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Structured Control of Affine Linear Parameter Varying Systems

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Abstract— **This paper presents a new procedure to design structured controllers for discrete-time affine linear parameter-varying systems (A-LPV). The class of control structures includes decentralized of any order, fixed-order output feedback, simultaneous plant-control design, among others. A parameter-varying non-convex condition for an upper bound on the induced $\mathcal{L}_2$-norm performance is solved by an iterative linear matrix inequalities (LMI) optimization algorithm. Numerical examples demostrate the effectiveness of the proposed approach.**

I. INTRODUCTION

Gain-scheduling techniques have been successfully applied on the control of nonlinear systems throughout the history of automatic control. The *classical* ad hoc approach begins by deriving a family of linear time invariant models (LTI), obtained from linearizations of a nonlinear model over different equilibrium points. An LTI controller is designed for each LTI model. When implemented, an interpolation scheme computes the actual controller as a function of the measured time-varying parameters. Due to a lack of solid theoretical background, the resulting closed-loop system has no guarantees of stability, performance, neither robustness. The *modern* design techniques arose as a result of the continous effort of the control reseach community to incorporate theoretical rigorousness on gain-scheduling [1]. It relies on synthesis conditions directly related to a linear parameter-varying (LPV) representation of the non-linear system and a parameter-dependent controller.

Most of the advances on LPV synthesis focus on further generalize the type of dependence of the system and stability criteria on the scheduling parameters. Basicaly, there are two distinct approaches to characterize LPV stability: Lyapunov theory is applied to parameter-dependent state-space systems, while small-gain theory is applied to systems with parameter-dependent linear fractional transformation. Although computationally simple, the synthesis conditions based on small gain theorem [2][3] or on the notion of quadratic stability [4][5] lead to conservative results, because they rely on parameter-independent Lyapunov functions. Several methods of gain-scheduled H_2 and $\mathcal{L}_2$ control based on parameter-dependent Lyapunov functions with/without bounds on the parameter rate of variation have been proposed to overcome such conservatism [6][7][8][9]. Affine, polytopic, and more recently polynomial [10] are among the types of parameter dependence.

Apparently, the same attention has not be given to the diversification of LPV controller structures. Static state feedback (SSF) and dynamic output feedback (DOF) are by far the most investigated ones. Nonetheless, some work on the design of static output feedback (SOF) LPV controllers can be found. The $\mathcal{L}_2$ PID controller design, which is a particular case of SOF, was studied in [11]. $\mathcal{L}_2$ SOF LPV controller design for polytopic LPV systems and Lyapunov function, considering bounded parameter rate of variation, is proposed in [12]. Another SOF synthesis condition were derived by extending the H_∞ loop-shaping to LPV systems [13]. A few works can be found on other controller strucures. In [14], a decentralized controller for polytopic systems is obtained by constraining the strucure of a parameter-varying SSF. $\mathcal{L}_2$ fixed-order controller design for single-input single-output polynomial LPV systems was recently studied in [15]. To-date, the theory of structured LPV control synthesis is far from being complete.

This paper presents a numerical procedure for discrete-time affine LPV (A-LPV) controller design. Decentralized of any order, fixed-order output feedback, and simultaneous plant-control design are among the possible control structures. Stability is assessed via an affine Lyapunov function, with varying parameters and their rate of variation contained in a polytope. A parameter-varying non-convex condition for an upper bound on the induced $\mathcal{L}_2$-norm performance is solved via an iterative LMI-based algorithm. The proposed optimization scheme can be seen as an extension of a convexifying algorithm for linear structured control design [16], further studied in [17][18], to linear parameter-varying systems.

The paper is organized as follows. Section II brings preliminaries about structured A-LPV and conditions for its parameter-dependant quadratic stability and induced $\mathcal{L}_2$-norm performance. The iterative LMI algorithm is presented in Section III. In Section IV, numerical examples illustrate the effectiveness of the proposed approach.

II. PRELIMINARIES

A. LPV Systems

An open-loop, discrete-time LPV system with state-space realization of the form,

$$\begin{aligned} x(k+1) &= A(\theta)x(k) + B_w(\theta)w(k) + B_u(\theta)u(k) \\ z(k) &= C_z(\theta)x(k) + D_{zw}(\theta)w(k) + D_{zu}(\theta)u(k) \quad (1) \\ y(k) &= C_y(\theta)x(k) + D_{yw}(\theta)w(k), \end{aligned}$$

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is considered for the purpose of synthesis, where $x(k) \in \mathbb{R}^n$ is the state vector, $w(k) \in \mathbb{R}^{n_w}$ is the vector of exogenous perturbation, $u(k) \in \mathbb{R}^{n_u}$ is the control input, $z(k) \in \mathbb{R}^{n_z}$ is the controlled output, and $y(k) \in \mathbb{R}^{n_y}$ is the measured output. $A(\cdot)$, $B(\cdot)$, $C(\cdot)$, $D(\cdot)$ are continuous functions of some time-varying parameter vector $\theta = [\theta_1, \ldots, \theta_{n\theta}]$. In the subclass of affine LPV systems (A-LPV), the open-loop system matrices are restricted to be affine functions of θ,

$$\begin{bmatrix} A(\theta) & B_w(\theta) & B_u(\theta) \\ C_z(\theta) & D_{zw}(\theta) & D_{zu}(\theta) \\ C_y(\theta) & D_{yw}(\theta) & 0 \end{bmatrix} = \begin{bmatrix} A_0 & B_{w,0} & B_{u,0} \\ C_{z,0} & D_{zw,0} & D_{zu,0} \\ C_{y,0} & D_{yw,0} & 0 \end{bmatrix} + \sum_{i=1}^{n_\theta} \theta_i \begin{bmatrix} A_i & B_{w,i} & B_{u,i} \\ C_{z,i} & D_{zw,i} & D_{zu,i} \\ C_{y,i} & D_{yw,i} & 0 \end{bmatrix}. \quad (2)$$

Assume θ ranges over a hyperrectangle denoted Θ,

$$\Theta = \left\{ \theta : \underline{\theta}_i \leq \theta_i \leq \overline{\theta}_i,\ i = 1, \ldots, n_\theta \right\}.$$

The rate of variation $\Delta\theta = \theta(k+1) - \theta(k)$ belongs to the hypercube,

$$\mathcal{V} = \left\{ \Delta\theta : |\Delta\theta_i| \leq v_i,\ i = 1, \ldots, n_\theta \right\}.$$

Also consider an LPV controller of the form,

$$\begin{aligned} x_c(k+1) &= \mathcal{A}_c(\theta)x_c(k) + \mathcal{B}_c(\theta)y(k) \\ u(k) &= \mathcal{C}_c(\theta)x_c(k) + \mathcal{D}_c(\theta)y(k), \end{aligned} \quad (3)$$

where $x_c(k) \in \mathbb{R}^{n_c}$ and matrices affine θ-dependent,

$$\begin{bmatrix} A_c(\theta) & B_c(\theta) \\ C_c(\theta) & D_c(\theta) \end{bmatrix} = \begin{bmatrix} A_{c,0} & B_{c,0} \\ C_{c,0} & D_{c,0} \end{bmatrix} + \sum_{i=1}^{n_\theta} \theta_i \begin{bmatrix} A_{c,i} & B_{c,i} \\ C_{c,i} & D_{c,i} \end{bmatrix}. \quad (4)$$

Representing the controller matrices in a compact way,

$$K(\theta) \stackrel{\text{def}}{=} \begin{bmatrix} D_c(\theta) & C_c(\theta) \\ B_c(\theta) & A_c(\theta) \end{bmatrix}, \quad (5)$$

the interconnection of system (1)-(2) and controller (3)-(4) leads to the following closed-loop LPV system,

$$\begin{aligned} x(k+1) &= \mathcal{A}(\theta, K(\theta))x_{cl}(k) + \mathcal{B}(\theta, K(\theta))w(k) \\ z(k) &= \mathcal{C}(\theta, K(\theta))x_{cl}(k) + \mathcal{D}(\theta, K(\theta))w(k), \end{aligned} \quad (6)$$

where the closed-loop matrices are [19],

$$\begin{aligned} \mathcal{A}(\theta, K(\theta)) &= \mathbf{A}(\theta) + \mathbf{B}(\theta)K(\theta)\mathbf{M}(\theta), \\ \mathcal{B}(\theta, K(\theta)) &= \mathbf{D}(\theta) + \mathbf{B}(\theta)K(\theta)\mathbf{E}(\theta), \\ \mathcal{C}(\theta, K(\theta)) &= \mathbf{C}(\theta) + \mathbf{H}(\theta)K(\theta)\mathbf{M}(\theta), \\ \mathcal{D}(\theta, K(\theta)) &= \mathbf{F}(\theta) + \mathbf{H}(\theta)K(\theta)\mathbf{E}(\theta), \end{aligned}$$

$$\begin{aligned} &\mathbf{A}(\theta) = \begin{bmatrix} A(\theta) & 0 \\ 0 & 0 \end{bmatrix} \quad \mathbf{B}(\theta) = \begin{bmatrix} B_u(\theta) & 0 \\ 0 & I \end{bmatrix}, \\ &\mathbf{M}(\theta) = \begin{bmatrix} C_y(\theta) & 0 \\ 0 & I \end{bmatrix} \quad \mathbf{E}(\theta) = \begin{bmatrix} D_{yw}(\theta) \\ 0 \end{bmatrix}, \\ &\mathbf{H}(\theta) = \begin{bmatrix} D_{zu}(\theta) & 0 \end{bmatrix} \quad \mathbf{D}(\theta) = \begin{bmatrix} B_w(\theta) \\ 0 \end{bmatrix}, \\ &\mathbf{C}(\theta) = \begin{bmatrix} C_z(\theta) & 0 \end{bmatrix} \quad \mathbf{F}(\theta) = D_{zw}(\theta). \end{aligned} \quad (7)$$

This general system structure can be particularized to some usual control topologies. For an *unconstrained* matrix $K(\theta)$, if $n_c = 0$, the problem becomes a static output feedback (SOF). The static state feedback (SSF) is a particular case of SOF, when the system output is a full rank linear transformation of the state vector $\forall\theta$. If $n = n_c$, the full-order dynamic output feedback arises. In a structured control context, more elaborate control systems can be designed by constraining $K(\theta)$. A fixed-order dynamic output feedback has $n_c < n$. For decentralized controllers of arbitrary order, $K(\theta)$ has a block diagonal structure $A_c(\theta) = \text{diag}(A_{ci}(\theta))$, ..., $D_c(\theta) = \text{diag}(D_{ci}(\theta))$.

B. Stabilizing LPV Controllers

The stability of LPV systems can be assessed by a parameter-dependent Lyapunov function. The next lemma brings a well-known stabilizability condition which assumes a quadratic parameter-dependent Lyapunov function.

Lemma 1: **(PDQ Stability)**. System (1) is parametrically-dependent quadratically (PDQ) stabilizable by a discrete-time LPV controller (3)-(4) if, and only if, there exist $K(\theta)$ and symmetric $\mathcal{P}(\theta) > 0$ such that,

$$\begin{bmatrix} \mathcal{P}(\theta) & \mathcal{A}(\theta, K(\theta))'\mathcal{P}(\theta(k+1)) \\ \star & \mathcal{P}(\theta(k+1)) \end{bmatrix} > 0, \quad (8)$$

is satisfied $\forall\ \theta \in \Theta$ and $\forall\ \Delta\theta \in \mathcal{V}$. The symbol $\star$ means inferred by symmetry.

An affine θ-dependent Lyapunov function,

$$\mathcal{P}(\theta) = P_0 + \sum_{i=1}^{n_\theta} \theta_i P_i, \quad (9)$$

is a natural choice for A-LPV systems, where $P_0 > 0$ and $P_i > 0$ are symmetric matrices $\in \mathbb{R}^n$.

C. Performance Level LPV Controllers

The design of a controller can also consider performance level specifications. Define $T_{zw}(\theta)$ as the input-output operator that provides the forced response of (6) to an input signal $w(k) \in \mathcal{L}_2$ for zero initial conditions. The next lemma states a condition for computing a controller with guaranteed upper-bound on the induced $\mathcal{L}_2$-norm of $T_{zw}(\theta)$ [20].

Lemma 2: **($\mathcal{L}_2$-norm performance)**[20]. $\|T_{zw}(\theta)\|_{i,2}^2 < \gamma$ holds, if there exist symmetric matrix $\mathcal{P}(\theta) > 0$ and $K(\theta)$ such that,

$$\begin{bmatrix} \mathcal{P}(\theta) & \mathcal{A}(\cdot)'\mathcal{P}(\theta(k+1)) & 0 & \mathcal{C}(\cdot)' \\ \star & \mathcal{P}(\theta(k+1)) & \mathcal{P}(\theta(k+1))\mathcal{B}(\cdot) & 0 \\ \star & \star & \gamma I & \mathcal{D}(\cdot)' \\ \star & \star & \star & I \end{bmatrix} > 0, \qquad (\cdot) = (\theta, K(\theta)), \quad (10)$$

is feasible $\forall\theta \in \Theta$ and $\forall\Delta\theta \in \mathcal{V}$.

D. Multi-Convexity

The multi-convexity property for matrix functions polynomially dependent on the parameters [22] is useful to turn untractable, infinite-dimensional LMI problems into tractable, finite-dimensional ones.

Lemma 3: **(Multi-convexity)**[22] Consider a polynomially θ-dependent LMI of the form,

$$\mathcal{F}(\theta, z) := \sum_{v \in J} \theta^{[v]} M_v(z) > 0,$$

where M_v denote symmetric matrix-valued linear functions of the decision variable z. The notation $[v]$ is the vector of partial degrees $[v] = [v_1, \ldots, v_N]$ associated with the lexicographically ordered term,

$$\theta^{[v]} = \theta_1^{v_1} \theta_2^{v_2} \ldots \theta_N^{[v_N]},$$

with the convention $\theta^{[0]} = 1$. J is a set of N-tuples of partial degrees describing the polynomial expansion. The symbols d_k and d designate the partial and total degrees in the matrix polynomial expansion. Then, the LMI condition,

$$\mathcal{F}(\theta, z) > 0, \quad \forall \theta \in \Theta,$$

hold for some z, whenever the finite set of LMI,

$$\mathcal{F}(\theta, z) > 0, \quad \forall \theta \in \text{Vert}\,\Theta, \tag{11a}$$

$$(-1)^m \frac{\partial^{2m}}{\partial \theta_{l_1}^2 \ldots \partial \theta_{l_m}^2} \mathcal{F}(\theta, z) \geq 0, \quad \forall \theta \in \text{Vert}\,\Theta, \tag{11b}$$

where $1 \leq l_1 \leq l_2 \leq \ldots \leq l_m \leq N$, $1 \leq m \leq d/2$, $2\sharp\{l_j = k : j \in \{1, \ldots, m\}\} \leq d_k$, $k = 1, 2, \ldots, N$.

III. MAIN RESULTS

In this section, some of the results are only developed for (8). The PDQ stability inequality appears as a sub-block of (10). Therefore, results for controllers with guaranteed performance level are obtained similarly. The next lemma states an equivalence between a stabilizing A-LPV controller synthesis condition and a non-convex inequality.

Lemma 4: Define

$$\mathcal{H}(\theta, \Delta\theta) := P_0 + \sum_{i=1}^{n_\theta} (\theta_i + \Delta\theta_i) P_i. \tag{12}$$

The following conditions are equivalent. An A-LPV system (1)-(2) is PDQ stabilizable by an A-LPV controller (3)-(4), certified by (9), if there exist $K(\theta)$, symmetric $P_0 > 0$ and symmetric $P_i > 0$ such that,

1) inequality (8),
2)
$$\begin{bmatrix} \mathcal{P}(\theta) & \mathcal{A}(\theta, K(\theta))' \\ \star & \mathcal{H}(\theta, \Delta\theta)^{-1} \end{bmatrix} > 0, \tag{13}$$

is satisfied $\forall\, \theta \in \Theta$ and $\forall\, \Delta\theta \in \mathcal{V}$.

Proof: Substitution of (9) in (8). The fact that $\theta(k+1) = \theta(k) + \Delta\theta$, and a direct application of the Schur complement results in (13). ∎

The main advantage of (13) is the absence of multiplication between $\mathcal{A}(\theta, K(\theta))$ and other matrix variables. The drawback of (13) is the non-convex matrix inverse located at the entry (2,2). Inspired by [16][17][18], the next lemma relates a parameter-dependent matrix inverse with a concave matrix functional. This lemma will be useful to find a relaxed stability condition.

Lemma 5: **(Convexifying Inequality)**. For symmetric matrix $\mathcal{H}(\theta, \Delta\theta) > 0$ and matrix $\mathcal{G}(\theta, \Delta\theta)$,

$$\mathcal{H}(*)^{-1} \geq -\mathcal{G}(*)'\mathcal{H}(*)\mathcal{G}(*) + \mathcal{G}(*)' + \mathcal{G}(*), \quad (*) = (\theta, \Delta\theta), \tag{14}$$

hold $\forall\, \theta \in \Theta$ and $\forall\, \Delta\theta \in \mathcal{V}$.

A relaxed PDQ stability condition can be derived from the concave inequality (14) and (13) as follows.

Lemma 6: The following conditions are equivalent.

1) There exist symmetric $P_0 > 0$, symmetric $P_i > 0$ and $K(\theta)$ such that (13) is satisfied $\forall \theta \in \Theta$ and $\forall \Delta\theta \in \mathcal{V}$.
2) There exist symmetric $P_0 > 0$, symmetric $P_i > 0$, $K(\theta)$, and full slack matrix $\mathcal{G}(\theta, \Delta\theta)$ such that (15) is satisfied $\forall\, \theta \in \Theta$ and $\forall\, \Delta\theta \in \mathcal{V}$.

$$\begin{bmatrix} \mathcal{P}(\theta) & \mathcal{A}(\theta, K(\theta))' \\ \star & -\mathcal{G}(*)'\mathcal{H}(*)\mathcal{G}(*) + \mathcal{G}(*)' + \mathcal{G}(*) \end{bmatrix} > 0 \quad (*) = (\theta, \Delta\theta) \tag{15}$$

Proof: A direct substitution of the $\mathcal{H}(\theta, \Delta\theta)^{-1}$ entry located at the (2,2) position of (13) by the right hand side of (14) results in (15). The equivalence occurs when $\mathcal{G}(\theta, \Delta\theta) = \mathcal{H}(\theta, \Delta\theta)^{-1}$. ∎

Conditions for the design of LPV controllers with garanteed performance level γ can be derived analogously.

Lemma 7: $\|T_{zw}(\theta)\|_{i,2}^2 < \gamma$ holds, if there exist symmetric $P_0 > 0$, symmetric $P_i > 0$, $K(\theta)$ and $\mathcal{G}(\theta, \Delta\theta)$, such that,

$$\begin{bmatrix} \mathcal{P}(\theta) & \mathcal{A}(\cdot)' & 0 & \mathcal{C}(\cdot)' \\ \star & -\mathcal{G}(*)'\mathcal{H}(*)\mathcal{G}(*) + \mathcal{G}(*)' + \mathcal{G}(*) & \mathcal{B}(\cdot) & 0 \\ \star & \star & \gamma I & \mathcal{D}(\cdot)' \\ \star & \star & \star & I \end{bmatrix} > 0, \quad (\cdot) = (\theta, K(\theta)), \quad (*) = (\theta, \Delta\theta), \tag{16}$$

holds $\forall\, \theta \in \Theta$ and $\forall\, \Delta\theta \in \mathcal{V}$.

Proof: The proof follows the same arguments of Lemma 6 and will be ommitted for brevity. ∎

The matrix $\mathcal{G}(\theta, \Delta\theta)$, hereafter also denoted slack matrix, is assumed affine $(\theta, \Delta\theta)$-dependent,

$$\mathcal{G}(\theta, \Delta\theta) = G_0 + \sum_{i=1}^{n_\theta} (\theta_i + \Delta\theta_i) G_i, \tag{17}$$

where G_0, G_i are full matrices $\in \mathbb{R}^n$.

All matrix functions previously shown are infinite dimensional in $(\theta, \Delta\theta)$, thus computationally untractables. Finite dimensional conditions, where inequalities are checked at the vertices, are described next.

Theorem 1: **(Finite-Dimensional PDQ Stability)**. The infinite-dimensional condition (15) hold $\forall\theta \in \Theta$, $\forall\Delta\theta \in \mathcal{V}$, whenever,

$$\begin{bmatrix} \mathcal{P}(\theta) & \mathcal{A}(\theta, K(\theta))' \\ \star & -\mathcal{G}(*)'\mathcal{H}(*)\mathcal{G}(*) + \mathcal{G}(*)' + \mathcal{G}(*) \end{bmatrix} > \sum_{i=1}^{n_\theta} \theta_i^2 \lambda_i I,$$

$$\lambda_i \geq 0, \quad \forall\theta \in \text{Vert}\,\Theta, \quad \forall\Delta\theta \in \text{Vert}\,\mathcal{V}, \tag{18a}$$

$$\begin{bmatrix} 0 & \Gamma_i' \\ \star & \Phi_i \end{bmatrix} \geq -\lambda_i I, \tag{18b}$$

$$\begin{aligned}
\Gamma_i &= 3\theta_i B_{u,i} D_{c,i} C_{y,i} \\
&+ \sum_{j=1, j\neq i}^{n_\theta} \big(B_{u,i} D_{c,i} C_{y,j} + B_{u,i} D_{c,j} C_{y,i}\big)\theta_j \\
&+ \big(B_{u,0} D_{c,i} C_{y,i} + B_{u,i} D_{c,0} C_{y,i} + B_{u,i} D_{c,i} C_{y,0}\big), \\
\Phi_i &= 3(\theta_i + \Delta\theta_i) G_i' P_i G_i \\
&+ \sum_{j=1, j\neq i}^{n_\theta} \big(G_i' P_i G_j + G_i' P_j G_i\big)(\theta_j + \Delta\theta_j) \\
&+ G_0' P_i G_i + G_i' P_0 G_i + G_i' P_i G_0, \\
&\quad i = 1, 2, \ldots, n_\theta, \quad \forall\theta \in \text{Vert}\,\Theta, \quad \forall\Delta\theta \in \text{Vert}\,\mathcal{V}.
\end{aligned} \tag{18c}$$

Proof: Inequality (18a) is a direct result of (11a) applied to a modified, more strict condition (15), where a term dependent on a scalar λ_i was added to the right hand side. It is of interest to note that (14)-(17) is cubically $(\theta, \Delta\theta)$-dependent ($d = 3$). Condition (11b) applied to (18a) results in the relaxed version (18b)-(18c).

■

To correct the indefiniteness of (18b) is the main reason for incorporating $\lambda_i I$ into the formulation. An obvious, alternative way to correct the indefiniteness is to take the right hand side of (18a) and of (18b) as 0 (zero) and $\lambda_i I$, respectively, with $\lambda_i > 0$. This approach can lead to excessively strict multi-convexity conditions (18b). By strengthening (11a), (11b) can be slightly relaxed [22].

Multi-convexity matrices can be greatly simplified if B_u, D_{zu}, C_y and D_{yw} are parameter-independent. In fact, (18b) reduces to,

$$\Phi_i \geq 0, \quad i = 1, 2, \ldots, n_\theta, \quad \forall\theta \in \text{Vert}\,\Theta, \forall\Delta\theta \in \text{Vert}\,\mathcal{V}.$$

If B_u and D_{zu} are parameter-dependent, they can be easily converted into constant matrices by pre-filtering the control input $u(k)$. If C_y and D_{yw} are parameter-dependent, they can be converted into constant matrices by post-filtering the measured variable $y(k)$ [21]. Therefore, the presented formulation can be simplified at the expense of extra states.

A finite-dimensional condition for controllers with guaranteed performance γ is obtained by using similar arguments. For the sake of simplicity, B_u, D_{zu}, C_y and D_{yw} are from now on considered parameter-independent.

Theorem 2: **(Finite-Dimensional Performance Level)**. The infinite-dimensional condition (16) hold $\forall\theta \in \Theta$, $\forall\Delta\theta \in \mathcal{V}$, whenever,

$$\begin{bmatrix} \mathcal{P}(\theta) & \mathcal{A}(\cdot)' & 0 & \mathcal{C}(\cdot)' \\ \star & -\mathcal{G}(*)'\mathcal{H}(*)\mathcal{G}(*) + \mathcal{G}(*)' + \mathcal{G}(*) & \mathcal{B}(\cdot) & 0 \\ \star & \star & \gamma I & \mathcal{D}(\cdot)' \\ \star & \star & \star & I \end{bmatrix}$$

$$> 0, \quad \forall\theta \in \text{Vert}\,\Theta, \forall\Delta\theta \in \text{Vert}\,\mathcal{V}, \tag{19a}$$

$$\Phi_i \geq 0, \quad i = 1, 2, \ldots, n_\theta, \quad \forall\theta \in \text{Vert}\,\Theta, \forall\Delta\theta \in \text{Vert}\,\mathcal{V}. \tag{19b}$$

A. Iterative Algorithm

Matrix inequalities (15), (16) are non-convex functions due to the product of P_0 and P_i with $\mathcal{G}(*)$. In order to make the problem computationally tractable, a sequential LMI-based optimization algorithm is here proposed.

In the proposed iteration scheme, the slack matrix $\mathcal{G}(*)$ is kept constant during an iteration. An *iteration* is here referred to as the solution of a LMI optimization. The value of the slack matrix is updated at each iteration. Thus, the iterative algorithm facilitates the use of $\mathcal{G}(*)$ as a *parameter-dependent slack variable*.

The update rule for $\mathcal{G}(*)$ is now presented. As [17] [18] suggested (to the LTI case), the concave inequality (14) can be interpreted as the parametrization of all Taylor expansions of the function $f(\mathcal{H}(\theta, \Delta\theta)) = \mathcal{H}(\theta, \Delta\theta)^{-1}$, The linearization at a particular point $H_0(\theta, \Delta\theta)$ arises when $\mathcal{G}(\theta, \Delta\theta) = H_0(\theta, \Delta\theta)^{-1}$. This fact governs the decision to an update rule of the form,

$$\begin{aligned}
G(\theta, \Delta\theta)^{\{j+1\}} &= \left(P_0^{\{j\}} + \sum_{i=1}^{n_\theta} (\theta_i + \Delta\theta_i)\, P_i^{\{j\}} \right)^{-1} \\
&\leq \left(P_0^{\{j\}\,-1} + \sum_{i=1}^{n_\theta} (\theta_i + \Delta\theta_i)\, P_i^{\{j\}\,-1} \right), \\
G_0^{\{j+1\}} &:= P_0^{\{j\}\,-1}, \quad G_i^{\{j+1\}} := P_i^{\{j\}\,-1},
\end{aligned} \tag{20}$$

where $^{\{\cdot\}}$ is the iteration index and j is the current iteration number. With the update rule at hand, the iterative algorithm can be conceptually described as follows.

Algorithm 1: **(Conceptual)**
Set an initial $G(\theta, \Delta\theta)^{\{0\}}$, $j = 0$ and start to iterate:

1) Find $P_0^{\{j\}} > 0$, $P_i^{\{j\}} > 0$, $\lambda_i^{\{j\}}$ and $K(\theta)^{\{j\}}$ that solve (18) or (19) with $\mathcal{G} = G(\theta, \Delta\theta)^{\{j\}}$.
2) If a stopping criterion is satisfied, exit. Otherwise, compute $G(\theta, \Delta\theta)^{\{j+1\}}$ according to (20). Update possible terms of interest. Increment $j = j + 1$ and go to step 1.

The general description of Algorithm 1 can be particularized to address *feasibility* as well as *optimization* problems. Due to page constraints, only the optimization problem of computing a controller with minimum performance level γ is addressed.

Algorithm 2: **(Performance level γ)**

Set an initial $G(\theta,\Delta\theta)^{\{0\}}$, a tolerance ϵ, $j = 0$ and start to iterate:

1) Find $P_0^{\{j\}}$, $P_i^{\{j\}}$, $K(\theta)^{\{j\}}$, $\lambda_i^{\{j\}}$, and $\gamma^{\{j\}}$ that solve, Minimize γ subject to (19) with $\mathcal{G}(*) = G(\theta,\Delta\theta)^{\{j\}}$.
2) If $\left|\gamma^{\{j\}} - \gamma^{\{j-1\}}\right| \le \epsilon$, stop. Otherwise, compute $G(\theta,\Delta\theta)^{\{j+1\}}$ according to (20), set $j = j+1$ and go to step 1.

IV. NUMERICAL EXAMPLES

A. Decentralized A-LPV Controller

Using two decentralized and strictly proper dynamic output feedback controllers, the control objective is to minimize the upper bound γ on the induced $\mathcal{L}_2$-norm from $w(k)$ to $z(k)$. A controller measures the first output and manipulates the first input, and another controller measures the second output and manipulates the second input. The discrete-time system matrices are,

$$A(\theta) = \begin{bmatrix} 0.7370 & 0.0777 & 0.0810 & 0.0732 \\ 0.2272 & 0.9030 & 0.0282 & 0.1804 \\ -0.0490 & 0.0092 & 0.7111 & -0.2322 \\ -0.1726 & -0.0931 & 0.1442 & 0.7744 \end{bmatrix} + \theta \begin{bmatrix} 0.0819 & 0.0086 & 0.0090 & 0.0081 \\ 0.0252 & 0.1003 & 0.0031 & 0.0200 \\ -0.0055 & 0.0010 & 0.0790 & -0.0258 \\ -0.0192 & -0.0103 & 0.0160 & 0.0860 \end{bmatrix},$$

$$B_w = \begin{bmatrix} 0.0953 & 0 & 0 \\ 0.0145 & 0 & 0 \\ 0.0862 & 0 & 0 \\ -0.0011 & 0 & 0 \end{bmatrix},\ B_u = \begin{bmatrix} 0.0045 & 0.0044 \\ 0.1001 & 0.0100 \\ 0.0003 & -0.0136 \\ -0.0051 & 0.0936 \end{bmatrix},$$

$$C_z = \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},\ C_y = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix},$$

$$D_{zu} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix},\ D_{yw} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},\ D_{zw} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Time-varying parameters are assumed to range over the hypercubes $-1 \le \theta \le 1$, $-0.01 \le \Delta\theta \le 0.01$. Plant dynamics differs significantly over the parameter space. For $\theta = 1$, the plant has an unstable mode in 1.0192, and for $\theta = -1$, the plant is stable.

The algorithm needs to be initialized with initial values $G(\theta,\Delta\theta)^{\{0\}}$ to each of the vertices of the parameter space. The nominal system ($\theta = 0$) in closed-loop with the following linear time-invariant decentralized controller borrowed from [16] is utilized in the well-known H_∞ LMI condition for discrete-time LTI systems. The resulting parameter-independent Lyapunov matrix P_{LTI} served as initial slack matrix $G(\theta,\Delta\theta)^{\{0\}} = P_{LTI}^{-1}$ to all vertices of the parameter space.

$$K = \begin{bmatrix} D_{c1} & 0 & C_{c1} & 0 \\ 0 & D_{c2} & 0 & C_{c2} \\ B_{c1} & 0 & A_{c1} & 0 \\ 0 & B_{c2} & 0 & A_{c2} \end{bmatrix},\ A_{c1} = \begin{bmatrix} 1.26 & -0.44 \\ 1.00 & 0 \end{bmatrix},$$

$$A_{c2} = \begin{bmatrix} -0.06 & 0.6552 \\ 1.00 & 0 \end{bmatrix},\ B_{c1} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}, B_{c2} = \begin{bmatrix} 0.25 \\ 0 \end{bmatrix},$$

$$C_{c1} = \begin{bmatrix} -1.1150 & 0.7582 \end{bmatrix},\ C_{c2} = \begin{bmatrix} -0.2400 & 0.1824 \end{bmatrix},$$

Convergence tolerance was set to $5 \cdot 10^{-2}$. A performance level of $\gamma^{\{46\}} = 4.78$ is reached after 46 iterations. The evolution of $\gamma^{\{j\}}$ during the course of the optimization is depicted on fig. 1.

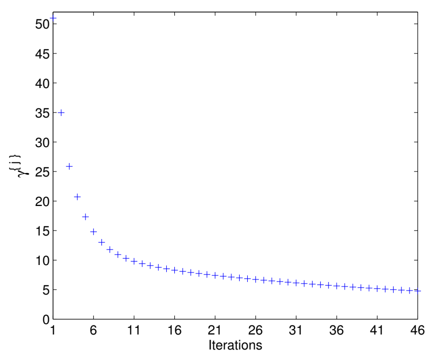


Fig. 1: Evolution of $\gamma^{\{j\}}$ for the decentralized A-LPV controller example.

B. Static Output Feedback A-LPV Controller

In this example, the aim is to minimize the upper bound γ on the induced $\mathcal{L}_2$-norm from $w(k)$ to $z(k)$ of the following system,

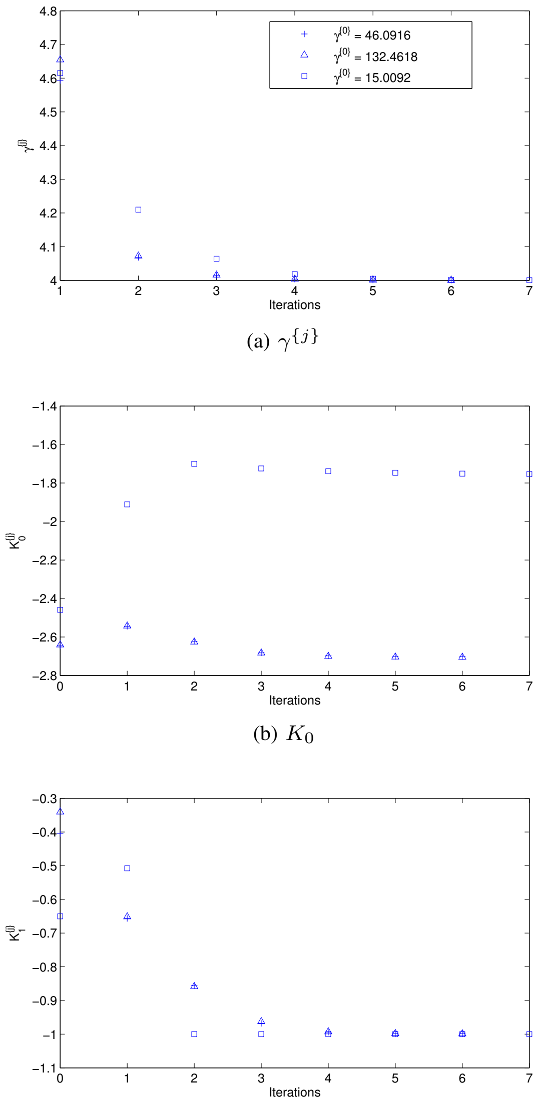
$$A(\theta) = \begin{bmatrix} 2 & 0 \\ 1 & 0.5 \end{bmatrix} + \theta \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix},\ B_u = [1 \quad 0]',$$

$$B_w = [0 \quad 1]',\ C_z = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix},\ C_y = [1 \quad 0],$$

$$D_{zw} = [0 \quad 0]',\ D_{zu} = [0 \quad 1]',\ D_{yw} = [0 \quad 1]',$$

controlled by an static output feedback. The time-varying parameter bounds are $-0.5 \le \theta \le 0.5$, $-0.01 \le \Delta\theta \le 0.01$.

Three different values of $G(\theta,\Delta\theta)^{\{0\}}$ will empirically show how sensitive γ is to different initial conditions. Convergence tolerance was set to $1 \cdot 10^{-3}$. For illustration purposes, the controller is represented as $K(\theta) = K_0 + K_1\theta$. Figures 2a to 2c depict the evolution of γ, K_0 and K_1. $\gamma^{\{0\}}$ is removed from the plot due to its large value. $\gamma^{\{0\}}$ values largely differs for each initial $G(\theta,\Delta\theta)^{\{0\}}$. Nonetheless, the algorithm reaches the same minimum of $\gamma = 4.0$. $\gamma^{\{1\}}$ values are quite similar, despite the considerably different initial conditions. K_1 converges to the same value irrespective of $G(\theta,\Delta\theta)^{\{0\}}$. The same does not happen to K_0.



(a) $\gamma^{\{j\}}$

(b) K_0

(c) K_1

Fig. 2: Evolution of variables for the A-LPV static output feedback example.

V. CONCLUSIONS

This paper proposed a procedure for structured control design of affine LPV systems in discrete-time, considering bounds on the rate of variation. A parameter-varying non-convex condition for the induced $\mathcal{L}_2$-norm performance level is solved via an iterative LMI algorithm. The value of a slack matrix is updated at each iteration, thus the algorithm facilitates the usage of parameter-dependent slack variables. Two numerical examples, a decentralized dynamic output and a static output feedback, illustrate the proposed approach.

The slack matrix was restricted to be a linearization point of the Taylor expansion of an inverse parameter-dependent matrix functional. The study of more general rules for the update of the slack matrix is a subject of future work. Algorithm's convergence and optimality properties are yet unknown and needs investigation. The generalization to other types of parameter dependencies (e.g. polynomial, LFT) is also of future interest.

VI. ACKNOWLEDGMENTS

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