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Nash Equilibrium Seeking with Infinitely-Many Players

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Abstract— **We introduce a non-model based approach for the stable attainment of a Nash equilibrium in noncooperative static games with infinitely-many (non-atomic) players. In classical game theory algorithms, each player employs the knowledge of the functional form of his payoff and of the other players' actions, whereas in the proposed algorithm, the players need to measure only their own payoff values. This strategy is based on the extremum seeking approach, which has previously been developed for standard optimization problems and employs sinusoidal perturbations to estimate the gradient of an unknown function. We consider games with quadratic payoff functions, proving convergence to a neighborhood of the Nash equilibrium, and provide simulation results for an example price game.**

I. Introduction

The development of algorithms to achieve convergence to a Nash equilibrium has been a focus of researchers for several decades. Advances in both theory and technology have helped continue this line of research as game theory has found applications across a wide-array of disciplines. We study here the problem of computing, in real time, the Nash equilibria of static noncooperative games with infinitely-many players by employing a non-model based approach. By utilizing extremum seeking with sinusoidal perturbations, the players achieve stable, local attainment of their Nash strategies without the need for any model information.

Most algorithms designed to achieve convergence to a Nash equilibrium require modeling information for the game, assume that the players can observe the actions of the other players, and are applied to games with a finite number of players. In [1], a gradient-like algorithm that requires knowledge of the game's modeling information is used to obtain an equilibrium point in convex games. A strategy known as fictitious play (employed in finite games) depends on the actions of the other players so that a player can devise a best response. A dynamic version of fictitious play and gradient response is developed in [2] and is shown to converge to a mixed-strategy Nash equilibrium in cases where previous algorithms did not converge. Distributed iterative algorithms are designed for the computation of equilibria in [3] for a general class of non-quadratic convex Nash games. In [4], a synchronous distributed learning algorithm, where players remember their own actions and utility values from the previous two time steps, is shown to converge in probability to the set of restricted Nash equilibria. An approach, which is similar to our Nash seeking method for games with finitely-many players (found in [5], [6], [7]), is studied in [8] to solve coordination problems in mobile sensor networks. A comprehensive treatment of static and dynamic noncooperative game theory can be found in [9].

The results of this work extend the methods of extremum seeking [10], [11], [12], [13], [14], [15], originally developed for standard optimization problems. Many works have used the extremum seeking method, which performs non-model based gradient estimation, for a variety of applications, such as steering vehicles toward a source in GPS-denied environments [16], [17], [18], optimizing the control of HCCI engines [19] and nonisothermal continuously stirred tank reactors [20], reducing the impact velocity of an electromechanical valve actuator [21], and controlling Tokamak plasmas [22].

In this work, uncountably-many players in a static noncooperative game with quadratic payoff functions employ extremum seeking to stably attain a Nash equilibrium. The key feature of our approach is that the players are not required to know the mathematical model of their payoff function or the game. The players only need to measure their own payoff values. The tradeoff is that convergence in this case is proved only locally (or at best semi-globally, see [11]). In games of this type, the action of a single player cannot affect the outcome of the game. Economic models with a continuum of players have been studied since the 1960s [23], [24], [25]. We present convergence results for two classes of quadratic payoff functions and provide an example price game with a continuum of players.

II. Nash Equilibrium Seeking

We consider static games with uncountably many (non-atomic) players that wish to maximize their quadratic payoff functions. For such games, we associate with each player a point x in the unit interval $[0,1]$ and denote the action of player x by $u(x)$ and its payoff value by $J(x)$.

By utilizing extremum seeking, which is a non-model based real-time optimization strategy, a player can stably attain a Nash equilibrium $u^*(x)$ by evolving its action $u(x)$ according to its measured payoff value $J(x)$. Specifically, the players employ the time-varying strategy

$$\frac{\partial}{\partial t}\hat{u}(x,t) = k(x)\mu(x,t)J(x,t), \tag{1}$$

$$u(x,t) = \hat{u}(x,t) + \mu(x,t), \tag{2}$$

where $\mu(x,t) = a(x)\sin(\omega(x)t + \varphi(x))$, $a(x)$ is measurable, positive, and bounded for all $x \in [0,1]$, and $\omega(x)$, $k(x) > 0$

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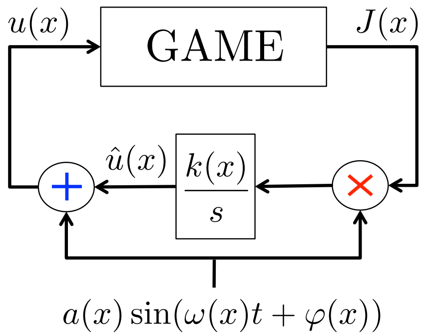

Fig. 1. Deterministic extremum seeking scheme employed by uncountably-many players with the Nash seeking loop for player $x \in [0,1]$ depicted.

for all $x \in [0,1]$. The strategy (1)–(2) requires player x to know only its payoff value $J(x)$. Knowledge of the mathematical form of the payoff functions or of the other players' actions is not needed. Figure 1 depicts a noncooperative game played by infinitely many players implementing the Nash seeking strategy (1)–(2).

We study two classes of quadratic payoff functions. First, we analyze payoff functions of the form

$$J(x,t) = -c(x)u^2(x,t) + d(x)u(x,t)\int_0^1 p(y)u(y,t)\,dy + q(x)u(x,t) + \int_0^1 r(x,y)u(y,t)\,dy + e(x), \tag{3}$$

where c, d, p, q, r, e are all measurable and bounded functions. Moreover, $c(x) > 0$ for all $x \in [0,1]$, and we assume $\int_0^1 \frac{p(y)d(y)}{c(y)}\,dy \neq 2$ because the Nash equilibrium for games with payoff functions of the form (3) is

$$u^*(x) = \frac{d(x)\int_0^1 \frac{p(y)q(y)}{c(y)}\,dy}{c(x)\left(4 - 2\int_0^1 \frac{p(y)d(y)}{c(y)}\,dy\right)} + \frac{1}{2}\frac{q(x)}{c(x)}. \tag{4}$$

The other class of quadratic payoff functions that we consider, which is not a subset of (3), is

$$J(x,t) = -u^2(x,t) + 2u(x,t)\int_0^1 r(x,y)u(y,t)\,dy, \tag{5}$$

where r is measurable, bounded, cannot be expressed as a product of two single-argument functions, i.e., $r(x,y) \neq g(x)h(y)$ in general. (Compare the second term of (5) to the second term of (3).) The payoff functions (5) yield

$$u^*(x) \equiv 0 \tag{6}$$

if $\sup_{x\in[0,1]} \int_0^1 |r(x,y)|^2\,dy < 1$.

III. Convergence Results

To state our convergence results, we introduce two sets of functions. We define Ω as the set of positive, bounded functions $\omega : [0,1] \to \mathbb{R}_+$ such that, at each element of the set $\omega([0,1]) \cup 2\omega([0,1])$, the level set of ω is of measure zero. Let Π be the set of positive functions $\nu(x)$ that are either strictly increasing or strictly decreasing. Then, $\Pi \subset \Omega$. Also contained in Ω are all bounded $C^1[0,1]$ positive functions whose derivative is zero on a set of measure zero.

For games with payoff functions of the form (3), we have the following result:

Theorem 1: Consider the system (1)–(2), along with (3) and (4), where $k(x) = \varepsilon K(x) = O(\varepsilon)$, ε is a small, positive constant, and $c(x) > \frac{1}{2}\left(\int_0^1 d^2(y)\,dy\right)^{1/2}\left(\int_0^1 p^2(y)\,dy\right)^{1/2}$ for all $x \in [0,1]$. There exists a constant $\bar{\varepsilon}$ such that for all $\varepsilon \in (0,\bar{\varepsilon})$ and functions $\omega \in \Omega$, if the $L_2[0,1]$ norm of $\Delta(x,0)$ is sufficiently small, then for all $t \geq 0$,

$$\int_0^1 \Delta^2(x,t)\,dx \leq Me^{-mt}\int_0^1 \Delta^2(x,0)\,dx + O\left(\varepsilon^2 + \max_{x\in[0,1]} a^2(x)\right), \tag{7}$$

where

$$\Delta(x,t) = u(x,t) - u^*(x), \tag{8}$$

$$M = \frac{\max_x\{k(x)a^2(x)\}}{\min_x\{k(x)a^2(x)\}}, \tag{9}$$

$$m = 2\min_{x\in[0,1]}\{\alpha(x)\}\min_{x\in[0,1]}\{k(x)a^2(x)\}, \tag{10}$$

$$\alpha(x) = c(x) - \frac{1}{2}\left(\int_0^1 d^2(y)\,dy\right)^{\frac{1}{2}}\left(\int_0^1 p^2(y)\,dy\right)^{\frac{1}{2}}. \tag{11}$$

Proof: Denote the error at time t relative to the Nash equilibrium as

$$\tilde{u}(x,t) = u(x,t) - \mu(x,t) - u^*(x). \tag{12}$$

By substituting (3) into (1)–(2), we obtain the error system

$$\frac{\partial}{\partial t}\tilde{u}(x,t) = \varepsilon K(x) G[\tilde{u}, u^*, c, d, p, q, r, e, \mu](x,t), \tag{13}$$

where the operator G (with the arguments suppressed) is defined as

$$\begin{aligned} G[\cdot](x,t) \triangleq \mu(x,t)\Big[&-c(x)(\tilde{u}(x,t) + u^*(x) + \mu(x,t))^2 \\ &+ d(x)(\tilde{u}(x,t) + u^*(x) + \mu(x,t)) \\ &\times \int_0^1 p(y)(\tilde{u}(y,t) + u^*(y) + \mu(y,t))\,dy \\ &+ q(x)(\tilde{u}(x,t) + u^*(x) + \mu(x,t)) \\ &+ \int_0^1 r(x,y)(\tilde{u}(y,t) + u^*(y) + \mu(y,t))\,dy \\ &+ e(x)\Big]. \end{aligned} \tag{14}$$

The form of (13) admits the application of general averaging theory [26] for stability analysis, and the average error system can be shown to be

$$\begin{aligned} \frac{\partial}{\partial t}\tilde{u}^{\text{ave}}(x,t) &= \lim_{T\to\infty}\frac{\varepsilon K(x)}{T}\int_0^T G[\cdot](x,t)\,dt, \\ &= -\varepsilon K(x)a^2(x)\Big(c(x)\tilde{u}^{\text{ave}}(x,t) \\ &\quad - \frac{1}{2}d(x)\int_0^1 p(y)\tilde{u}^{\text{ave}}(y,t)\,dy\Big). \end{aligned} \tag{15}$$

(Details of computing (15) are shown in the appendix.)

Let $V(t)$ be a Lyapunov functional defined as

$$V(t) = \frac{1}{2\varepsilon}\int_0^1 \frac{1}{K(x)a^2(x)}(\tilde{u}^{\text{ave}})^2(x,t)\,dx \tag{16}$$

and bounded from both sides by

$$V(t) \geq \frac{1}{2\varepsilon \max_x\{K(x)a^2(x)\}}\int_0^1 \left(\tilde{u}^{\text{ave}}\right)^2(x,t)\,dx, \tag{17}$$

$$V(t) \leq \frac{1}{2\varepsilon \min_x\{K(x)a^2(x)\}}\int_0^1 \left(\tilde{u}^{\text{ave}}\right)^2(x,t)\,dx. \tag{18}$$

Taking the time derivative, substituting (15), and applying the Cauchy-Schwarz inequality yields

$$\begin{aligned}\dot{V} = &-\int_0^1 c(x)\left(\tilde{u}^{\text{ave}}\right)^2(x,t)\,dx\\ &+\frac{1}{2}\int_0^1 d(x)\tilde{u}^{\text{ave}}(x,t)\,dx\int_0^1 p(y)\tilde{u}^{\text{ave}}(y,t)\,dy,\\ \leq &-\int_0^1 c(x)\left(\tilde{u}^{\text{ave}}\right)^2(x,t)\,dx\\ &+\frac{1}{2}\left(\int_0^1 |d(x)|^2\,dx\right)^{\frac{1}{2}}\left(\int_0^1 |\tilde{u}^{\text{ave}}(x,t)|^2\,dx\right)^{\frac{1}{2}}\\ &\times\left(\int_0^1 |p(y)|^2\,dy\right)^{\frac{1}{2}}\left(\int_0^1 |\tilde{u}^{\text{ave}}(y,t)|^2\,dy\right)^{\frac{1}{2}}.\end{aligned} \tag{19}$$

Collecting terms and substituting the bound (18) gives

$$\begin{aligned}\dot{V} &\leq -\min_{x\in[0,1]}\{\alpha(x)\}\int_0^1\left(\tilde{u}^{\text{ave}}\right)^2(x,t)\,dx,\\ &\leq -2\varepsilon\min_{x\in[0,1]}\{\alpha(x)\}\min_{x\in[0,1]}\{K(x)a^2(x)\}V.\end{aligned} \tag{20}$$

From the Comparison Lemma [27] and the bounds (17), (18), we obtain

$$\int_0^1 \left(\tilde{u}^{\text{ave}}\right)^2(x,t)\,dx \leq Me^{-mt}\int_0^1\left(\tilde{u}^{\text{ave}}\right)^2(x,0)\,dx. \tag{21}$$

From [26, Theorem 3.6], the error system (13) retains the stability properties of the average system (15). Specifically,

$$\int_0^1 \tilde{u}^2(x,t)\,dx \leq Me^{-mt}\int_0^1 \tilde{u}^2(x,0)\,dx + O(\varepsilon^2). \tag{22}$$

Noting $u(x,t)-u^*(x) = \tilde{u}(x,t)+\mu(x,t)$ and $\mu(x,t)$ is $O(\max_x a(x))$ completes the proof. ■

Similarly, for payoff functions of the form (5), we have the following result:

Theorem 2: Consider the system (1)–(2), along with (5) and $u^*(x)\equiv 0$, where $k(x)=\varepsilon K(x)=O(\varepsilon)$, ε is a small, positive constant, and $\sup_{x\in[0,1]}\int_0^1 |r(x,y)|^2\,dy < 1$. There exists a constant $\bar{\varepsilon}$ such that for all $\varepsilon\in(0,\bar{\varepsilon})$ and functions $\omega\in\Omega$, if the $L_2[0,1]$ norm of $u(x,0)$ is sufficiently small, then for all $t\geq 0$,

$$\begin{aligned}\int_0^1 u^2(x,t)\,dx \leq Me^{-\sigma t}\int_0^1 u^2(x,0)\,dx\\ + O\left(\varepsilon^2+\max_{x\in[0,1]}a^2(x)\right),\end{aligned} \tag{23}$$

where

$$\sigma = 2\beta\min_{x\in[0,1]}\{k(x)a^2(x)\}, \tag{24}$$

$$\beta = 1-\sup_{x\in[0,1]}\left(\int_0^1 |r(x,y)|^2\,dy\right)^{\frac{1}{2}}, \tag{25}$$

and M is given by (9).

Proof: Following the proof of Theorem 1, we obtain the average error system,

$$\begin{aligned}\frac{\partial}{\partial t}\tilde{u}^{\text{ave}}(x,t) = -\varepsilon K(x)a^2(x)\bigg(\tilde{u}^{\text{ave}}(x,t)\\ -\int_0^1 r(x,y)\tilde{u}^{\text{ave}}(y,t)\,dy\bigg),\end{aligned} \tag{26}$$

and using the Lyapunov functional (16), we have

$$\begin{aligned}\dot{V} = &-\int_0^1\left(\tilde{u}^{\text{ave}}\right)^2(x,t)\,dx\\ &+\int_0^1 \tilde{u}^{\text{ave}}(x,t)\int_0^1 r(x,y)\tilde{u}^{\text{ave}}(y,t)\,dy\,dx,\\ \leq &-\int_0^1\left(\tilde{u}^{\text{ave}}\right)^2(x,t)\,dx + \sup_{x\in[0,1]}\left(\int_0^1 |r(x,y)|^2\,dy\right)^{\frac{1}{2}}\\ &\times\left(\int_0^1 |\tilde{u}^{\text{ave}}(x,t)|^2\,dx\right)^{\frac{1}{2}}\left(\int_0^1 |\tilde{u}^{\text{ave}}(y,t)|^2\,dy\right)^{\frac{1}{2}},\\ \leq &-2\varepsilon\beta\min_{x\in[0,1]}\{K(x)a^2(x)\}V,\end{aligned} \tag{27}$$

where we have applied the Cauchy-Schwarz inequality and substituted the bound (18). The remainder of the proof directly follows after noting $u^*(x)\equiv 0$. ■

The structural differences between these two classes of quadratic payoff functions are manifested in their respective convergence rates—specifically, in the terms (11) and (25). To understand more intuitively how these convergence results are achieved, one may also compute the average of the $\hat{u}(x,t)$-system (1), which reveals that this Nash seeking method is, on average, a gradient descent algorithm.

IV. Uncountably-Many Player Price Game

For an example game with uncountably-many players, we consider a price game where the players, indexed by $x\in[0,1]$, wish to maximize payoff functions of the form

$$J(x,t) = s(x,t)\left(u(x,t)-m(x)\right), \tag{28}$$

where $s(x,t)$ is the sales volume of player x, $u(x,t)$ is its price, and $m(x)$ is its marginal cost.

The sales volume $s(x,t)$ is modeled as

$$s(x,t) = \frac{R_{||}}{R(x)}\left(S-\frac{u(x,t)}{R_{||}}+\int_0^1\frac{u(y,t)}{R(y)}\,dy\right), \tag{29}$$

$$R_{||} = \left(\int_0^1\frac{dy}{R(y)}\right)^{-1}, \tag{30}$$

where S is the total sales volume and $R(x)$ is the consumer's "resistance" toward buying the product of player x. This resistance $R(x)$ is measurable, positive, and bounded for

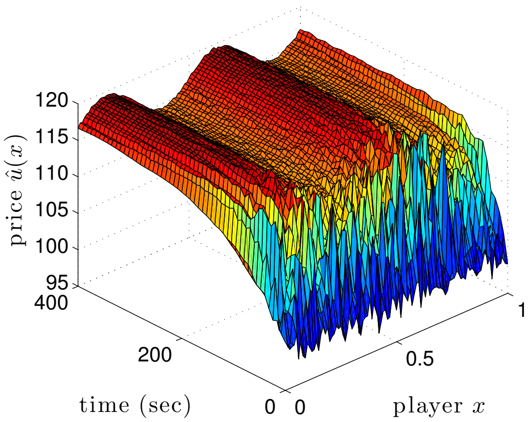

Fig. 2. Price evolution of the uncountably-many player oligopoly price game.

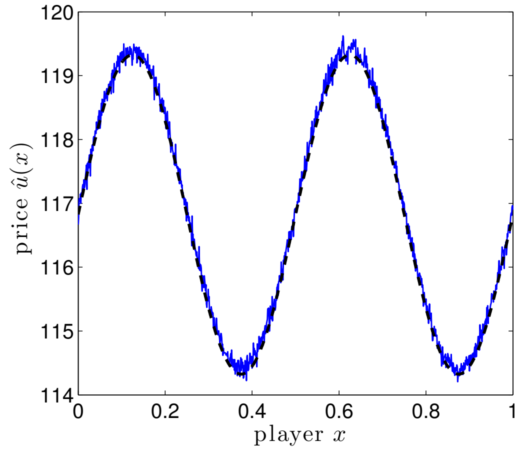

Fig. 3. Nash equilibrium $u^*(x)$ (dashed) of the oligopoly price game with the players' price (blue) at $t = 400$ sec superimposed.

all $x \in [0,1]$. We assume the players do not know the mathematical structure of (28) as it is difficult to know the consumer preference and how it enters the sales model (29).

Substituting (29) into (28) leads to

$$\begin{aligned} J(x,t) = &-\frac{u^2(x,t)}{R(x)} + \frac{R_{||}}{R(x)} u(x,t) \int_0^1 \frac{u(y,t)}{R(y)}\, dy \\ &+ \left(\frac{R_{||} S}{R(x)} + \frac{m(x)}{R(x)} \right) u(x,t) \\ &- \frac{R_{||} m(x)}{R(x)} \int_0^1 \frac{u(y,t)}{R(y)}\, dy - \frac{R_{||} S m(x)}{R(x)}, \end{aligned} \tag{31}$$

which yields the Nash equilibrium prices,

$$u^*(x) = R_{||} \left(S + \frac{1}{2}\frac{m(x)}{R_{||}} + \frac{1}{2}\int_0^1 \frac{m(y)}{R(y)}\, dy \right). \tag{32}$$

Because (31) has the form of (3), the conditions of Theorem 1 are satisfied when the players implement the Nash seeking strategy (1)–(2). Hence, the players achieve local, exponential convergence to Nash equilibrium $u^*(x)$. For this specific example, one can obtain

$$\begin{aligned} \int_0^1 \Delta^2(x,t)\, dx \le M e^{-\xi t} \int_0^1 \Delta^2(x,0)\, dx \\ + O\left(\varepsilon^2 + \max_{x\in[0,1]} a^2(x) \right), \end{aligned} \tag{33}$$

where Δ, M are given by (8), (9). The convergence rate $\xi = \frac{\min_x \{k(x)a^2(x)\}}{\max_x\{R(x)\}}$ is found by performing Lyapunov analysis on the average error system using the Lyapunov functional (16).

For a numerical example, we choose $S = 100$, $m(x) = 20 + 5\sin(4\pi x)$, and $R(x) = 1 + \cos(2\pi x)/4$, which result in the Nash equilibrium

$$u^*(x) = 25\sqrt{15} + 20 + \frac{5}{2}\sin(4\pi x), \tag{34}$$

and the corresponding sales volume

$$s^*(x) = \frac{100\sqrt{15} - 10\sin(4\pi x)}{4 + \cos(2\pi x)}. \tag{35}$$

To approximate the uncountably-many player game, we discretize the interval $[0,1]$ using $N = 1001$ points, representing N players, and use the trapezoidal numerical integration scheme to approximate the integral terms in (31). For these N players, we select the Nash seeking parameters $k(x)$, $a(x)$, and $\omega(x)$ by a random draw from a uniform distribution. Namely, $k(x)$ is sampled from the distribution $U^k(1,5)$, $a(x)$ from $U^a(0.1, 0.2)$, and $\omega(x)$ from $U^\omega(30,60)$, where $U(a,b)$ denotes the uniform distribution with probability density function

$$f(x) = \begin{cases} \frac{1}{b-a} & \text{if } a \le x \le b, \\ 0 & \text{otherwise} \end{cases}. \tag{36}$$

The players are initialized at $u(x,0) = S + v(x)$ where v is distributed normally with zero-mean and unity variance.

Figure 2 depicts the evolution of the players' prices as they converge to a neighborhood of $u^*(x)$. Figure 3 shows $u^*(x)$ with $\hat{u}(x,t)$ at $t = 400$ sec. We show $\hat{u}(x,t)$ to highlight the players' convergence to $u^*(x)$ since $u(x,t)$ contains the additive signal $\mu(x,t)$. Figure 4 depicts the evolution of the players' payoff values.

V. Conclusions

We have introduced a non-model based approach to solve noncooperative games with uncountably-many players that possess quadratic payoff functions. A player can stably attain its Nash equilibrium by measuring only the value of its payoff function. No other information about the game is needed. Such an approach may be used to negotiate prices in electronic markets in real time as the supply and demand fluctuates.

Appendix

To obtain the average error system (15), we compute the average of the operator G (14), which requires averaging terms of the following forms: $\mu(x,t)$, $\mu^2(x,t)$, $\mu^3(x,t)$, $\mu(x,t)\int_0^1 \rho(y)\mu(y,t)dy$, and $\mu^2(x,t)\int_0^1 \rho(y)\mu(y,t)dy$, where ρ is bounded and measurable.

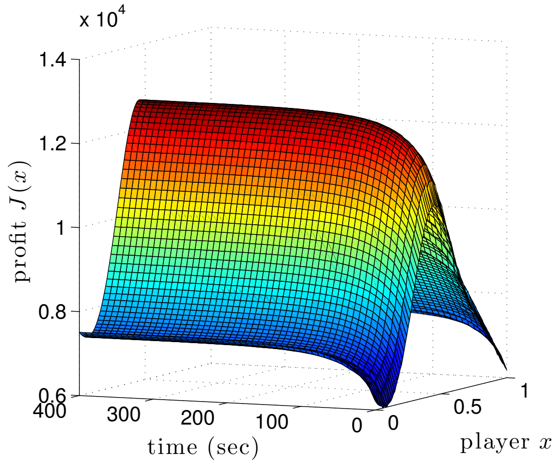

Fig. 4. Payoff evolution of the uncountably-many player oligopoly price game.

Computing the average of the first three terms is straightforward:

$$\begin{aligned}
&\lim_{T\to\infty}\frac{1}{T}\int_0^T \mu(x,t)\,dt\\
&\quad = a(x)\lim_{T\to\infty}\frac{\cos\varphi(x)-\cos(\omega(x)T+\varphi(x))}{T\omega(x)},\\
&\quad = 0, \qquad (37)\\
&\lim_{T\to\infty}\frac{1}{T}\int_0^T \mu^2(x,t)\,dt\\
&\quad = \lim_{T\to\infty}\frac{a^2(x)}{2T}\int_0^T [1-\cos(2\omega(x)t+2\varphi(x))]\,dt,\\
&\quad = \frac{a^2(x)}{2}, \qquad (38)\\
&\lim_{T\to\infty}\frac{1}{T}\int_0^T \mu^3(x,t)\,dt\\
&\quad = \lim_{T\to\infty}\frac{3a^3(x)}{4T}\int_0^T \sin(\omega(x)t+\varphi(x))\,dt\\
&\qquad -\lim_{T\to\infty}\frac{a^3(x)}{4T}\int_0^T \sin(3\omega(x)t+3\varphi(x)))\,dt,\\
&\quad = 0. \qquad (39)
\end{aligned}$$

Calculating the averages of $\mu(x,t)\int_0^1 \rho(y)\mu(y,t)dy$ and $\mu^2(x,t)\int_0^1 \rho(y)\mu(y,t)dy$ requires more care. We start by computing the average of $\mu(x,t)\int_0^1 \rho(y)\mu(y,t)dy$,

$$\begin{aligned}
&\lim_{T\to\infty}\frac{1}{T}\int_0^T \mu(x,t)\int_0^1 \rho(y)\mu(y,t)dy\,dt\\
&\quad = a(x)\lim_{T\to\infty}\int_0^1 \frac{a(y)\rho(y)}{T}\int_0^T \sin(\omega(x)t+\varphi(x))\\
&\qquad \times\sin(\omega(y)\tau+\varphi(y))\,dt\,dy, \qquad (40)
\end{aligned}$$

where we have switched the order of integration and substituted for $\mu(x,t)$ and $\mu(y,t)$. Next, we compute the inner integral over t on the last line of (40). Specifically,

$$\begin{aligned}
&\frac{1}{T}\int_0^T \sin(\omega(x)t+\varphi(x))\sin(\omega(y)t+\varphi(y))\,dt\\
&\quad = \frac{1}{2T}\int_0^T [\cos((\omega(x)-\omega(y))t+\varphi(x)-\varphi(y))\\
&\qquad -\cos((\omega(x)+\omega(y))t+\varphi(x)+\varphi(y))]\,dt,\\
&\quad = \frac{1}{2}\Bigg(\frac{\sin((\omega(x)-\omega(y))T+\varphi(x)-\varphi(y))}{(\omega(x)-\omega(y))T}\\
&\qquad -\frac{\sin(\varphi(x)-\varphi(y))}{(\omega(x)-\omega(y))T}+\frac{\sin(\varphi(x)+\varphi(y))}{(\omega(x)+\omega(y))T}\\
&\qquad -\frac{\sin((\omega(x)+\omega(y))T+\varphi(x)+\varphi(y))}{(\omega(x)+\omega(y))T}\Bigg),\\
&\quad = \delta_0(x,y,T), \qquad (41)
\end{aligned}$$

where we have used the fact that for any given x, the set $\{y\in[0,1]|\omega(x)=\omega(y)\}$ is of measure zero since $\omega\in\Omega$.

To switch the order of the limit and the integration over y in (40), we apply the dominated convergence theorem, which requires that the integrand $a(y)\rho(y)\delta_0(x,y,T)$ be bounded by a function $\eta_0(x,y)$ and that $\int_0^1 \eta_0(x,y)\,dy$ be finite. Using the sum of angles trigonometric identity, we have

$$\begin{aligned}
&|\delta_0(x,y,T)|\\
&\quad \le \frac{1}{2}\Bigg(|\cos(\varphi(x)-\varphi(y))|\left|\frac{\sin((\omega(x)-\omega(y))T)}{(\omega(x)-\omega(y))T}\right|\\
&\qquad +|\sin(\varphi(x)-\varphi(y))|\left|\frac{\cos((\omega(x)-\omega(y))T)-1}{(\omega(x)-\omega(y))T}\right|\\
&\qquad +|\cos(\varphi(x)+\varphi(y))|\left|\frac{\sin((\omega(x)+\omega(y))T)}{(\omega(x)+\omega(y))T}\right|\\
&\qquad +|\sin(\varphi(x)+\varphi(y))|\left|\frac{\cos((\omega(x)+\omega(y))T)-1}{(\omega(x)+\omega(y))T}\right|\Bigg),\\
&\quad \le 2, \qquad (42)
\end{aligned}$$

which implies the following bound on the integrand,

$$|a(y)\rho(y)\delta_0(x,y,T)|\le 2\max_{y\in[0,1]}\{a(y)\rho(y)\}. \qquad (43)$$

Clearly, $\int_0^1 2\max_{y\in[0,1]}\{a(y)\rho(y)\}\,dy < \infty$, which with (43), allows the dominated convergence theorem to be applied to (40). Thus,

$$\begin{aligned}
&\lim_{T\to\infty}\frac{1}{T}\int_0^T \mu(x,t)\int_0^1 \rho(y)\mu(y,t)dy\,dt\\
&\quad = a(x)\int_0^1 \lim_{T\to\infty} a(y)\rho(y)\delta_0(x,y,T)dy = 0, \qquad (44)
\end{aligned}$$

since $\lim_{T\to\infty}\delta_0(x,y,T)=0$.

We compute the average of $\mu^2(x,t)\int_0^1 \rho(y)\mu(y,t)dy$ in a similar manner. We have

$$\begin{aligned}
&\lim_{T\to\infty}\frac{1}{T}\int_0^T \mu^2(x,t)\int_0^1 \rho(y)\mu(y,t)dy\,dt\\
&\quad = a^2(x)\lim_{T\to\infty}\int_0^1 \frac{a(y)\rho(y)}{T}\int_0^T \sin^2(\omega(x)t+\varphi(x))\\
&\qquad \times\sin(\omega(y)t+\varphi(y))\,dt\,dy, \qquad (45)
\end{aligned}$$

and computing the inner integral on the last line of (45) gives

$$\begin{aligned}
&\frac{1}{T}\int_0^T \sin^2(\omega(x)t+\varphi(x))\sin(\omega(y)t+\varphi(y))\,dt \\
&\quad= \frac{1}{2T}\int_0^T [\sin(\omega(y)t+\varphi(y)) \\
&\quad\quad -\cos(2\omega(x)t+2\varphi(x))\sin(\omega(y)t+\varphi(y))]\,dt, \\
&\quad= \frac{1}{4T}\int_0^T [2\sin(\omega(y)t+\varphi(y)) \\
&\quad\quad -\sin((2\omega(x)+\omega(y))t+2\varphi(x)+\varphi(y)) \\
&\quad\quad +\sin((2\omega(x)-\omega(y))t+2\varphi(x)-\varphi(y))]\,dt, \\
&\quad= \frac{1}{4}\Bigg(\frac{2\cos(\varphi(y))-2\cos(\omega(y)T+\varphi(y))}{\omega(y)T} \\
&\quad\quad +\frac{\cos((2\omega(x)+\omega(y))T+2\varphi(x)+\varphi(y))}{(2\omega(x)+\omega(y))T} \\
&\quad\quad -\frac{\cos(2\varphi(x)+\varphi(y))}{(2\omega(x)+\omega(y))T}+\frac{\cos(2\varphi(x)-\varphi(y))}{(2\omega(x)-\omega(y))T} \\
&\quad\quad -\frac{\cos((2\omega(x)-\omega(y))T+2\varphi(x)-\varphi(y))}{(2\omega(x)-\omega(y))T}\Bigg), \\
&\quad= \delta_1(x,y,T), \qquad (46)
\end{aligned}$$

where we have used the fact that for any given x, the set $\{y\in[0,1] \mid 2\omega(x)=\omega(y)\}$ is of measure zero since $\omega\in\Omega$.

As before, to switch the order of the limit and the integration over y in (45), we apply the dominated convergence theorem. Bounding $\delta_1(x,y,T)$ leads to

$$\begin{aligned}
&|\delta_1(x,y,T)| \\
&\le \frac{1}{4}\Bigg(2\,|\cos(\varphi(y))|\left|\frac{1-\cos(\omega(y)T)}{\omega(y)T}\right| \\
&\quad +2\,|\sin\varphi(y))|\left|\frac{\sin(\omega(y)T)}{\omega(y)T}\right| \\
&\quad +|\cos(2\varphi(x)+\varphi(y))|\left|\frac{\cos((2\omega(x)+\omega(y))T)-1}{(2\omega(x)+\omega(y))T}\right| \\
&\quad +|\sin(2\varphi(x)+\varphi(y))|\left|\frac{\sin((2\omega(x)+\omega(y))T)}{(2\omega(x)+\omega(y))T}\right| \\
&\quad +|\cos(2\varphi(x)-\varphi(y))|\left|\frac{1-\cos((2\omega(x)-\omega(y))T)}{(2\omega(x)-\omega(y))T}\right| \\
&\quad +|\sin(2\varphi(x)-\varphi(y))|\left|\frac{\sin((2\omega(x)-\omega(y))T)}{(2\omega(x)-\omega(y))T}\right|\Bigg), \\
&\le 2, \qquad (47)
\end{aligned}$$

which implies the bound on the integrand

$$|a(y)\rho(y)\delta_1(x,y,T)| \le 2\max_{y\in[0,1]} a(y)\rho(y). \qquad (48)$$

Thus, the dominated convergence theorem applies and (45) becomes

$$\begin{aligned}
&\lim_{T\to\infty}\frac{1}{T}\int_0^T \mu^2(x,\tau)\int_0^1 \rho(y)\mu(y,t)\,dt \\
&\quad= a^2(x)\int_0^1 \lim_{T\to\infty} a(y)\rho(y)\delta_1(x,y,T)dy = 0, \qquad (49)
\end{aligned}$$

since $\lim_{T\to\infty}\delta_1(x,y,T)=0$. From (37), (38), (39), (44), and (49), we obtain the average system (15).

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