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Model Predictive Control Formulation for a Class of Time-Varying Linear Parabolic PDEs

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***Abstract*— This paper considers the model predictive control (MPC) formulation for a class of discrete time-varying linear state-space model representations of parabolic partial differential equations (PDEs) with time-dependent parameters. The time-dependence of the parameters are due to the changes in physical properties or operating conditions of the system such as phase transformation, reactor catalyst fouling, and/or domain deformations which arise in many industrial processes. The MPC formulation is constructed for the low dimensional discrete finite-dimensional state space representation of the PDE system and constraints on input and infinite-dimensional state evolution are incorporated in the convex optimization algorithm. The underlying MPC synthesis is utilizing the appropriately defined model representation of the PDE and yields convex quadratic optimization problem which includes input and PDE state constraints. Using the illustrative example of a crystal growth process in which the time-varying property is associated with the evolution of grown crystal, the proposed time-varying MPC formulation is implemented for the optimal crystal temperature regulation problem under the presence of input and state constraints.**

I. INTRODUCTION

Partial differential equations (PDEs) with time-varying parameters represent an important class of physical process models. In particular, tubular reactors and packed bed reactors, and the (CZ) crystal growth process, are prime examples of distributed parameter systems in which parameters vary in time, for example the process of catalyst deactivation is the time-varying process which needs to be incorporated in the underlying PDE model [1], see Fig.1, while the time varying nature in the CZ crystal growth process is due to the growth of the domain described by the parabolic PDE, [2], [3]. However, even when the model accounts for the time varying nature of the system dynamics the set of tools available for analysis and subsequent control of time-varying dissipative systems described by the parabolic PDEs is significantly smaller relative to those available for time invariant models.

The distributive parameter systems modelled by the time varying parabolic PDEs are represented in the time-varying infinite dimensional state-space setting in [4]. The fundamental solution to the autonomous infinite dimensional time-varying differential equations (e.g. $\dot{x}(t) = \mathscr{A}(t)x(t)$) is given as the approximation of the fundamental solutions corresponding to the piecewise constant generators, see [4], [5]. In essence, the solution can been represented by the system generator consisting of a time-invariant generator and time-varying perturbation term [6]. Along these lines of work, the

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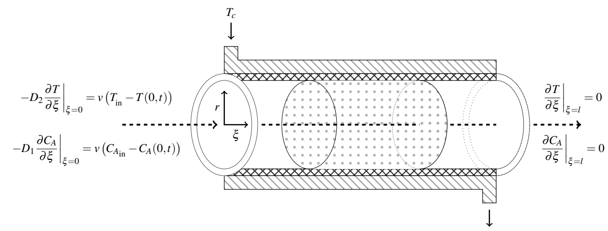

Fig. 1. Packed bed reactor

issue of stabilization of time varying systems is given in [7], for bounded ($\mathscr{A}(t)$) and in [8] for unbounded. A general stability theory has been provided in Hinrichsen and Pritchard [9] which also introduced the notion of largest bound such that the stability of system is preserved for all perturbations of norm less than the bound in a given perturbation set. Along the stabilization theme, the optimal control regulator synthesis for the time-varying infinite dimensional systems has been addressed by Curtain and Pritchard [10] and Lions [11]. However, in the time-varying infinite dimensional case there are few works which explore the Ricatti equation and general optimal control problem. On the other side, the linear time-varying discrete infinite dimensional systems were rarely consider in the model predictive control framework in which the optimality, input and/or state of PDE constraints satisfaction is required. The previous contributions to the model predictive control of time invariant parabolic PDEs addressed the low dimensional parabolic PDE representation that is given in the formulation of the cost functional while the infinite dimensional state satisfaction was incorporated by feasibility of the optimization,[12]. Motivated by the previous works, we look at the model predictive control approach to the optimal control of general linear parabolic PDE systems given through their representation as infinite-dimensional state space systems $\Sigma(\mathscr{A}, \mathscr{B}, -)$ on separable Hilbert spaces $\mathscr{X}$ and $\mathscr{U}$ with state evolution $x \in \mathscr{X}$ and input $u \in \mathscr{U}$, where $\mathscr{A}$ is an unbounded spectral operator with discrete spectrum $\sigma(\mathscr{A}) = \{\lambda_n, n \in \mathbb{N}\}$, normalized eigenvectors $\{\phi_n, n \in \mathbb{N}\}$ and input operator $\mathscr{B} \in \mathscr{L}(\mathscr{U}, \mathscr{X})$, see [13]. As an example, the parabolic PDE model:

$$\frac{\partial f}{\partial t}(\xi,t) = \alpha(t)\left(\frac{\partial^2 f}{\partial \xi^2}(\xi,t)\right) + Kf(\xi,t) + Q(\xi,t)$$
$$\frac{\partial f}{\partial t}(0,t) = 0, \quad \frac{\partial f}{\partial t}(l,t) = 0, \qquad f(\xi,0) = f_0 \qquad (1)$$

gives the dynamics of a reaction-diffusion process in which the diffusivity $0 < \alpha(t) < \infty$ of the domain $0 < \xi < l$ changes over a period of time, with zero-flux boundary conditions, and where K is the linearized generation term and $Q(\xi,t)$ is a spatially distributed input to the system with f_0 denoting the initial condition. The spatial operator of Eq.1 is given by the following:

$$\mathscr{A}\phi = \alpha(t)\frac{d^2\phi}{d\xi^2} + K\phi \tag{2}$$

with $\mathscr{D}(\mathscr{A}) := \{\phi \in \mathbb{L}_2(0,l)\,|\,\phi, \frac{d\phi}{d\xi}$ abs. cont. $\frac{d^2\phi}{d\xi^2} \in \mathbb{L}_2(0,l)$ and $\frac{d\phi}{d\xi}(0) = 0,\ \frac{d\phi}{d\xi}(l) = 0\}$ and has eigenvalues $\{\lambda_n, n \in \mathbb{N}\}$ and associated eigenvectors $\{\phi_n, n \in \mathbb{N}\}$ given by the following:

$$\lambda_n = -\alpha(t)\left(\frac{n\pi}{l}\right)^2 + K, \quad \phi_n(\xi) = \sqrt{\frac{2}{l}}\cos\left(\frac{n\pi}{l}\right) \tag{3}$$

with $\langle\phi_n, \phi_m\rangle = \delta_{nm}$ at each time $t > 0$. One can note that $\alpha(t)$ appears in the expression of the eigenvalues which are associated with the dynamics of a PDE system and consequently are influenced by the evolution of $\alpha(t)$. The input $Q(\xi,t) := b(\xi)u(t)$ where $u(t) \in \mathbb{R}$ so that $\mathscr{B} = \langle b(\xi), \phi_n\rangle$. Then for the state evolution $x(\cdot,t) = \{x(\xi,t), 0 \leq \xi \leq l\}$ with $x \in \mathbb{L}_2(0,l)$ for all $t \geq 0$, the infinite dimensional state space representation of Eq.1 is given by:

$$\dot{x}(t) = \mathscr{A}(t)x(t) + \mathscr{B}(t)u(t) \tag{4}$$

The optimal control of systems in the form of Eq.4 have been realized by the minimization of an associated quadratic cost function which, in some cases, for which closed form solutions for the optimal control law can be determined, see [14], [13]. However, the introduction of state and input constraints produce a level of complexity which does not permit analytic expressions for the optimal regulator and the model predictive control synthesis for this class of systems needs to be addressed.

Since, the model predictive control is discrete regulator synthesis the analogous discrete-time state space models originating from the modal infinite-dimensional state space system representation of the PDE can be obtained under the conditions of existence of an evolution system $(\Phi(t,s))$ generated by the continuous family of operators $\mathscr{A}(t)$. Therefore, the discretized version of Eq.4 emerges. Provided the above model formulation, one can consider the MPC synthesis for the following power stable infinite-dimensional discrete time-varying dynamical system, $\Sigma(A_k, B_k, -)$ given by:

$$x_{k+1} = A_k x_k + B_k u_k, \quad k = \{0,1,2,\dots\} \tag{5}$$

in which u_k is the vector of inputs, x_k is the vector of states.

Although, the Eq.5 refers to the discrete infinite dimensional system, we invoke the high level approximation of it in the construction of the subsequent constrained regulator formulation. In particular, we account for the low dimensional model representation in the construction of the cost functional while the infinite dimensional state is accounted in PDE state constraints satisfaction. In practical control applications the PDE system is typically approximated via modal decomposition utilizing appropriate basis functions where the dominant dynamics of the infinite dimensional system are captured within a low dimensional finite set of the first n modes, see [15]. We consider the MPC formulation as an extension of the work by *Muske & Rawlings (1993)* to time-varying systems, see [16], since we want to utilize the structural benefits of MPC algorithm in a case of decoupled state dynamics as in the case of Eq.. The MPC algorithm gives an input profile of N future control moves which is based on the minimization of the following infinite horizon open-loop quadratic objective function at time k:

$$\min_{u^N} \sum_{j=0}^{\infty} \left(x_{k+j}^T Q x_{k+j} + u_{k+j}^T R u_{k+j}\right) \tag{6}$$

where x_k state is low dimensional state representation of the PDE state, Q is symmetric positive semidefinite penalty matrix on states, R is symmetric positive definite penalty matrix on inputs, u_{k+j} is input vector at time j in the open-loop objective function, [17], [16]. The regulator calculates the vector u^N which contains the N future open-loop control moves starting from time $k = 0$:

$$u^N = \begin{pmatrix} u_k & u_{k+1} & \cdots & u_{k+N-1} \end{pmatrix}^T \tag{7}$$

which optimize the objective function in Eq.6. The first input u_k is sent to the plant and u^N is recalculated at each time step with $u_{k+j} = 0$ for all $j \geq N$. The infinite-horizon open-loop objective in Eq.6 function can be expressed as the finite horizon open-loop objective function:

$$\min_{u^N} \Phi_k = x_{k+N}^T \tilde{Q} x_{k+N} + \sum_{j=0}^{N-1} \left(x_{k+j}^T Q x_{k+j}^T + u_{k+j}^T R u_{k+j}\right) \tag{8}$$

The following section is motivated to provide a general MPC regulator formulation for a time-varying discrete linear system for which the time-dependence of the operators A_k and B_k are known functions of time with some estimate of the settling time index s such that $A_k = A_s$ is constant for all time $k \geq s$. This condition is required in the algorithmic realization of the MPC since, as it will be demonstrated, the expression for the terminal state penalty matrix $\tilde{Q}$ depends on the settling time s relative to the horizon N and the knowledge of s is necessary to compute the contribution of the infinite time horizon terms to $\tilde{Q}$. Given the example PDE system in Eq.1, the condition implies that the time varying parameter $\alpha(t)$ is known such that $\{\lambda_n, n \in \mathbb{N}\}$ and $\{\phi_n, n \in \mathbb{N}\}$ can be determined for $t \geq 0$.

II. Model Predictive Control (MPC) synthesis for time-varying infinite-dimensional system

The following results provide synthesis for a general class of discrete time-varying linear systems $\Sigma(A_k, B_k, -)$ arising from the discretization of Eq.4. In particular, we decompose the system's dynamics in low-dimensional and infinite dimensional complement by introducing the projection operator $\mathscr{P}$ such that $\tilde{A} = \mathscr{P}A$ and $\tilde{A}_f = (I - \mathscr{P})A$, $B = \mathscr{P}B$, $B_f = (I - \mathscr{P})B$. In other words, we obtain the state of parabolic system decomposed in finite $\tilde{x}$ and infinite subsystem $\tilde{x}_f$, so that the time-varying dynamics are given by:

$$\tilde{x}_{(k+1)} = \tilde{A}_{(k)}\tilde{x}_{(k)} + B_{(k)}u_{(k)}, \quad k = \{0,1,2,\dots\} \tag{9}$$

$$\tilde{x}_{f(k+1)} = \tilde{A}_{f(k)}\tilde{x}_{f(k)} + B_{f(k)}u_k, \tag{10}$$

The ensuing formulation of the MPC synthesis that deals with the construction of the cost functional deals only with the finite low dimensional state given by Eq.9.

A. MPC cost functional formulation

Theorem 2.1: For $s > k$ the terminal state penalty matrix $\tilde{Q}$ in Eq.8 is given by:

$$\tilde{Q} = \tilde{Q}^{(a)} + M\tilde{Q}^{(b)}M^T \tag{11}$$

where for L and M defined as:

$$L = \prod_{j=N}^{i} \tilde{A}^T_{k+j}, \quad M = \prod_{j=N}^{s-1} \tilde{A}^T_{k+j} \tag{12}$$

the matrices $\tilde{Q}^{(a)}$ and $\tilde{Q}^{(b)}$ are given by:

$$\tilde{Q}^{(a)} = Q + \sum_{i=k+N}^{s-2} LQL^T, \qquad \tilde{Q}^{(b)} = \sum_{i=0}^{\infty} \tilde{A}_s^{T^i} Q \tilde{A}_s^i \tag{13}$$

such that $\tilde{Q}^{(b)}$ is determined from the solution of the discrete Lyapunov equation:

$$\tilde{Q}^{(b)} = Q + \tilde{A}_s^T \tilde{Q}^{(b)} \tilde{A}_s \tag{14}$$

Proof: From Eq.8, the series expansion with respect to $\tilde{x}$ gives:

$$\tilde{x}^T_{k+N}\tilde{Q}\tilde{x}_{k+N} = \sum_{j=N}^{\infty}\left(\tilde{x}^T_{k+j}Q\tilde{x}_{k+j}\right) =$$

$$\tilde{x}^T_{k+N}Q\tilde{x}_{k+N} + \tilde{x}^T_{k+N+1}Q\tilde{x}_{k+N+1} + \tilde{x}^T_{k+N+i}Qx_{k+N+i} + \cdots$$

$$+ \tilde{x}^T_{s-1}Q\tilde{x}_{s-1} + \tilde{x}^T_s Q\tilde{x}_s + \tilde{x}^T_{s+1}Q\tilde{x}_{s+1} + \cdots$$

Since $\tilde{x}_{k+N+r} = \tilde{A}_{k+N+r-1}\tilde{A}_{k+N+r-2}\cdots\tilde{A}_{k+N}\tilde{x}_{k+N}$:

$$\tilde{x}^T_{k+N}\tilde{Q}\tilde{x}_{k+N} = \sum_{j=N}^{\infty}\left(\tilde{x}^T_{k+j}Q\tilde{x}_{k+j}\right)$$

$$= \tilde{x}^T_{k+N}\Big\{Q + \tilde{A}^T_{k+N}Q\tilde{A}_{k+N} + \tilde{A}^T_{k+N}\tilde{A}^T_{k+N+1}Q\tilde{A}_{k+N+1}\tilde{A}_{k+N} + \cdots$$
$$+ \tilde{A}^T_{k+N}\cdots\tilde{A}^T_{k+N+i-1}Q\tilde{A}_{k+N+i-1}\cdots\tilde{A}_{k+N} + \cdots$$
$$+ \tilde{A}^T_{k+N}\cdots\tilde{A}^T_{s-2}Q\tilde{A}_{s-2}\cdots\tilde{A}_{k+N} + \cdots$$
$$+ \tilde{A}^T_{k+N}\cdots\tilde{A}^T_{s-1}\left(Q + \tilde{A}^T_sQ\tilde{A}_s + \tilde{A}^T_s\tilde{A}^T_sQ\tilde{A}_s\tilde{A}_s + \right.$$
$$\left.\tilde{A}^T_s\tilde{A}^T_s\tilde{A}^T_sQ\tilde{A}_s\tilde{A}_s\tilde{A}_s + \cdots\right)\tilde{A}_{s-1}\cdots\tilde{A}_{k+N}\Big\}\tilde{x}_{k+N}$$

Collecting terms gives:

$$\tilde{x}^T_{k+N}\tilde{Q}\tilde{x}_{k+N} = \tilde{x}^T_{k+N}\left\{Q + \sum_{i=k+N}^{s-2}\left(\prod_{j=N}^{i}\tilde{A}^T_{k+j}\right)Q\left(\prod_{j=N}^{i}\tilde{A}^T_{k+j}\right)^T\right.$$
$$\left. + \left(\prod_{j=N}^{s-1}\tilde{A}^T_{k+j}\right)\left(\sum_{i=0}^{\infty}\tilde{A}_s^{T^i}Q\tilde{A}_s^i\right)\left(\prod_{j=N}^{s-1}\tilde{A}^T_{k+j}\right)^T\right\}x_{k+N}$$

which yields Eq.11 using the identities defined in Eqs.12-13. ■

Theorem 2.2: The general form of the quadratic program for u^N of Eq.8 is given by:

$$\min_{u^N}\Phi_k = (u^N)^T F u^N + 2(u^N)^T G\tilde{x}_k \tag{15}$$

where for $s > k$ the matrices F and G in Eq.15 have the following structures:

$$F = \begin{pmatrix} B_k^T\tilde{Q}_{[1]}B_k + R & B_k^T H_{[1]}\tilde{Q}_{[2]}B_{k+1} & \cdots & B_k^T H_{[N-2]}\tilde{Q}_{[N]}B_{k+N-2} & B_k^T H_{[N-1]}\tilde{Q}_{[N]}B_{k+N-1} \\ B^T_{k+1}\tilde{Q}_{[2]}H^T_{[1]}B_k & B^T_{k+1}\tilde{Q}_{[2]}B_{k+1} + R & \cdots & B^T_{k+1}H_{[N-2]}\tilde{Q}_{[N]}B_{k+N-2} & B^T_{k+1}H_{[N-1]}\tilde{Q}_{[N]}B_{k+N-1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ B^T_{k+N-1}\tilde{Q}_{[N]}H^T_{[N-1]}B_k & B^T_{k+N-1}\tilde{Q}_{[N]}H^T_{[N-2]}B_{k+1} & \cdots & B^T_{k+N-1}\tilde{Q}_{[N]}H^T_{[N-1]}B_{k+N-2} & R + B^T_{k+N-1}\tilde{Q}_{[N]}B_{k+N-1} \end{pmatrix}$$

$$G = \begin{pmatrix} B_k^T\tilde{Q}_{[1]}H^T_{[1]}\tilde{A}_k \\ B^T_{k+1}\tilde{Q}_{[2]}H^T_{[2]}\tilde{A}_k \\ \vdots \\ B^T_{k+N-1}\tilde{Q}_{[2]}H^T_{[N-1]}\tilde{A}_k \end{pmatrix} \tag{16}$$

For the following identities defined as:

$$L_{[p]} = \prod_{j=p}^{i}\tilde{A}^T_{k+j},\ M_{[p]} = \prod_{j=p}^{s-1}\tilde{A}^T_{k+j},\ H_{[q]} = \prod_{j=q}^{N-1}\tilde{A}^T_{k+j} \tag{17}$$

the elements of F are given by:

$$F_{mn} = \begin{cases} B^T_{k+q}\tilde{Q}_{[p]}B_{k+q} + R, & m = n \\ B^T_{k+q}\tilde{Q}_{[p]}H^T_{[q]}B_{k+q}, & m \neq n \end{cases} \tag{18}$$

for $\{m,n\} \in \{1,2,\ldots N\}$ and where $p = \{1,2,\ldots,N\}$ and $q = \{0,1,\ldots,N-1\}$ such that F is symmetric with $F^T_{mn} = F_{nm} = B^T_{k+q}H_{[q]}\tilde{Q}_{[p]}B_{k+q}$. and the state penalty matrix is given by:

$$\tilde{Q}_{[p]} = \tilde{Q}^{(a)}_{[p]} + M_{[p]}\tilde{Q}^{(b)}M^T_{[p]} \tag{19}$$

with components $\tilde{Q}^{(a)}_{[p]}$ and $\tilde{Q}^{(b)}$ defined as:

$$\tilde{Q}^{(a)}_{[p]} = Q + \sum_{i=k+p}^{s-2} L_{[p]}QL^T_{[p]}, \qquad \tilde{Q}^{(b)} = \sum_{i=0}^{\infty}\tilde{A}_s^{T^i}Q\tilde{A}_s^i \tag{20}$$

Proof: Algebraic manipulation of Eq.8 leads to the matrix structures of F and G with Eqs.19-20 determined from the recursion of the identities given in Eq.12 to those in Eq.17. ■

Remark 1: The quadratic minimization formulation in Theorem 2.2 is given for $s > k$. The structures of F, G and the state penalty matrix $\tilde{Q}_{[p]}$ are determined by the underlying dynamical nature of the system $\Sigma(\tilde{A}_k, B_k, -)$ in Eq.9. In particular, one can notice that the terminal state penalty matrix $\tilde{Q}$ is comprised of two components: $\tilde{Q}^{(a)}$ which corresponds to the contribution of A_k for time instances $r < s$, and $\tilde{Q}^{(b)}$ which corresponds to the contribution of $\tilde{A}_s$ for $r \geq s$. Since u^N is recalculated at each time step k, the interval of N control moves is shifted forward whereas the settling time remains fixed at time $k = s$ such that $\tilde{Q}_{[p]}$ is also time-varying. As $r \to s$, the contribution of $\tilde{Q}^{(a)}$ is calculated over a shorter time interval. This change in the state penalty matrix is demonstrated by examining more specific cases of Theorem 2.2 in the following set of Lemmas.

Lemma 2.1: For $s \leq k+N$ the terminal state penalty matrix $\tilde{Q}$ in Eq.8 is given by:

$$\tilde{Q} = \tilde{Q}^{(b)} \tag{21}$$

where $\tilde{Q}^{(b)}$ is determined from the solution of the discrete Lyapunov equation in Eq.14.

Proof: If $s \leq k+N$, $\tilde{A}_r = \tilde{A}_s$ for all $r \geq k+N$ and the expansion of the infinite sum in Eq.8 gives,

$$\sum_{j=N}^{\infty}\left(\tilde{x}^T_{k+j}Q\tilde{x}^T_{k+j}\right) =$$
$$\tilde{x}^T_{k+N}\left(Q + \tilde{A}^T_sQ\tilde{A}_s + \tilde{A}^T_s\tilde{A}^T_sQ\tilde{A}_s\tilde{A}_s + \cdots\right)\tilde{x}_{k+N}$$

which immediate yields $Q^{(b)}$ in Eq.13. ■

This result leads to the simplification of the element structures of F and G in quadratic program for u^N in Eq.15 which is given in the following Lemma.

Lemma 2.2: For $s \leq k+N$, the quadratic program for u^N is given in Eq.15. The matrices F and G in Eq.16 with element structures given in Eq.18 and the state penalty matrix $\tilde{Q}_{[p]}$ in Eq.19 with components $\tilde{Q}^{(a)}_{[p]}$ and $\tilde{Q}^{(b)}_{[p]}$ given by:

$$\tilde{Q}^{(a)}_{[p]} = Q + \sum_{i=k+p}^{N-2} L_{[p]} Q L^T_{[p]}, \qquad \tilde{Q}^{(b)} = \sum_{i=0}^{\infty} \tilde{A}_s^{T^i} Q \tilde{A}_s^i \tag{22}$$

are defined for the following identities:

$$L_{[p]} = \prod_{j=p}^{i} \tilde{A}^T_{k+j}, \quad M_{[p]} = \prod_{j=p}^{N-1} \tilde{A}^T_{k+j}, \quad H_{[q]} = \prod_{j=q}^{N-1} \tilde{A}^T_{k+j} \tag{23}$$

Proof: Algebraic manipulation of Eq.8 with $\tilde{Q} = \tilde{Q}^{(b)}$ yields the matrix structures of F and G with the components given in Eq.22 and identities given in Eq.23. ■

Remark 2: One can note that the quadratic program in Eq.15 can be made to satisfy Lemma 2.2 by selecting an appropriately long control horizon. That is, choosing $N \geq s$ ensures that the condition $s \leq k+N$ is satisfied which is feasible since $s < \infty$ and less computationally intensive for small s. Similarly, one can determine $\tilde{Q}_{[p]}$ for $r < s < r+N$ to further verify the change in the state penalty matrix as the interval of control moves N shifts forward. Moreover, the transition in $\tilde{Q}_{[p]}$ as $\tilde{A}_k \rightarrow \tilde{A}_s$ is captured by the quadratic program structure for u^N in Theorem 2.2 as can be observed by determining the quadratic program for the time $r \geq s$. In this case, state penalty matrix is given by $\tilde{Q}_{[p]} = \tilde{Q}^{(b)}$ and the matrices F and G of the quadratic program in Eq.15 are given by:

$$F = \begin{pmatrix} B_r^T \tilde{Q}^{(b)} B_r + R & B_r^T \tilde{A}_s^T \tilde{Q}^{(b)} B_r & \cdots & B_r^T \tilde{A}_s^{T^{N-1}} \tilde{Q}^{(b)} B_r \\ B_r^T \tilde{Q}^{(b)} \tilde{A}_s B_r & B_r^T \tilde{Q}^{(b)} B_r + R & \cdots & B_r^T \tilde{A}_s^{T^{N-2}} \tilde{Q}^{(b)} B_r \\ \vdots & \vdots & \ddots & \vdots \\ B_r^T \tilde{Q}^{(b)} \tilde{A}_s^{N-1} B_r & B_r^T \tilde{Q}^{(b)} \tilde{A}_s^{N-2} B_r & \cdots & R + B_r^T \tilde{Q}^{(b)} B_r \end{pmatrix}$$

$$G = \begin{pmatrix} B_r^T \tilde{Q}^{(b)} \tilde{A}_s \\ B_r^T \tilde{Q}^{(b)} \tilde{A}_s^2 \\ \vdots \\ B_r^T \tilde{Q}^{(b)} \tilde{A}_s^N \end{pmatrix} \tag{24}$$

which complies with the quadratic program for a time-invariant linear system in the standard MPC formulation.

B. Constraints

The quadratic program in Eq.15 is subject to the following set of constraints:

$$\begin{aligned} u_{\min} \leq u_{k+j} \leq u_{\max}, &\quad j = 0, 1, \ldots, N-1 \\ x_{\min} \leq x_{k+j} \leq x_{\max}, &\quad j = j_1, j_1+1, \ldots, j_2 \end{aligned} \tag{25}$$

where state constraints are applied from time $k+j$ up to time $k+j_2$ with $j_1 \geq 1$ and $j_2 \geq j_1$. The choice of j_1 is made to ensure that the state constraints are feasible at time k and j_2 is chosen such that feasibility of the state constraints up to time $k+j_2$ guarantees the feasibility of the state constraints on the infinite time horizon. One can construct the following set of matrices which relates the constraint on input u_k to the states of the future N future control moves under closed loop state feedback is given by:

$$T = \begin{pmatrix} B_k & 0 & \cdots & 0 \\ W^T_{[2,1]} B_k & B_{k+1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ W^T_{[N-1,1]} B_k & W^T_{[N-1,2]} B_{k+1} & \cdots & B_{k+N} \end{pmatrix}$$

where for $i = \{1, 2, \ldots, N\}$ and $j = \{1, 2, \ldots, N\}$ the elements $W_{[i,j]}$ are defined as:

$$W_{[i,j]} = \begin{cases} \prod_{j=p}^{i-1} \tilde{A}^T_{k+p} & i > j \\ 0 & i \leq j \end{cases} \tag{26}$$

Then the constraints in Eq.25 can be expressed as:

$$\begin{pmatrix} I \\ -I \\ T \\ -T \end{pmatrix} u^N \leq \begin{pmatrix} I_1 \\ I_2 \\ T_1 \\ T_2 \end{pmatrix} \tag{27}$$

where I_1, I_2, T_1 and T_2 are given by:

$$I_1 = \begin{pmatrix} u_{\max} \\ \vdots \\ u_{\max} \end{pmatrix}, \quad I_2 = \begin{pmatrix} -u_{\min} \\ \vdots \\ -u_{\min} \end{pmatrix} \tag{28}$$

The infinite dimensional state constraints show up through the following expression $x_k = \tilde{x}_k + \tilde{x}_{f(k)}$ in the Eq.25:

$$T_1 = \begin{pmatrix} x_{\max} - \tilde{A}_k \tilde{x}_k - \tilde{x}_{f(k)} \\ x_{\max} - Z^T_{[1]} \tilde{A}_k \tilde{x}_k - \tilde{x}_{f(k)} \\ \vdots \\ x_{\max} - Z^T_{[N-1]} \tilde{A}_k \tilde{x}_k - \tilde{x}_{f(k)} \end{pmatrix} \tag{29}$$

$$T_2 = \begin{pmatrix} -x_{\min} + \tilde{A}_k \tilde{x}_k + \tilde{x}_{f(k)} \\ -x_{\min} + Z^T_{[1]} \tilde{A}_k \tilde{x}_k + \tilde{x}_{f(k)} \\ \vdots \\ -x_{\min} + Z^T_{[N-1]} \tilde{A}_k \tilde{x}_k + \tilde{x}_{f(k)} \end{pmatrix} \tag{30}$$

where the element $Z_{[i]}$ is given by:

$$Z_{[i]} = \prod_{p=i}^{N-1} \tilde{A}^T_{k+p} \tag{31}$$

and where $\tilde{x}_{f(k)}$ dynamics is bounded due to the power spectral stability of infinity dimensional $\tilde{A}_{f(k)}$ and boudness of the input due to the existence of feasible optimization solution.

C. Unstable systems

The receding horizon regulator in Section-II-A was formulated for stable systems. In the case of unstable systems, which are assumed to be stabilizable with the number of control moves being greater or equal than the number of unstable modes, i.e. $N \geq n_u$, the approach to the regulator synthesis is by the partitioning of the Jordan form of $\tilde{A}_k$ into stable parts and unstable parts, J_s and J_u respectively, see [16]:

$$A_k = VJV^{-1} = \begin{pmatrix} V_k^u & V_k^s \end{pmatrix} \begin{pmatrix} J_k^u & 0 \\ 0 & J_k^s \end{pmatrix} \begin{pmatrix} \tilde{V}_k^u \\ \tilde{V}_k^s \end{pmatrix} \quad (32)$$

Then for $\begin{pmatrix} z_k^u \\ z_k^s \end{pmatrix} = \begin{pmatrix} \tilde{V}_k^u \\ \tilde{V}_k^s \end{pmatrix} x_k$ the transformed discrete state space system is given by:

$$\begin{pmatrix} z_{k+1}^u \\ z_{k+1}^s \end{pmatrix} = \begin{pmatrix} J_k^u & 0 \\ 0 & J_k^s \end{pmatrix} \begin{pmatrix} z_k^u \\ z_k^s \end{pmatrix} + \begin{pmatrix} \tilde{V}_u \\ \tilde{V}_s \end{pmatrix} B_k u_k \quad (33)$$

and the objective function in Eq.8 is subject to the equality constraint $z_{k+N}^u = 0$ such that u^N stabilizes the unstable modes at time $k+N$. The state and input constraints of Section-II-B are applied in the regulator formulation for the system in Eq.33.

Remark 3: The time-varying nature of the system Eq.5 produces complexity for the synthesis of MPC for the transformed system in Eq.33 such as the case if the instability of the system is due to a finite number of time-varying process parameters, e.g. $\alpha(t) = \{\alpha_1(t), \ldots, \alpha_j(t)\}$. Even if the trajectory of $\alpha(t)$ is known, the condition that $N \geq n_u$ must be satisfied to ensure that $z_{k+N}^u = 0$ is feasible for every x_k. If, for example, the trajectory of $\alpha(t)$ is such that the modes alternate between stable and unstable regions at various intervals of time, then N must be adjusted to accommodate this variation. Moreover, for all cases considered, the nominal stability of the regulator formulation is ensured by the evaluation of the state penalty on the infinite horizon, see [17]. For time-varying unstable systems, the requirement for evaluating $\tilde{Q}$ is that $\tilde{A}_k \to \tilde{A}_s$ in a finite time index s and remains there for all $k \geq s$ and also that $\tilde{A}_s$ is stable. The regulator formulation of this class of unstable time varying discrete systems is not considered in this work.

III. Example

In this section we consider the annealing process depicted in Figure 2 in which a mechanically driven pulling arm draws a slab from a melt according to a prespecified schedule which determines the length of the domain $l(t)$ and the velocity of the boundary $w(t)$.

A. System representation

The PDE model of the temperature dynamics in the slab domain $0 < \xi < l(t)$ is given by:

$$\frac{\partial f}{\partial t}(\xi,t) = D_0 \frac{\partial^2 f}{\partial \xi^2}(\xi,t) - w(t)\frac{\partial f}{\partial \xi}(\xi,t) + Q(\xi,t)$$
$$\frac{\partial f}{\partial t}(0,t) = 0, \quad \frac{\partial f}{\partial t}(l(t),t) = 0, \quad f(\xi,0) = f_0 \quad (34)$$

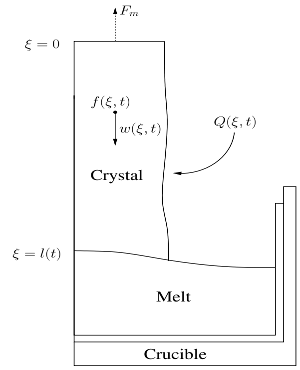

Fig. 2. CZ crystal growth process where $f(\xi,t)$ represents the temperature. The boundary $\xi = l(t)$ is moving with velocity $w(t)$ such that the domain is $0 < \xi < l(t)$ and where $Q(\xi,t)$ is the heat input to the system.

where D_0 is the diffusivity constant, $Q(\xi,t)$ is the heat input to the system and f_0 is the initial temperature distribution of the slab, see [14], [18]. For each $t \in [0,\infty)$, and admissible function $\phi(\xi) \in \mathbb{L}_2(0,l)$, the time-dependent spatial operator of Eq.34 is given by:

$$\mathscr{A}\phi := D_0 \frac{d^2\phi}{d\xi^2} - w\frac{d\phi}{d\xi}$$
$$\mathscr{D}(\mathscr{A}) := \Big\{ \phi \in \mathbb{L}_2(0,l) : \phi, \frac{d\phi}{d\xi} \text{ abs. cont.},$$
$$\frac{d^2\phi}{d\xi^2} \in \mathbb{L}_2(0,l), \text{and } \frac{d\phi}{d\xi}(0) = 0, \frac{d\phi}{d\xi}(l) = 0 \Big\} \quad (35)$$

The eigenvectors $\{\phi_n, n \in \mathbb{N}\}$ and eigenvalues $\{\lambda_n, n \in \mathbb{N}\}$ of the operator defined in Eq.35 are given by:

$$\phi_n(\xi) = B_n e^{\frac{w}{2D_0}\xi} \left(\cos\left(\frac{n\pi}{l}\xi\right) - \frac{w}{2D_0\left(\frac{n\pi}{l}\right)} \sin\left(\frac{n\pi}{l}\xi\right) \right)$$
$$\lambda_n = -D_0 \left(\frac{n\pi}{l}\right)^2 - \frac{1}{2D_0}\frac{w^2}{2}, \quad n \geq 1 \quad (36)$$

The procedure given in Section-I to obtain the discrete time-varying linear state space system representation of Eq.34 is employed. The eigenvectors in Eq.36 that vary with $w(t)$ and $l(t)$ are utilized for the exact modal decomposition of Eq.35 for the first $n = 10$ modes which gives the finite-dimensional evolutionary state space system representation of Eq.34 for each time t. Identifying $\lambda_{\max} = \sup_{n,t} |\lambda_n(t)|$, the sampling time t_s is selected such that $t_s < \lambda_{\max}^{-1}$ and the system is discretized which yields the discrete time-varying finite dimensional linear state space system $\Sigma(A_k, B_k, C_k)$ of the form in Eq.5. The system is stable in the absence of generation terms and the receding horizon regulator formulation with terminal state penalty matrix given in Theorem 2.1 and quadratic program in Theorem 2.2 is employed.

B. Simulation and numerical results

The finite dimensional discrete time-varying system representation of Eq.34 given by $\Sigma(A_k, B_k, C_k)$ as determined in the previous section was simulated with the regulator formulation of Section-II-A employed which determined the minimizing vector u_N of the quadratic function Eq.15 subject

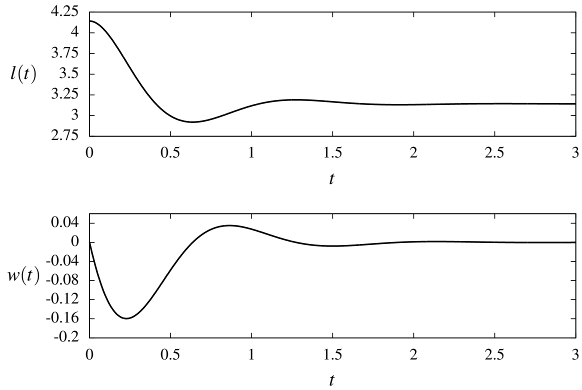

Fig. 3. Domain evolution

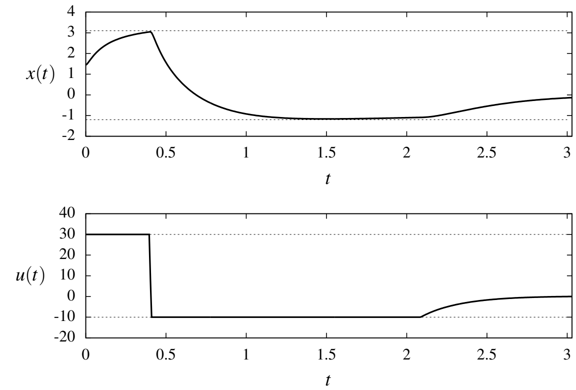

Fig. 5. *Upper*: Constrained state evolution $x^{\min} \leq x(t) \leq x^{\max}$, at input location $\xi = 2.1$. *Lower*: Constrained input evolution $u^{\min} \leq u(t) \leq u^{\max}$.

to the input constraints, $u^{\min} \leq u \leq u^{\max}$, given in Section-II-B. The evolution of the domain is depicted in Figure 3 which settles at time $t_s \cong 2$ to a steady state value of $l_{ss} = \pi$ at which time $\tilde{A}_k = \tilde{A}_s$ for all $k \geq s$. The control horizon was set at $N = 30$ discrete time instances k. At each k the vector u^N of $k+N$ future control moves was determined and input u_k was injected into the closed loop system which resulted in the temperature profile evolution of the slab with moving boundary depicted in Figure 4. The minimizing input u_k stabilized the temperature of the domain within $t = 3$ time instances from an initial perturbation temperature distribution of $f(\xi,0) = f_0$. The state and input profiles are shown in Fig.5 which demonstrates that the respective constraints are satisfied as the temperature of the slab is stabilized.

IV. Conclusions

In this paper, we considered the model predictive control of the discrete time-varying system representation of a class of linear partial differential equations with time-varying coefficients. The infinite-time horizon MPC was developed by the construction of the quadratic program for the time varying system $\Sigma(A_k, B_k, -)$ which accounts for the low dimensional cost function representation while the input and state constraints account for the infinite dimensional PDE state representation. A parabolic PDE with time-varying coefficient described by the evolution of the spatial domain was simulated using the discrete modal system representation of the system. Numerical simulation results of the closed loop system demonstrated that the regulator formulation stabilized the PDE system subject to the imposed state and input constraints.

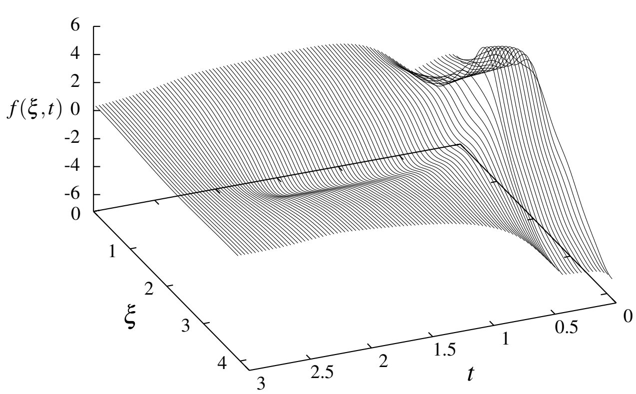

Fig. 4. Closed-loop temperature evolution in domain with slab conductivity $D_0 = 1.5$ and initial condition $f_0 = 5\sin(\xi)$ with heat input determined by constrained receding horizon regulator formulation with control parameters $Q_{nn} = 10$ and $R = 0.01$.

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