a necessary condition for stability of the closed-loop system.

Substituting this result into the upper-right hand (or lower-left hand) equality yields the following equality:

$$Z_i(n) = \frac{R \left( 1 + e^{-j \frac{2\pi}{N} n_i} \right)}{\hat{G}(n) - \alpha} + \frac{d + \sum_{l=1}^d \cos \frac{2\pi}{N} n_l}{\hat{G}(n) (\hat{G}(n) - \alpha)} \hat{F}_i(n). \quad (12)$$

Finally, substituting both of these results into the lower-right hand equality yields the following set of equalities for the diagonal elements of  $Y(n)$ :

$$\begin{aligned}Y_{ii}(n) &= \frac{1}{\alpha} \left[ \frac{R^2}{2} + \frac{R (1 + \cos \frac{2\pi}{N} n_i)}{\hat{G}(n) - \alpha} \hat{F}_i(n) \right. \\ &\quad \left. + \frac{d + \sum_{l=1}^d \cos \frac{2\pi}{N} n_l}{\hat{G}(n) (\hat{G}(n) - \alpha)} \hat{F}_i^2(n) \right].\end{aligned}\quad (13)$$

Multiplying by  $\beta^2$  and summing over  $i$  yields  $J_n$ . Further dividing by  $N^d$  and summing over  $n$  yields  $J$ .  $\blacksquare$

*Corollary 1:* The open-loop system's performance is

$$J_{open} = \frac{\beta^2 R^2 d}{2\alpha}.$$

*Proof:* Simply substitute  $\hat{F}_i = \hat{G} = 0$ . ■

### B. Fundamental Limits of Performance

We are interested in understanding the limits of disturbance rejection for the DC power grid. To this end, we seek to develop analytic bounds for  $J$  in terms of the grid's parameters. The following theorem provides a bound for performance that is independent of the feedback control law used, linear or otherwise.

*Theorem 1:* For any feedback control law,

$$J \geq \begin{cases} \frac{\beta^2 R^2}{2\alpha} \left[ \frac{(d-1)^2 N^d - 1}{N^d} + \frac{d}{N^d} \right], & N \text{ even} \\ \frac{\beta^2 R^2 (d-1)^2}{2\alpha d}, & N \text{ odd} \\ \geq \frac{\beta^2 R^2 (d-1)^2}{2\alpha d}. \end{cases}$$

*Proof:* To compute the lower bound, we solve for the optimal  $V^*(t)$  that minimizes  $\|z(t)\|_2$  subject to  $z(t) = \bar{C} * [V^*(t) \ i(t)]^T$ . We start by taking the DFT of the output equation:

$$\begin{aligned} \hat{z}(n) &= \hat{C}(n) [\hat{V}(n) \ \hat{i}(n)]^T \\ &= \sum_{l=1}^d e_l \left( 1 + e^{j\frac{2\pi}{N} n_l} \right) \hat{V}(n) - R \hat{i}(n). \end{aligned}$$

The optimizing  $\hat{V}^*(n)$  that minimizes  $\|\hat{z}(n)\|_2$  can be solved for by taking  $\hat{z} = 0$  and applying the Moore-Penrose inverse:

$$\hat{V}^*(n) = \frac{R \sum_{l=1}^d \left( 1 + e^{-j\frac{2\pi}{N} n_l} \right) e_l^T \hat{i}(n)}{2 \sum_{l=1}^d 1 + \cos \frac{2\pi}{N} n_l}.$$

The optimal output voltage is  $\hat{z}(n) = \hat{C}'(n) \hat{i}(n)$  where

$$\hat{C}'(n) = \begin{cases} R \left( \frac{1}{\sum_{i=1}^d m_{n_i}} \sum_{l=1}^d m_{n_l} e_l e_l^T - I \right), & n \neq \frac{\bar{N}}{2} \\ -RI, & n = \frac{\bar{N}}{2}. \end{cases}$$

where  $m_{n_l} = 1 + \cos \left( \frac{2\pi}{N} n_l \right)$  and  $\bar{N} = [N, N, \dots, N]^T$ . Note that if  $N$  is odd, the case  $n = \frac{\bar{N}}{2}$  does not occur. We now seek to analyze the  $\mathcal{H}_2$  gain of the system

$$\begin{aligned} \frac{d}{dt} \hat{i}(n) &= -\alpha I_d \hat{i} + \beta I_d \hat{w} \\ \hat{z}(n) &= \hat{C}'(n) \hat{i}(n). \end{aligned}$$

The solution  $\hat{P}(n)$  to the corresponding Lyapunov equation satisfies the following equality along its diagonal:

$$[\hat{P}(n)]_{ii} = \begin{cases} \frac{R^2}{2\alpha} \left[ 1 - 2 \frac{m_{n_i}}{\sum_{l=1}^d m_{n_l}} + \frac{m_{n_i}^2}{\left( \sum_{l=1}^d m_{n_l} \right)^2} \right], & n \neq \frac{\bar{N}}{2} \\ \frac{R^2}{2\alpha}, & n = \frac{\bar{N}}{2}. \end{cases}$$

Therefore,  $J_n = \beta^2 \text{Trace} \left( \hat{P}(n) \right)$  is

$$J_n = \begin{cases} \frac{\beta^2 R^2}{2\alpha} \left[ d - 2 + \frac{\sum_{i=1}^d m_{n_i}^2}{\left( \sum_{l=1}^d m_{n_l} \right)^2} \right], & n \neq \frac{\bar{N}}{2} \\ \frac{\beta^2 R^2}{2\alpha} d, & n = \frac{\bar{N}}{2}. \end{cases}$$

Applying Jensen's Inequality to the final term in  $n \neq \frac{\bar{N}}{2}$  case yields

$$J_n \geq \begin{cases} \frac{\beta^2 R^2}{2\alpha} \frac{(d-1)^2}{d}, & n \neq \frac{\bar{N}}{2} \\ \frac{\beta^2 R^2}{2\alpha} d, & n = \frac{\bar{N}}{2}. \end{cases}$$

Now, summing over  $n \in \mathbb{Z}_N^d$  gives us

$$\sum_{n \in \mathbb{Z}_N^d} J_n \geq \begin{cases} \frac{\beta^2 R^2}{2\alpha} \left( \frac{(d-1)^2}{d} (N^d - 1) + d \right), & N \text{ even} \\ \frac{\beta^2 R^2}{2\alpha} \frac{(d-1)^2}{d} N^d, & N \text{ odd}. \end{cases}$$

Dividing by  $N^d$  yields the bound for  $J$ . The second lower bound is obtained by using the fact that  $J_n \geq \frac{\beta^2 R^2 (d-1)^2}{2\alpha d}$  for all  $N$ . ■

The performance bounds are slightly worse if we restrict ourselves to linear feedback controllers. The following theorem provides the performance limits for this important case.

*Theorem 2:* For any stabilizing linear feedback control law,

$$J \geq \begin{cases} \frac{\beta^2 R^2 d}{4\alpha}, & 1 \leq d \leq 3 \\ \frac{\beta^2 R^2 (d-1)^2}{2\alpha d}, & d > 3. \end{cases} \quad (14)$$

*Proof:* First, since  $\hat{G}(n) < 0$  and  $\alpha > 0$ , we have the inequality  $\hat{G}(n)(\hat{G}(n) - \alpha) < \hat{G}^2(n)$ . Letting  $\hat{H}_i(n) = \frac{\hat{F}_i(n)}{\hat{G}(n)}$  gives us

$$\begin{aligned} J &> \frac{\beta^2}{\alpha} \frac{1}{N^d} \left[ \frac{R^2 d N^d}{2} + \sum_{n \in \mathbb{Z}_N^d} \sum_{i=1}^d R \left( 1 + \cos \frac{2\pi}{N} n_i \right) \hat{H}_i(n) \right. \\ &\quad \left. + \sum_{n \in \mathbb{Z}_N^d} \sum_{i=1}^d \left( d + \sum_{l=1}^d \cos \frac{2\pi}{N} n_l \right) \hat{H}_i^2(n) \right] \\ &= \frac{\beta^2}{\alpha} \frac{1}{N^d} \left[ \frac{R^2 d N^d}{2} + \sum_{n \in \mathbb{Z}_N^d} \sum_{i=1}^d R \left( 1 + \cos \frac{2\pi}{N} n_i \right) \hat{H}_i(n) \right. \\ &\quad \left. + \sum_{n \in \mathbb{Z}_N^d} \sum_{i=1}^d \left( 1 + \cos \frac{2\pi}{N} n_i \right) \hat{H}_i^2(n) \right. \\ &\quad \left. + \sum_{n \in \mathbb{Z}_N^d} \sum_{i=1}^d \sum_{l=1, l \neq i}^d \left( 1 + \cos \frac{2\pi}{N} n_l \right) \hat{H}_i^2(n) \right]. \end{aligned}$$

Because  $1 + \cos(x) \geq 0$ , we can drop the last term, yielding

$$J > \frac{\beta^2}{\alpha} \frac{1}{N^d} \left[ \frac{R^2 d N^d}{2} + \sum_{n \in \mathbb{Z}_N^d} \sum_{i=1}^d \left( 1 + \cos \frac{2\pi}{N} n_i \right) \left( R \hat{H}_i(n) + \hat{H}_i^2(n) \right) \right].$$

We can lower bound the expression by optimizing over each  $\hat{H}_i(n)$  individually. The optimizing value is  $\hat{H}_i(n) = -\frac{R}{2}$ . Substituting yields

$$J > \frac{\beta^2}{\alpha} \frac{1}{N^d} \left[ \frac{R^2 d N^d}{2} - \frac{R^2}{4} \sum_{n \in \mathbb{Z}_N^d} \sum_{i=1}^d \left( 1 + \cos \frac{2\pi}{N} n_i \right) \right] = \frac{\beta^2}{\alpha} \frac{1}{N^d} \left[ \frac{R^2 d N^d}{2} - \frac{R^2}{4} N^d d \right] = \frac{\beta^2 R^2 d}{4\alpha}.$$

where the first equality follows from the fact that the  $\cos(\cdot)$  term sums to zero. The remainder of the claim follows from comparing the two bounds. ■

### C. Performance under Finite Control Efforts

We now derive an analytic expression for the performance of a linear stabilizing control law. We proceed with presenting an upper bound on the performance.

*Theorem 3:* Let  $\gamma > 0$  and define

$$\hat{G}(n) = -\gamma\alpha, \quad \forall n \in \mathbb{Z}_N^d \quad (15)$$

$$\hat{F}_i(n) = \frac{R\gamma\alpha}{2d}, \quad \forall n \in \mathbb{Z}_N^d, \quad 1 \leq i \leq d. \quad (16)$$

Let  $V^{opt}$  be the optimal normalized  $\mathcal{H}_2$  gain of the closed loop system. Then

- (i) If  $d > 1$ , then the matrix  $P(n)$  defined by (9)–(13) and (15)–(16) is positive definite for all  $n \in \mathbb{Z}_N^d$ , and

$$V^{opt} \leq \frac{\beta^2 R^2 d}{4\alpha} \left( 2 - \frac{\gamma}{d(1+\gamma)} \right). \quad (17)$$

- (ii) If  $d = 1$ , then  $P(n)$  defined by (9)–(13) and (15) and  $\hat{F}_{\theta i}(n) = \theta \hat{F}_i(n)$  is positive definite for  $\theta \in (0, 1)$ , and

$$V^{opt} \leq \frac{\beta^2 R^2 d}{4\alpha} \left( 2 - \frac{\gamma(2\theta - \theta^2)}{d(1+\gamma)} \right) \quad (18)$$

*Proof:* For brevity, we prove the theorem for  $d > 1$ . We will make clarifying comments for the  $d = 1$  case. An upperbound for  $V^{opt}$  can be obtained by searching over all stabilizing controllers satisfying (15) and (16). Note that these also satisfy Assumption 1. Substituting  $\hat{G}(n)$  and  $\hat{F}_i(n)$  in Lemma 1 yields

$$V^{opt} \leq \frac{\beta^2}{\alpha N^d} \left[ \frac{R^2 d N^d}{2} - \frac{R^2 N^d \gamma}{2(\gamma+1)} + \frac{R^2 \gamma}{4d^2(\gamma+1)} \sum_{n \in \mathbb{Z}_N^d} \sum_{i=1}^d \sum_{l=1}^d \left( 1 + \cos \left( \frac{2\pi}{N} n_l \right) \right) \right].$$

Since

$$\sum_{n \in \mathbb{Z}_N^d} \sum_{i=1}^d \sum_{l=1}^d \left( 1 + \cos \left( \frac{2\pi}{N} n_l \right) \right) = N^d d^2,$$

we have

$$V^{opt} \leq \frac{\beta^2}{\alpha N^d} \left[ \frac{R^2 d N^d}{2} - \frac{R^2 N^d \gamma}{2(\gamma+1)} + \frac{R^2 N^d \gamma}{4(\gamma+1)} \right] = \frac{\beta^2 R^2 d}{4\alpha} \left[ 2 - \frac{\gamma}{d(\gamma+1)} \right],$$

which is exactly the bound claimed in (17). When  $d = 1$ , the upper bound (18) can be obtained by substituting  $\hat{G}(n)$  and  $\theta \hat{F}_i(n)$  in Lemma 1 and simplifying the expressions as done for the  $d > 1$  case. It remains to prove that  $P(n)$  is indeed positive definite. This is proven in Proposition 1 for  $d > 1$ , followed by remarks on the proof for the  $d = 1$  case. ■

*Proposition 1:* Let  $\gamma > 0$ , and  $\hat{G}(n)$  and  $\hat{F}_i(n)$  be defined as in (15) and (16). Then there exists  $\bar{\gamma} > 0$ , and a family of matrices  $P(n)$  satisfying (9)–(13) and

$$P(n) > 0, \quad \forall \gamma \geq \bar{\gamma}, \quad n \in \mathbb{Z}_N^d.$$

*Proof:* Without loss of generality we can restrict  $P(n)$  to the class of matrices with diagonal lower right block  $Y(n)$ . Let  $u = \sum_{i=1}^d 1 + \cos(2\pi n_i/N)$ ,  $t_i = (1 + \cos(2\pi n_i/N))/u$ , and  $s_i = \sin(2\pi n_i/N)/u$ , where, for convenience in notation, we have dropped the dependence of  $u$ ,  $t_i$ , and  $s_i$  on  $n$ . It can be then verified that

$$X(n) = \frac{u}{\gamma\alpha} \quad (19)$$

$$Z_i(n) = \frac{Ru(0.5d^{-1} - t_i + js_i)}{\alpha(1+\gamma)} \quad (20)$$

$$Y_{ii}(n) = \frac{R^2\gamma}{2\alpha(1+\gamma)} \left( \frac{u}{2d^2} - \frac{t_i u}{d} + \frac{\gamma+1}{\gamma} \right) \quad (21)$$

$$Y_{ir} = 0, \quad i \neq r. \quad (22)$$

The matrix  $P(n)$  defined by (19)–(22) is positive definite if and only if

$$\tilde{P}(n) = K^T P(n) K > 0$$

where,

$$K = \sqrt{\frac{\alpha(1+\gamma)}{\gamma u}} \begin{bmatrix} \gamma & 0 \\ 0 & 2R^{-1}dI_d \end{bmatrix}.$$

Let  $\tilde{X}(n)$ ,  $\tilde{Z}_i(n)$ , and  $\tilde{Y}(n)$  be the corresponding block elements of  $\tilde{P}(n)$ . Then

$$\tilde{X}(n) = 1 + \gamma$$

$$\tilde{Z}_i(n) = 1 - 2dt_i + 2jds_i$$

$$\tilde{Y}_{ii}(n) = 1 - 2dt_i + 2d^2u^{-1}(1 + \gamma^{-1})$$

$$\tilde{Y}_{ir} = 0, \quad i \neq r.$$

Note that  $\tilde{P}(n)$  is a complex matrix. Hence,  $\tilde{P}(n) > 0$  if and only if

$$W = \begin{bmatrix} \operatorname{Re}\{P(n)\} & \operatorname{Im}\{P(n)\} \\ -\operatorname{Im}\{P(n)\} & \operatorname{Re}\{P(n)\} \end{bmatrix} > 0.$$

After a permutation of certain rows and columns of  $W$ , we obtain  $\tilde{P}(n) > 0$  if and only if

$$\begin{bmatrix} 1+\gamma & 0 & T & S \\ 0 & 1+\gamma & -S & T \\ T^T & -S^T & \Gamma + \bar{T} & 0 \\ S^T & T^T & 0 & \Gamma + \bar{T} \end{bmatrix} > 0$$

where

$$S = [s_1 \cdots s_d] 2d,$$

$$T = \mathbf{1}_d^T - [t_1 \cdots t_d] 2d$$

$$\bar{T} = \operatorname{diag}(T), \quad \Gamma = 2d^2 u^{-1} (1 + \gamma^{-1}) \mathbf{I}_d.$$

Since  $t_i \in [0, 2u^{-1}]$  and  $u \in [0, 2d]$  a sufficient condition for positive definiteness is

$$\begin{bmatrix} 1+\gamma & 0 & T & S \\ 0 & 1+\gamma & -S & T \\ T^T & -S^T & \eta(\gamma, d) \mathbf{I}_d & 0 \\ S^T & T^T & 0 & \eta(\gamma, d) \mathbf{I}_d \end{bmatrix} > 0$$

where  $\eta(\gamma, d) = d\gamma^{-1} + d - 1$ . Taking the Schur Complement and using the fact that

$$\begin{bmatrix} T & S \\ -S & T \end{bmatrix} \begin{bmatrix} T & S \\ -S & T \end{bmatrix}^T \leq 8d^3 \mathbf{I}_d$$

(which follows from  $S_i^2 + T_i^2 \leq 8d^2$ , and subsequently  $TT^T + SS^T \leq 8d^3$ ), a sufficient condition is obtained as

$$1 + \gamma > \frac{8d^3}{d\gamma^{-1} + d - 1} \quad (23)$$

It can be seen that for any  $d > 1$ , (23) can be satisfied for a sufficiently large  $\gamma$ , as the left hand side goes to infinity whereas the right hand side remains bounded. In particular, it can be verified that  $\gamma > 3d$  is sufficient (when  $d > 1$ ). The threshold  $\bar{\gamma}$  is an increasing function of  $d$ . ■

*Remark 1:* If  $d = 1$ , condition (23) becomes

$$1 + \gamma > \frac{8}{\gamma^{-1} + h(\theta)} \quad (24)$$

where  $h(\theta)$  is a polynomial in  $\theta$  which is positive for  $\theta \in (0, 1)$  and zero for  $\theta = 1$ . Note that  $\theta < 1$  is necessary for existence of  $\gamma$  satisfying (24) and subsequently (18).

*Remark 2:* Although we did not discuss the computation of optimal control laws for spatially-invariant systems in general, it is appropriate to note how such controllers can be designed. The results are well known and were first described in [7]. Essentially, optimal controllers can be constructed by (a) computing an optimal feedback control law at each spatial frequency  $n$ , and (b) performing the inverse DFT to

recover the spatial-domain controller, which will be spatially-invariant. An interesting fact about this class of optimal controllers is that the dependence of the control signal on the network diminishes exponentially with distance. Using this fact, controllers using only localized information can be designed by, for example, truncating the control law to a finite radius.

## V. SUMMARY

In this paper, we explored the fundamental limits of voltage regulation on a spatially-invariant DC power grid model by developing upper and lower bounds for the (normalized)  $\mathcal{H}_2$  gain of the network. The bounds reveal fundamental performance limits of the network that cannot be overcome using feedback control. The research has immediate applications in understanding the fundamental limits in integration of renewable energy resources with variable output into the main power grid. Future work will include additional control elements including reactive loads and the limits of control using only local information.

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