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A Note to Robustness Analysis of the Hybrid Consensus Protocols

Haopeng Zhang, Sean R. Mullen, and Qing Hui

Abstract— **The averaging problem for networked systems has attracted a significant amount of research interest these days. Averaging protocol design is much more challenging, whereas hybrid protocols seem to display some advantages, such as relatively fast consensus in finite time. As is well known, noise exists at nearly every stage of the control process, so it is necessary to consider the noise effects on consensus protocols. In this paper, we study the effect of noise on two types of hybrid consensus protocols, which turn out to exhibit strong robustness. Noise as a constant is investigated in detail, and a hybrid formation control protocol is proposed.**

I. INTRODUCTION

Motivated by behaviors found commonly in nature, such as flocks of birds and schools of fish that can achieve virtually flawless consensus formations via their information interaction, more and more research is focusing on the study of multi-agent behavior following the same idea [2]. Consensus for a networked system means that, for each agent of the system, through information interaction with its neighbors, the state of each agent can achieve the same value [1]. The wide array of applications of networked systems includes sensor-networked systems, mobile ground vehicles systems, and autonomous underwater vehicles (AUV's). The primary challenge is to design a proper consensus protocol that leads to consensus for the networked system–e.g., within a double integrator system investigated in [4], which is based on Newtonian mechanics, the velocity and displacement of each agent achieve consensus as time approaches infinity. [12] proposes some consensus protocols for general networked systems, and formation control is considered there as well. Nonlinear systems are also considered for the consensus problem. [8] presents a nonlinear consensus protocol which is based on system thermodynamic theory. In that paper, the finite-time property and semistability are investigated for the system as well.

In addition to deterministic systems, stochastic systems are studied widely as well. [13] develops a gossip algorithm for the network to achieve consensus, and in this paper the communication link between each pair of agents is randomly selected. A quantized gossip algorithm is proposed in [10], [11], and an upper bound for the convergence time is also presented therein. Furthermore, [19] proposes a gossip algorithm via a quantized communication, which is quite useful for broad practical applications. Meanwhile, ergodic theory has a close relationship with the consensus problem [14], [15], and covers a much more general case for stochastic systems.

Another main research interest is investigation into the effect of some non-ideal conditions, like time delay, quantization, and noise disturbances for networked systems. [8] develops a quantized consensus protocol under which the system can achieve near-consensus for continuous-time systems. However, it is a weaker requirement than exact-consensus, and the state of each agent of the system is bounded. Discrete-time consensus protocols are also developed in [9] and two different quantized protocols are proposed, where near-consensus and exact-consensus are achieved, respectively. Average consensus on networks is investigated with quantized communication and two different encoding/decoding strategies are presented in [18]. Moreover, since noise disturbance exists at nearly every stage of the control process, noise effects on consensus protocols are investigated extensively [20]–[23].

A hybrid consensus protocol is proposed in [17], which develops a novel framework for solving the fast consensus problem, in particular, the averaging problem. In this paper, we are attempting to study the robustness of this hybrid consensus protocol under the effect of certain kinds of noise. Based on linear systems theory and Lyapunov theory, the system displays a strong robustness quality in which the difference between the state of each agent is bounded. Additionally, a special case of constant scalar noise is considered in the paper, with which the states of the agents end up having errors between them. From a formation control point of view, we can design proper constant values for the hybrid protocol so that the dynamical system achieves the desired formation.

The organization of the paper is as follows. In Section II, some basic background on the graph topology and a brief review of the hybrid protocol in [17] is given. The analysis of the continuous part of the hybrid system is contained in Section III, with ideal conditions and noise disturbances, respectively. The effect of constant noise is emphasized. Together with the jump process, the robustness of the hybrid system is investigated. Moreover, a hybrid formation control protocol is proposed. Some simulation results are provided in Section IV, and finally, Section V concludes the paper and states further works.

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H. Zhang, S. R. Mullen, and Q. Hui are with the Department of Mechanical Engineering, Texas Tech University, Lubbock, TX 79409-1021, USA (haopeng.zhang@ttu.edu; sean.mullen@ttu.edu; qing.hui@ttu.edu).

II. Mathematical Background and Literature Review

Graph theory is a powerful tool for investigating networked control systems. In this paper, we use graph-related notation to describe our network model. More specifically, let $\mathscr{G} = (\mathscr{V}, \mathscr{E}, \mathscr{A})$ denote an undirected graph with the set of vertices $\mathscr{V} \doteq \{v_1, v_2, v_3, ...\}$ and $\mathscr{E} \subseteq \mathscr{V} \times \mathscr{V}$ represents the set of edges. The matrix $\mathscr{A}$ with nonnegative adjacency elements $a_{i,j}$ serves as the weighted adjacency matrix. The node index of $\mathscr{G}$ is denoted by a finite index set $\mathscr{N} = \{1, 2, 3, ...\}$. An edge of $\mathscr{G}$ is denoted by $e_{i,j} = (V_i, V_j)$ and the adjacency elements associated with the edges are positive. We assume $e_{i,j} \in \mathscr{E} \Leftrightarrow a_{i,j} = 1$ and $a_{i,i} = 0$ for all $i \in \mathscr{N}$.

If there is a path from any node to any other node in the graph, then we call the graph *connected*. Next, we define the connectivity matrix C for the graph.

Definition 2.1:

$$C_{i,j} \triangleq \begin{cases} 0, & \text{if } (i,j) \notin \mathscr{E}, \\ 1, & \text{otherwise}, \end{cases} \quad i \neq j, \quad i,j = 1, \ldots, q, \tag{1}$$

$$C_{i,i} \triangleq -\sum_{k=1,\, k\neq i}^{q} C_{i,k}, \quad i = 1, \ldots, q, \tag{2}$$

where q is the number of agents.

In this paper, we always assume that the graph topology of the multi-agent system is connected.

The consensus problem for networked systems has attracted more and more attention from many research fields, such as mathematics, engineering, and computer science, and many consensus protocols are proposed for the networked system. The hybrid consensus framework presented in [17] addresses the fast consensus-seeking problem for networked systems. A unique feature of the proposed framework is that the proposed controller architectures are hybrids and appear to achieve finite-time coordination, and hence, significantly improving the transient performance of the closed-loop system.

The hybrid consensus protocol we consider in this paper is given by

$$\begin{aligned}
\dot{x}_{ci}(t) &= -\sum_{j=1,j\neq i}^{q} C_{i,j}(x_{ci}(t) - x_{cj}(t)) \\
&\quad -\sum_{j=1,j\neq i}^{q} C_{i,j}(x_i(t) - x_j(t) - w_{i,j}) \\
&\quad (x_i(t), \bar{x}_i(t), x_{ci}(t), \bar{x}_{ci}(t)) \notin Z_i \\
&\quad x_{ci}(0) = x_{ci0}, \quad t \geq 0 \\
\dot{x}_i(t) &= \sum_{j=1,j\neq i}^{q} C_{i,j}(x_{ci}(t) - x_{cj}(t)) \\
x_{ci}(t^+) &= \mathrm{argmin}_{x_{ci}(t)} \sum_{j=1,j\neq i}^{q} C_{i,j} \parallel x_{ci}(t) - x_{cj}(t) \parallel_2^2 \\
&\quad (x_i(t), \bar{x}_i(t), x_{ci}(t), \bar{x}_{ci}(t)) \in Z_i
\end{aligned} \tag{3}$$

where the resetting set Z_i is given by

$$\begin{aligned} Z_i = \{(x_i, \bar{x}_i, x_{ci}, \bar{x}_{ci}) : \frac{d}{dt} L_i(x_i, \bar{x}_i) = 0 \\ L_i(x_{ci}, \bar{x}_{ci}) > \min_{x_{ci}} L_i(x_{ci}, \bar{x}_{ci})\} \end{aligned} \tag{4}$$

or

$$\begin{aligned} Z_i = \{(x_i, \bar{x}_i, x_{ci}, \bar{x}_{ci}) : \frac{d}{dt} L_i(x_{ci}, \bar{x}_{ci}) = 0 \\ L_i(x_{ci}, \bar{x}_{ci}) > \min_{x_i} L_i(x_{ci}, \bar{x}_{ci})\} \end{aligned} \tag{5}$$

where $L_i(x_i, \bar{x}_i) = \sum_{j=1,j\neq i}^{q} C_{i,j} \parallel x_i(t) - x_j(t) \parallel_2^2$ and $L_i(x_{ci}, \bar{x}_{ci}) = \sum_{j=1,j\neq i}^{q} C_{i,j} \parallel x_{ci}(t) - x_{cj}(t) \parallel_2^2$.

The above one is a state-dependent hybrid consensus protocol proposed in [17], in which a time-dependent hybrid consensus protocol is also proposed. Under those hybrid consensus protocols, it is shown that the network achieves the average fast and even in finite time [17].

III. Main Result

In this section, we intend to investigate the robustness of the hybrid consensus protocols in Section II with the presence of noise.

A. Ideal Conditions for the Hybrid Consensus Protocols

To investigate the asymptotic behavior of the hybrid system, we first study the continuous-time linear system between each jump. The continuous-time part of the protocol (3) can be represented in a vector form. Put $X = \begin{bmatrix} x_{c1} & \cdots & x_{cq} & x_1 & \cdots & x_q \end{bmatrix}^{\mathrm{T}}$, then the continuous-time system becomes

$$\dot{X} = \Phi \times X \tag{6}$$

where $\Phi = \begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix} \otimes L$ and L is the Laplacian matrix for the graph topology of the networked system. Then, we arrive at the following theorem for state averaging.

Theorem 3.1: For a connected networked system, each agent achieves the average consensus under the protocol (6).

To prove Theorem 3.1, the following lemma is needed.

Lemma 3.1: For a connected graph topology, the matrix Φ has the following properties:

- The matrix Φ has two 0 eigenvalues and the real parts of other eigenvalues are less than 0.
- The corresponding eigenvectors for the two eigenvalues 0 are $w_1 = \begin{bmatrix} \frac{1}{\sqrt{2q}}\mathbf{1}^{\mathrm{T}} & \frac{-1}{\sqrt{2q}}\mathbf{1}^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}}$ and $w_2 = \begin{bmatrix} \frac{-1}{\sqrt{2q}}\mathbf{1}^{\mathrm{T}} & \frac{1}{\sqrt{2q}}\mathbf{1}^{\mathrm{T}} \end{bmatrix}^{T}$ respectively.

Proof: First, we diagonalize matrix $\begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix}$ as follows:

$$\begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix} = v \times d \times v^{-1} \tag{7}$$

where v is an orthogonal matrix and d is a diagonal matrix. Furthermore,

$$\begin{aligned} (v \otimes I) \times \Phi \times (v^{-1} \otimes I) &= (v \times A \times v^{-1}) \otimes L \\ &= d \otimes L \end{aligned} \tag{8}$$

Since matrix L has a 0 eigenvalue and other eigenvalues' real parts are negative, the first item of the lemma has been proven.

Moreover,

$$\begin{aligned}\Phi \times w_1 &= \begin{bmatrix}-1 & -1\\ 1 & 0\end{bmatrix} \otimes L \times \begin{bmatrix}\frac{1}{\sqrt{2q}}\mathbf{1}^{\mathrm{T}} & \frac{-1}{\sqrt{2q}}\mathbf{1}^{\mathrm{T}}\end{bmatrix}^T\\ &= \begin{bmatrix}-L & -L\\ L & 0\end{bmatrix} \times \begin{bmatrix}\frac{1}{\sqrt{2q}}\mathbf{1} & \frac{-1}{\sqrt{2q}}\mathbf{1}\end{bmatrix}\\ &= \begin{bmatrix}\frac{-1}{\sqrt{2q}}L\mathbf{1} + \frac{1}{\sqrt{2q}}L\mathbf{1}\\ L\frac{1}{\sqrt{2q}}L\mathbf{1}\end{bmatrix}\\ &= \mathbf{0} \end{aligned} \tag{9}$$

The proof for w_2 is similar.

Also,

$$\begin{aligned} w_1^T \times w_1 &= \begin{bmatrix}\frac{1}{\sqrt{2q}}\mathbf{1}^{\mathrm{T}} & \frac{-1}{\sqrt{2q}}\mathbf{1}^{\mathrm{T}}\end{bmatrix} \times \begin{bmatrix}\frac{1}{\sqrt{2q}}\mathbf{1}\\ \frac{-1}{\sqrt{2q}}\mathbf{1}\end{bmatrix}\\ &= \frac{1}{2} + \frac{1}{2}\\ &= 1 \end{aligned} \tag{10}$$

Together,

$$\begin{aligned} w_2^T \times w_2 &= \begin{bmatrix}\frac{-1}{\sqrt{2q}}\mathbf{1}^{\mathrm{T}} & \frac{1}{\sqrt{2q}}\mathbf{1}^{\mathrm{T}}\end{bmatrix} \times \begin{bmatrix}\frac{-1}{\sqrt{2q}}\mathbf{1}\\ \frac{1}{\sqrt{2q}}\mathbf{1}\end{bmatrix}\\ &= \frac{1}{2} + \frac{1}{2}\\ &= 1 \end{aligned} \tag{11}$$

This ends the proof of the second conclusion. ■

Proof: [Proof of Theorem 3.1] According to linear systems theory, we can obtain the solution of our system (6) as follows:

$$\begin{aligned} X(t) &= e^{\Phi t}X_0\\ &= Pe^{\Delta t}P^{-1}X_0 \end{aligned} \tag{12}$$

Let $t \to \infty$, we have

$$\begin{aligned} X(\infty) &= \begin{bmatrix}w_1 & w_2\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}w_1^T\\ w_2^T\end{bmatrix}X_0\\ &= \begin{bmatrix}\frac{1}{q}\times\mathbf{1}\mathbf{1}^{\mathrm{T}} & 0\\ 0 & \frac{1}{q}\times\mathbf{1}\mathbf{1}^{\mathrm{T}}\end{bmatrix}X_0 \end{aligned} \tag{13}$$

which completes the proof of Theorem 3.1. ■

B. Noise Disturbance

In this subsection, we study the noise effect on the system (6). Before we introduce the result, the following assumptions are needed.

Assumption 1: Suppose the disturbances for node i is same or positive at any time t.

Theorem 3.2: For a connected networked system (6) with any noise disturbance satisfying $Assumption 1$, $x_i = c$, c is some constant real number, furthermore, $x_{cj} - x_{ci} = 0$ for $i, j = 1 : q$

Proof: The system dynamics with noise disturbance we consider here is given by

$$\begin{aligned} \dot{x}_{ci}(t) &= -\sum_{j=1,j\neq i}^{q} C_{i,j}(x_{ci}(t) - x_{cj}(t))\\ &\quad -\sum_{j=1,j\neq i}^{q} C_{i,j}(x_i(t) - x_j(t) - w_{i,j})\\ \dot{x}_i &= \sum_{j=1,j\neq i}^{q} C_{i,j}(x_{ci}(t) - x_{cj}(t)) \end{aligned} \tag{14}$$

where $i \in \{1, \cdots, q\}$, and $w_{i,j}$ is the noise generated when communication between x_i and x_j occurs.

Consider the nonnegative function

$$\begin{aligned} V(X) &= \frac{1}{4}\sum_{i=1}^{q}\sum_{j=1,j\neq i}^{q} \| x_j - x_i \|_2^2\\ &\quad + \frac{1}{4}\sum_{i=1}^{q}\sum_{j=1,j\neq i}^{q} \| x_{cj} - x_{ci} \|_2^2 \end{aligned} \tag{15}$$

where $X = \left[x_1^T, \cdots, x_q^T, x_{c1}^T, \cdots, x_{cq}^T\right]^T$. The derivative of the function $V(X)$ along the trajectories of the closed-loop dynamics is given by

$$\begin{aligned} \dot{V}(X) &= \frac{1}{2}\sum_{i=1}^{q}\sum_{j=1,j\neq i}^{q}(C_{i,j} + C_{j,i})(x_i - x_j)^T\dot{x}_i\\ &\quad \frac{1}{2}\sum_{i=1}^{q}\sum_{j=1,j\neq i}^{q}(C_{i,j} + C_{j,i})(x_{ci} - x_{cj})^T\dot{x}_{ci}\\ &= -\sum_{i=1}^{q}\left[\sum_{i=1}^{q}C_{i,j}(x_{ci} - x_{cj})\right]^T\\ &\quad \times\left[\sum_{i=1}^{q}C_{i,j}(x_{ci} - x_{cj})\right]\\ &\quad + \frac{1}{2}\sum_{i=1}^{q}\sum_{j=1,j\neq i}^{q}(C_{i,j} + C_{j,i})(x_{ci} - x_{cj})^T\\ &\quad \times\sum_{j=1,j\neq i}^{q}C_{i,j}(-w_{i,j})\\ &\leq -\sum_{i=1}^{q}\left[\sum_{i=1}^{q}C_{i,j}(x_{ci} - x_{cj})\right]^T\\ &\quad \times\left[\sum_{i=1}^{q}C_{i,j}(x_{ci} - x_{cj})\right]\\ &\leq 0 \end{aligned} \tag{16}$$

Considering (16), $\dot{V} = 0$ if and only if $\sum_{i=1}^{q} C_{i,j}(x_{ci} - x_{cj}) = 0$, and furthermore, $L \times X_c = \mathbf{0}$, thus, $x_{cj} - x_{ci} = 0$. If $x_{cj} - x_{ci} = 0$, then $\dot{x}_i = 0$, and $\sum \dot{x_{ci}} = \sum w_{ij}$, since $x_{cj} = x_{ci}$, then $\dot{x}_{ci} = w_{i,j}$, and furthermore, $\sum_{i=1}^{q} C_{i,j}(x_i - x_j) = 0$, and $x_i = x_j$. Therefore, $x_{ci} = x_{ci}(t_0) + \int_{t=0}^{\infty}(w(s)_{i,j})ds$ ■

Next, in this subsection, we will find the effect of constant noises on (6). First, we can rewrite the system in vector form:

$$\dot{X} = \Phi X + \Psi \tag{17}$$

where

$$\Psi = \begin{bmatrix} -\sum_{j\in N_1} a_{1,j} w_{1,j} \\ -\sum_{j\in N_2} a_{2,j} w_{2,j} \\ \vdots \\ -\sum_{j\in N_q} a_{q,j} w_{q,j} \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

The following lemma gives an equivalent vector form of Ψ:

Lemma 3.2: Given the networked system (6), the matrix Ψ can be represented as $\begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix} \otimes L \times \begin{bmatrix} 0 & w_{12} & \cdots & w_{1q} & 0 & \cdots & 0 \end{bmatrix}^T$.

Proof: Suppose the noise $w_{ij} = w_{1j} - w_{1i}$, for any $i = 1, 2, \cdots, q$. Then we have

$$\begin{aligned} &\textstyle\sum_{j\in N_i} a_{i,j} w_{i,j} = \sum_{j\in N_i} a_{ij}(w_{1,i} - w_{1,j}) \\ &= \begin{bmatrix} a_{i,1} & \cdots & \sum_{j=1, j\neq i}^{q} a_{i,j} & a_{i,i+1} & \cdots & a_{i,q} \end{bmatrix} \\ &\quad \times \begin{bmatrix} 0 & w_{1,2} & \cdots & w_{1,q} & 0 & \cdots & 0 \end{bmatrix}^T \\ &= -L_i \times \begin{bmatrix} 0 & w_{1,2} & \cdots & w_{1,q} & 0 & \cdots & 0 \end{bmatrix}^T \end{aligned} \tag{18}$$

where L_i is the ith row of the Laplacian matrix L. Hence,

$$\Psi = \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix} \otimes L \times \begin{bmatrix} 0 & w_{12} & \cdots & w_{1q} & 0 & \cdots & 0 \end{bmatrix}^T$$

■

Based on linear systems theory, we can write the solution for the system as

$$X(t) = \exp^{\Phi t} X_0 + \int_0^t \exp^{\Phi(t-\tau)} \Psi(\tau) d\tau \tag{19}$$

As a result of Theorem 3.1,

$$\exp^{\Phi t} X_0 = \begin{bmatrix} \frac{1}{q} \times \mathbf{I} & 0 \\ 0 & \frac{1}{q} \times \mathbf{I} \end{bmatrix} X_0 \tag{20}$$

Lemma 3.3: For the noise corrupted system (17), the other part of the solution has the following form:

$$\begin{aligned} &\textstyle\lim_{t\to\infty} \int_0^t \exp^{\Phi(t-\tau)} \Psi(\tau) d\tau \\ &= \begin{bmatrix} 0 & \cdots & 0 & 0 & w_{12} & \cdots & w_{1q} \end{bmatrix}^T \\ &\quad -\tfrac{1}{q} \textstyle\sum_{j=2}^{q} w_{1,j} \otimes \begin{bmatrix} \mathbf{0} \\ \mathbf{1} \end{bmatrix} \end{aligned} \tag{21}$$

Proof:

$$\begin{aligned} &\textstyle\lim_{t\to\infty} \int_0^t \exp^{\Phi(t-\tau)} \Psi(\tau) d\tau \\ &= P \textstyle\lim_{t\to\infty} \int_0^t \exp^{D(t-\tau)} P^{-1} \\ &= \begin{bmatrix} w_1 & w_2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} w_1^T \\ w_2^T \end{bmatrix} \\ &+ \begin{bmatrix} w_3 & \cdots & w_{2q} \end{bmatrix} diag\left(\tfrac{1}{\lambda_{\Phi,3}}, \cdots, \tfrac{1}{\lambda_{\Phi,2q}}\right) \begin{bmatrix} e_3 \\ \vdots \\ e_{2q} \end{bmatrix} \end{aligned} \tag{22}$$

where $w_3, w_4, \ldots$ are the 3rd, 4th,...and $2q$th columns for the matrix Φ, and $e_3, e_4, \ldots$ are the 3rd, 4th,..., and $2q$th rows of matrix P. Obviously,

$$\begin{bmatrix} w_1 & w_2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} w_1^T \\ w_2^T \end{bmatrix} = \frac{1}{q} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \otimes \mathbf{1}\mathbf{1}^T \tag{23}$$

And,

$$\begin{aligned} &\begin{bmatrix} w_3 & \cdots & w_{2q} \end{bmatrix} diag\left(\frac{1}{\lambda_{\Phi,3}}, \cdots, \frac{1}{\lambda_{\Phi,2q}}\right) \begin{bmatrix} e_3 \\ \vdots \\ e_{2q} \end{bmatrix} \\ &\qquad = \left(\Phi - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \otimes \mathbf{1}\mathbf{1}^T\right)^{-1} \end{aligned} \tag{24}$$

Next, we will prove

$$\left(\Phi - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \otimes \mathbf{1}\mathbf{1}^T\right)^{-1} = \begin{bmatrix} -\frac{1}{q}\mathbf{1}\mathbf{1}^T & L_1 \\ -L_1 & L_M \end{bmatrix}$$

where $LL_1 = I - \frac{1}{q}\mathbf{1}\mathbf{1}^T$, $LL_M = -\frac{1}{q}\mathbf{1}\mathbf{1}^T - I$, $L_1 \times \frac{1}{q}\mathbf{1}\mathbf{1}^T = \mathbf{0}$ and $-\frac{1}{q}\mathbf{1}\mathbf{1}^T \times (-L_M) = \frac{1}{q}\mathbf{1}\mathbf{1}^T$

To see this, note that

$$\begin{aligned} &\left(\Phi - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \otimes \mathbf{1}\mathbf{1}^T\right) \begin{bmatrix} -\frac{1}{q}\mathbf{1}\mathbf{1}^T & L_1 \\ -L_1 & L_M \end{bmatrix} \\ &= \begin{bmatrix} -L - \frac{1}{q}\mathbf{1}\mathbf{1}^T & -L \\ L & -\frac{1}{q}\mathbf{1}\mathbf{1}^T \end{bmatrix} \begin{bmatrix} -\frac{1}{q}\mathbf{1}\mathbf{1}^T & L1 \\ -L1 & LM \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{q}\mathbf{1}\mathbf{1}^T + LL_1 & LL_1 + LL_M \\ \mathbf{0} & LL_1 + \frac{1}{q}\mathbf{1}\mathbf{1}^T \end{bmatrix} \\ &= I \end{aligned} \tag{25}$$

Next,

$$\begin{aligned} &\textstyle\lim_{t\to\infty} \int_0^t \exp^{\Phi(t-\tau)} \Psi(\tau) d\tau \times \Psi \\ &= P \textstyle\lim_{t\to\infty} \int_0^t \exp^{D(t-\tau)} P^{-1} \Psi \\ &= \begin{bmatrix} w_1 & w_2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} w_1^T \\ w_2^T \end{bmatrix} \times \Psi \\ &+ \begin{bmatrix} w_3 & \cdots & w_{2q} \end{bmatrix} diag\left(\tfrac{1}{\lambda_{\Phi,3}}, \cdots, \tfrac{1}{\lambda_{\Phi,2q}}\right) \begin{bmatrix} e_3 \\ \vdots \\ e_{2q} \end{bmatrix} \times \Psi \\ &= \mathbf{0} + \begin{bmatrix} -\frac{1}{q}\mathbf{1}\mathbf{1}^T & L_1 \\ -L_1 & L_M \end{bmatrix} \times \Psi \\ &= \begin{bmatrix} -\frac{1}{q}\mathbf{1}\mathbf{1}^T & L_1 \\ -L_1 & L_M \end{bmatrix} \times (\begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix} \otimes L \\ &\quad \times \begin{bmatrix} 0 & w_{12} & \cdots & w_{1q} & 0 & \cdots & 0 \end{bmatrix}^T) \\ &= \begin{bmatrix} 0 & 0 \\ LL_1 & 0 \end{bmatrix} \begin{bmatrix} 0 & w_{12} & \cdots & w_{1q} & 0 & \cdots & 0 \end{bmatrix}^T \\ &= \begin{bmatrix} 0 & \cdots & 0 & 0 & w_{12} & \cdots & w_{1q} \end{bmatrix}^T \\ &\quad -\tfrac{1}{q} \textstyle\sum_{j=2}^{q} w_{1,j} \otimes \begin{bmatrix} \mathbf{0} \\ \mathbf{1} \end{bmatrix} \end{aligned}$$

■

Based on the previous result, we have the following theorem.

Theorem 3.3: For the connected system (17), the state of each agent of the system has a distance between the other agents as constant noise is presented and

$$X(\infty)=\begin{bmatrix}\frac{1}{q}\times\mathbf{I} & 0\\ 0 & \frac{1}{q}\times\mathbf{I}\end{bmatrix}X_0$$
$$+\begin{bmatrix}0 & \cdots & 0 & 0 & w_{12} & \cdots & w_{1q}\end{bmatrix}^T$$
$$-\frac{1}{q}\sum_{j=2}^{q}w_{1,j}\otimes\begin{bmatrix}\mathbf{0}\\ \mathbf{1}\end{bmatrix} \qquad (26)$$

C. Robustness of the Hybrid Consensus Protocols

In this subsection, we investigate the jump process's effect on the continuous-time linear system (17).

Theorem 3.4: Under the hybrid consensus protocol (3), the system is Lyapunov stable with the disturbance satisfying $Assumption 1$.

Proof: Since

$$\begin{aligned}\Delta V(X) &= \frac{1}{4}\sum_{i=1}^{q}\Big[\min\sum_{j=1,j\neq i}^{q}C_{i,j}\parallel x_{ci}(t)-x_{cj}(t)\parallel\Big]\\ &\quad -\frac{1}{4}\sum_{i=1}^{q}\sum_{j=1,j\neq i}qC_{i,j}\parallel x_{ci}(t)-x_{cj}(t)\parallel<0\\ &\quad X\in Z, \end{aligned}\qquad (27)$$

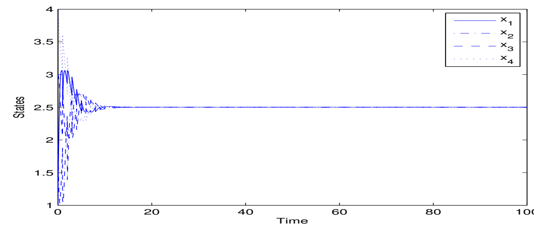
for $i,j=1,\cdots,q,i\neq j$, it follows that the set $D=\{x_i-x_j,x_{ci}-x_{cj}:V(X)\leq c\}$, where $c>0$, is a compact positively invariant set. According to Corollary 1 in [24], together with (16) and (27), the unconnected hybrid system is Lyapunov stable. ■

Furthermore, from a formation control point of view, we propose the following hybrid formation control protocol for the networked system. Here, the purpose of a formation control for the system is to control the distance between each agent of the system and then to control the formation of the entire system. In particular,

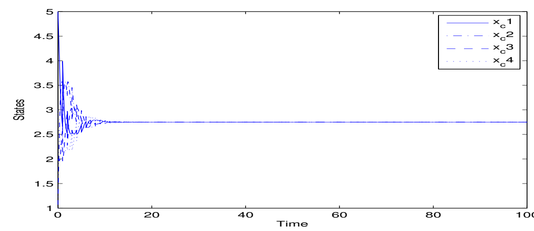
$$\begin{aligned}\dot{x}_{ci}(t) &= -\sum_{j=1,j\neq i}^{q}C_{i,j}(x_{ci}(t)-x_{cj}(t))\\ &\quad -\sum_{j=1,j\neq i}^{q}C_{i,j}(x_i(t)-x_j(t)-l_{i,j})\\ &\quad (x_i(t),\bar{x}_i(t),x_{ci}(t),\bar{x}_{ci}(t))\notin Z_i\\ &\quad x_{ci}(0)=x_{ci0},\quad t\geq 0\\ \dot{x}_i &= \sum_{j=1,j\neq i}^{q}C_{i,j}(x_{ci}(t)-x_{cj}(t))\\ x_{ci}(t^+) &= \mathrm{argmin}_{x_{ci}(t)}\sum_{j=1,j\neq i}^{q}C_{i,j}\parallel x_{ci}(t)-x_{cj}(t)\parallel_2^2\\ &\quad (x_i(t),\bar{x}_i(t),x_{ci}(t),\bar{x}_{ci}(t))\in Z_i \end{aligned}\qquad (28)$$

and

$$\begin{aligned}Z_i=\{(x_i,\bar{x}_i,x_{ci},\bar{x}_{ci}):\frac{d}{dt}L_i(x_{ci},\bar{x}_{ci})=0\\ L_i(x_{ci},\bar{x}_{ci})>\min_{x_i}L_i(x_{ci},\bar{x}_{ci})\}\end{aligned}\qquad (29)$$

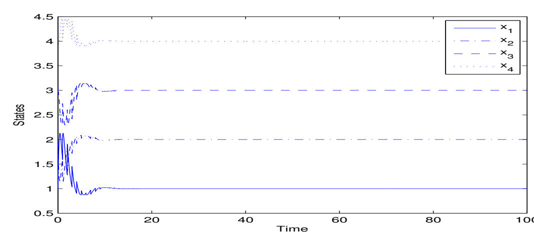


(a) Practical states

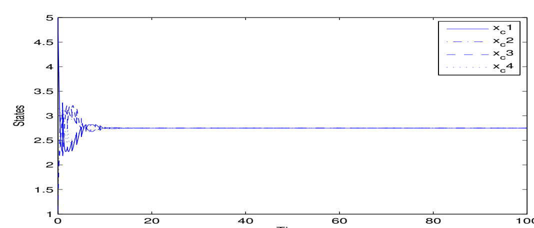


(b) Observation states

Fig. 1. Hybrid consensus protocol with ideal condition



(a) Practical states



(b) Observation states

Fig. 2. Hybrid consensus protocol with constant disturbance

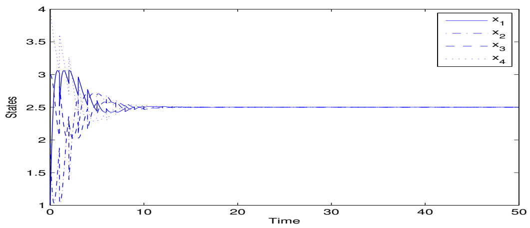
Define $d_{i,j}=x_j-x_i$, then we have the following corollary.

Corollary 3.1: For the connected system (28) and (29), as time approaches infinity, the system's formation becomes

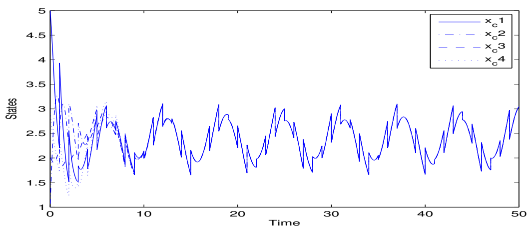
$$d_{i,j}(\infty)=l_{i,j} \qquad (30)$$

IV. Simulation

A four-agent system is considered in the simulation and the following figures display the system's behavior subject to different kinds of noises. Firstly, Fig. 1 shows the system evaluation under ideal condition. The noises are constant, sinusoidal, and exponential, respectively in Fig. 2, 3, 4.

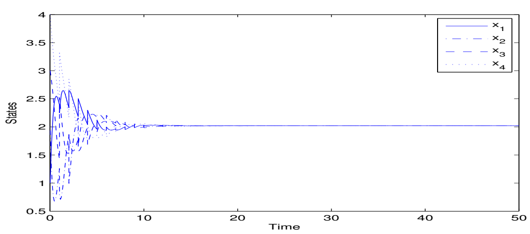


(a) Practical states

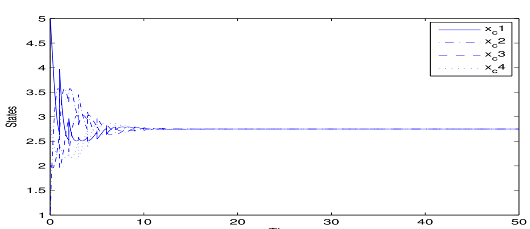


(b) Observation states

Fig. 3. Hybrid consensus protocol with sin disturbance



(a) Practical states



(b) Observation states

Fig. 4. Hybrid consensus protocol with $exponential$ disturbance

V. Conclusions

In this paper, the robustness of the hybrid consensus protocols in [17] is investigated. Using linear systems theory and Lyapunov theory, the difference between the state of each agent of the hybrid system is shown to be bounded. Certain types of noise are discussed in the paper–in particular, the constant scalar noise is studied in detail. Applying the constant noise results in a gap between the final state values, and this result has applications in formation control. Further works to be done on these hybrid protocols include a detailed investigation of the effects of white noise.

References

[1] R. Olfati-Saber, J. A. Fax, and R. M. Murray, "Consensus and cooperation in networked multi-agent systems," *Proc. IEEE*, vol. 95, 2007, pp. 215-233.

[2] X. Ma, Q. Liu, "An artificial fish swarm algorithm for Steiner tree problem," in *IEEE Int. Conf. Fuzzy Syst.*, South Korea, 2009, pp. 20-24.

[3] R. Olfati-Saber and R. M. Murray, "Consensus problems in networks of agents with switching topology and time-delays," *IEEE Trans. Autom. Control*, vol. 49, 2004, pp. 1520-1533.

[4] G. Xie and L. Wang, "Consensus control for a class of networks of dynamic agents," *Int. J. Robust Nonlin. Control*, vol. 17, 2006, pp. 941-959.

[5] D. Zheng, *Linear Systems Theory*, Beijing, China: Tsinghua Univ. Press, 2002.

[6] R. A. Horn and C. R. Johnson, *Matrix Analysis*, Cambridge, U.K.: Cambridge Univ. Press, 1990.

[7] Q. Hui, "Quantized near-consensus via quantized communication links," in *2010 Amer. Control Conf.*, Baltimore, MD, 2010, pp. 247-252.

[8] Q. Hui, W. M. Haddad, and S. P. Bhat, "Finite-time semistability and consensus for nonlinear dynamical networks," *IEEE Trans. Autom. Control*, vol. 53, no. 8, pp. 1887-1900, 2008.

[9] H. Zhang and Q. Hui, "Distributed consensus under limited information," *Proc. 19th Int. Symp. Math. Theory Networks Syst.*, Budapest, Hungary, 2010, pp. 2253-2257.

[10] J. Lavaei and R. M. Murray, "On quantized consensus by means of gossip algorithm–Part I: Convergence proof," in *2009 Amer. Control Conf.*, St. Louis, MO, 2009, pp. 394-401.

[11] J. Lavaei and R. M. Murray, "On quantized consensus by means of gossip algorithm–Part II: Convergence time," in *Proc. 2009 Amer. Control Conf.*, St. Louis, MO, 2009, pp. 2958-2965.

[12] W. Ren and R. W. Beard, *Distributed Consensus in Multi-vehicle Cooperative Control Theory and Applications*, London, U.K.: Springer-Verlag, 2008.

[13] S. Boyd, A. Ghosh, B. Prabhakar, and D. Shah, "Randomized gossip algorithms", *IEEE Trans. Info. Theory,* vol. 52, no. 6, 2006, pp. 2508-2530.

[14] A. Tahbaz-Salehi and A. Jadbabaie, "Necessary and sufficient conditions for consensus consensus over random independent and identically distributed switching graphs," *Proc. 46th IEEE Conf. Decision Control*, New Orleans, LA, 2007, pp. 791-795.

[15] Q. Hui and H. Zhang, "Ergodicity of flocking systems for infinite-dimensional multi-agent coordination" in *Proc. 49th IEEE Conf. Decision Control*, Atlanta, GA, 2010, pp. 5750-5755.

[16] M. Zavlanos, A. Jadbabaie, and G. J. Pappas, "Flocking while preserving network connectivity," in *Proc. 46th IEEE Conf. Decision Control*, New Orleans, LA, 2007, pp. 2919-2924.

[17] Q. Hui, "Hybrid consensus protocols: An impulsive dynamical system approach" *Int. J. Control*, vol. 83, no 6, 2010, pp. 1107-1116.

[18] R. Carli, F. Bullo1, and S. Zampieri, "Quantized average consensus via dynamic coding/decoding schemes," *Int. J. Robust Nonlin. Control*, vol. 20, pp. 156-175, 2010.

[19] R. Carli, P. Frasca, F. Fagnani, and S. Zampieri, "Gossip consensus algorithms via quantized communication," *Automatica*, vol. 46, pp. 70-80, 2010.

[20] L. Wang and Z. Liu, "Robust consensus of multi-agent systems with noise," *Sci. China Series F: Info. Sci.*, vol. 52, no 5, pp. 824-834, 2009.

[21] P. Lin, Y. Jia, and L. Li, "Distributed robust H_∞ consensus control in directed networks of agents with time-delay," *Syst. Control Lett.*, vol. 57, pp. 643-653, 2008.

[22] X. Lin, S. Boyd and S. Lall, "A scheme for robust distributed sensor fusion based on average consensus," in *Proc. Int. Symp. Info. process. sensor networks*, Los Angeles, California, 2005, pp. 63-70.

[23] J. Cortes, S. Martinez, and F. Bullo, "Robust rendezvous for mobile autonomous agents via proximity graphs in arbitrary dimensions," *IEEE Trans. Autom. Control*, vol. 51, no. 8, pp. 1289-1298, 2006.

[24] S. Pettersson, B. Lennartson, "Stability and robustness for hybrid systems," *Proc. 35th IEEE Conf. Decision Control*, Kobe, Japan, 1996, pp. 1202-1207.