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On the Efficiency of Equilibria in Mean-field Oscillator Games

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***Abstract*— A key question in the design of engineered competitive systems has been that of the efficiency of the associated equilibria. Yet, there is little known in this regard in the context of stochastic dynamic games in a large population regime. In this paper, we revisit a class of noncooperative games, arising from the synchronization of a large collection of homogeneous oscillators. In [1], we derived a PDE model for analyzing the associated equilibria in large population regimes through a *mean field approximation*. Here, we examine the efficiency of the associated mean-field equilibria with respect to a related welfare optimization problem. We construct variational problems both for the noncooperative game and its centralized counterpart and employ these problems as a vehicle for conducting this analysis. Using a bifurcation analysis, we analyze the variational solutions and the associated efficiency loss. An expression for the efficiency loss is obtained. Finally, our results are validated through detailed numerics.**

I. INTRODUCTION

Computation of optimal or approximate control laws in large populations of coupled heterogeneous nonlinear systems is of interest in networked multiagent systems. Traditionally, the characterization of the aggregate behavior of a large number of interacting particles is core to statistical mechanics. Through *mean-field approximation* techniques, the interactions of an agent with the ensemble may be reduced to that of the individual with the mass. The key to such an approximation lies in assuming that the fluctuations of the mass influence on the individual are "averaged out" when the number of agents grow to be infinite and importantly this mass-influence can effectively be viewed as a deterministic function. Consequently, this allows for any agent to make decisions based on on its state and the deterministic mass influence. Together with a consistency requirement imposed by the mass-influence, the resulting problem in an infinite population setting can be reduced to a set of coupled PDEs. Such avenues have been employed in modeling networked resource allocation problems [2], industry dynamics [3], amongst others.

In an effort to obtain distributed control laws, Huang et al. [2] considered the associated game-theoretic problem corresponding to the original multiagent control problem. Crucial to such an approach is the need to quantify the efficiency loss in considering the associated game. It may be recalled that if an equilibrium to this game leads to no loss in social welfare, then this game is said to be *efficient* [4], [5]. Huang et al. [2] address the efficiency question in the context of mean-field games with linear dynamics. In [1], we considered the large-population game associated with the synchronization of a homogeneous population of oscillators, a class of mean-field games with nonlinear dynamics. In this paper, we study the efficiency of precisely such a class of games. Our setting remains consistent with that of our preceding work [1] in that we consider the synchronization of a large population of oscillators and employ a mean-field approximation that allows for examining the system when the number of oscillators grows to infinity. An oblivious control [3] strategy is obtained and represents an ε-Nash equilibrium for the finite population game. It must be emphasized that our efficiency analysis is carried out in the large population regime. Quantification of efficiency loss in finite player games has been studied in the context of routing [5], resource allocation [4] and congested markets [6].

In this paper, we construct two variational problems; of these, the first provides a solution to the coupled set of PDEs corresponding to the mean-field equilibrium while the second provides a solution to the centralized welfare optimization problem in the infinite-player regime. Through an examination of the associated nonlinear eigenvalue problems (Euler-Lagrange equations) for the variational formulation, we relate the solutions to the optimization problem and the game. An expression for the efficiency loss associated with the game is provided and its validity is supported through a numerical example.

The second part of the paper pertains to the analysis of phase transitions, a recurring characteristic of nonlinear systems. These transitions are of relevance across a range of applications; for example, in thalamocortical circuits in the brain, transition to the synchronized state is associated with diseased brain states such as epilepsy [7], [8]. We use the method from bifurcation analysis to obtain the critical point where phase transition starts. Additionally, this analysis leads to locally valid bounds on the efficiency loss.

This paper is organized as follows. We provide a brief background to our problem and define the game-theoretic problem and its associated centralized optimization problem in Sec. II. We present two variational problems in Sec. III whose solutions provide us with system behavior in the infinite population limit. This paves the way for investigating the solutions to the two nonlinear eigenvalue problems through the methods of bifurcation theory in Sec. IV. Finally, we provide some numerical results in Sec. V.

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II. Preliminaries & problem statement

We begin with a review of the game-theoretic problem and the associated centralized or welfare optimization problem.

A. Game-theoretic problem

We consider a set of N homogeneous oscillators, denoted by $\mathcal{N} := \{1,\ldots,N\}$. The dynamics for the i^{th} oscillator is described by the stochastic differential equation (SDE):

$$\mathrm{d}\theta_i(t) = (1+u_i(t))\,\mathrm{d}t + \sigma\,\mathrm{d}\xi_i(t), \tag{1}$$

where $\theta_i(t)$ is the phase of the i^{th} oscillator at time t, $u_i(t)$ is the control input, and $\{\xi_i(t),\ i\in\mathcal{N}\}$ are mutually independent standard Wiener processes.

We consider an $N-$player noncooperative game, denoted by $\mathcal{G}_N$, where we assume that the i^{th} oscillator minimizes its own performance objective, given the decisions of (competing) oscillators:

$$\eta_i^{\text{(POP)}}(u_i;u_{-i}) = \lim_{T\to\infty}\frac{1}{T}\int_0^T [c(\theta_i;\theta_{-i}) + \tfrac{1}{2}Ru_i^2]\,\mathrm{d}s, \tag{2}$$

$u_{-i} = (u_j)_{j\neq i}$, $R>0$ denotes the control penalty, $\theta_{-i} = (\theta_j)_{j\neq i}$, $c(\cdot)$ is the cost function of the following separable form:

$$c(\theta_i;\theta_{-i}) := \frac{1}{N}\sum_{j\neq i} c^{\bullet}(\theta_i,\theta_j(t)), \tag{3}$$

and the following assumption is made for $c^{\bullet}$:

Assumption (A1) The function $c^{\bullet}$ introduced in (3) is assumed to be a bounded function that is

1) spatially invariant, i.e., $c^{\bullet}(\vartheta,\theta) = c^{\bullet}(\vartheta-\theta)$,
2) 2π-periodic, i.e., $c^{\bullet}(\theta) = c^{\bullet}(\theta+2\pi)$,
3) non-negative, i.e., $c^{\bullet}(\theta)\geq 0$,
4) even, i.e., $c^{\bullet}(\theta) = c^{\bullet}(-\theta)$. ■

The form of the function c and the value of R are assumed to be common to the entire population. A *Nash equilibrium* in control policies is given by $\{u_i^*\}_{i\in\mathcal{N}}$ such that u_i^* minimizes $\eta_i^{\text{(POP)}}(u_i;u_{-i}^*)$ for $i=1,\ldots,N$.

Our interest in this paper on the regime where $N\to\infty$. We refer to the infinite-player counterpart of the dynamic game by $\mathcal{G}_\infty$. As shown in [1], a mean-field approximation leads to the following PDE-based characterization of the equilibria:

$$\partial_t h + \partial_\theta h = \frac{1}{2R}(\partial_\theta h)^2 - \bar{c}(\theta,t) + \eta^* - \frac{\sigma^2}{2}\partial^2_{\theta\theta}h, \tag{4}$$

$$\partial_t p + \partial_\theta p = \frac{1}{R}\partial_\theta\,[p(\partial_\theta h)] + \frac{\sigma^2}{2}\partial^2_{\theta\theta}p, \tag{5}$$

$$\bar{c}(\theta,t) = \int_0^{2\pi} c^{\bullet}(\theta,\vartheta)p(\vartheta,t)\,\mathrm{d}\vartheta, \tag{6}$$

where $h(\theta,t)$ is the relative value function, $p(\theta,t)$ is intended to approximate probability density of the random variable $\theta_i(t)$, evolving according to the SDE (1) with the optimal control law

$$u_i = -\frac{1}{R}\partial_\theta h(\theta_i,t). \tag{7}$$

The equation (6) represents the mean-field approximation:

$$c(\theta;\theta_{-i}) = \frac{1}{N}\sum_{j\neq i} c^{\bullet}(\theta,\theta_j(t)) \approx \int_0^{2\pi} c^{\bullet}(\theta,\vartheta)p(\vartheta,t)\,\mathrm{d}\vartheta.$$

We conclude this section by defining the *mean field equilibrium*, an equilibrium of $\mathcal{G}_\infty$.

Definition 1: A triple (p^*,h^*,η^*) is said to be a *mean-field equilibrium* (MFE) of $\mathcal{G}_\infty$ if it is a solution of (4)–(6).

B. Welfare optimization problem

Given our interest in the efficiency of the equilibrium, we consider the $N-$player social welfare optimization problem, denoted by $\mathcal{W}_N$, given succinctly by a related centralized optimization problem. Under the validity of the interchange, this objective of this problem may be expressed as follows.

$$\begin{aligned}\eta^{\text{(OPT)}}(u) &:= \frac{1}{N}\sum_{i=1}^N \eta_i^{\text{(POP)}}(u_i;u_{-i})\\ &= \lim_{T\to\infty}\frac{1}{T}\int_0^T \frac{1}{N}\sum_{i=1}^N \left[c(\theta_i;\theta_{-i}) + \tfrac{1}{2}Ru_i^2\right]\mathrm{d}s\end{aligned} \tag{8}$$

The welfare optimization problem requires the minimization of $\eta^{\text{(OPT)}}(u)$ over the control input vector $u=(u_i)_{i=1}^N$. If u is an equilibrium of $\mathcal{G}_N$, then it is said to be *efficient* if $\eta^{\text{(OPT)}}(u) = \frac{1}{N}\sum_{i=1}^N \eta_i^{\text{(POP)}}$; if not, then the loss in efficiency is captured by the difference. We emphasize that our interest lies in examining the efficiency loss of equilibria to $\mathcal{G}_\infty$ with respect to the associated welfare optimization problem $\mathcal{W}_\infty$.

C. Specific solutions

In this paper, we restrict our attention to solutions of the following type:

$$p(\theta,t) = p(\theta-t), \quad h(\theta,t) = h(\theta-t), \tag{9}$$

giving us two special cases: If $p(\cdot),h(\cdot)$ are constant functions, we refer to the solution as an *equilibrium* solution; If they are 2π-periodic, we refer to the solution as the traveling wave solution with wave speed $a=1$. The equilibrium and traveling wave solutions are important in that they potentially represent the incoherence and synchrony solutions described in the coupled oscillators literature [9], [10].

Using the assumption that u_i in (7) is 2π-periodic, the control may be obtained in terms of the density function p from FPK equation (5) through the following lemma.

Lemma 1: Suppose (p,h) is a 2π-periodic solution of the type (9). Then the control input (see (7)) is given by

$$u = \frac{\sigma^2}{2}\partial_\theta\ln(p). \tag{10}$$

Proof: See the proof of Prop. 1 in Sec. III-A. ■

Next, we develop variational problems for characterizing solutions of $\mathcal{G}_\infty$ and $\mathcal{W}_\infty$.

III. Variational problems and efficiency loss

A. Variational formulation of $\mathcal{G}_\infty$

Consider the following variational problem:

$$\eta^g(v;\bar{c}) := \int_0^{2\pi}[\bar{c}(\theta)v^2(\theta) + \frac{R\sigma^4}{2}(\partial_\theta v)^2(\theta)]\,\mathrm{d}\theta \tag{11}$$

$$\text{s.t.}\quad 1 = \int_0^{2\pi} v^2(\theta)\,\mathrm{d}\theta. \tag{12}$$

The solutions of this problem are characterized by the following:

Lemma 2: Suppose v is a critical point of (11)-(12). Then v is a solution of

$$\partial^2_{\theta\theta} v + \frac{2}{\sigma^4 R}\left(\lambda - \bar{c}(\theta)\right) v = 0, \tag{13}$$

$$\int v^2(\theta)\,\mathrm{d}\theta - 1 = 0, \tag{14}$$

where λ is the Lagrange multiplier associated with the constraint (12).

Proof: The result follows from consideration of the first variation of (11)-(12). ■

Our interest lies in the *constrained variational problem*; more specifically, we are interested in critical points to (11)–(12) satisfying the additional requirement

$$\bar{c}(\theta) = \int_0^{2\pi} c^{\bullet}(\theta - \vartheta) v^2(\vartheta)\,\mathrm{d}\vartheta =: \mathscr{C}(v)(\theta). \tag{15}$$

We refer to the constrained variational problem as $(\mathrm{V}^{\mathscr{G}}_\infty)$.

Let $\mathbf{X} := C^2_{2\pi}([0,2\pi],\mathbb{R}^+)$, the space of twice continuously differentiable nonnegative real-valued periodic functions on $[0,2\pi]$. In the following definition, $\mathbf{V}$ denotes as the subspace of functions $v(\in \mathbf{X})$ that satisfy the density constraint (12).

From Lemma 2, we obtain the necessary conditions of optimality. Furthermore, the Lagrange multiplier is seen to be exactly equal to the optimal value of the variational problem.

Lemma 3: Suppose $(\bar{c}^*, v^*)$ is a solution of $(V^{\mathscr{G}}_\infty)$, corresponding to (11)-(12), and λ^* is the corresponding Lagrange multiplier. Then $(\bar{c}^*, v^*, \lambda^*)$ is a solution of the problem (13)-(15) and $\lambda^* = \eta^*_g := \eta^g(v^*;\bar{c}^*) = \min_v \eta^g(v;\bar{c}^*)$.

Proof: Omitted. ■

We first show that an MFE of $\mathscr{G}_\infty$ is a solution to$(\mathrm{V}^{\mathscr{G}}_\infty)$.

Theorem 1: Suppose $((v^*)^2, h^*, \eta^*)$ is an MFE of the dynamic game $\mathscr{G}_\infty$, under the mean field approximation and the traveling wave ansatz (9). $\bar{c}^*$ is the corresponding function given by (6). Then $(\bar{c}^*, v^*)$ is a solution to $(V^{\mathscr{G}}_\infty)$.

Proof: The two cost terms in the integrand of (16), given by

$$\eta_i^{(\mathrm{POP})}(u_i;u_{-i}) = \lim_{T\to\infty} \frac{1}{T}\int_0^T [c(\theta_i;\theta_{-i}) + \tfrac{1}{2}Ru_i^2]\,\mathrm{d}s, \tag{16}$$

can be simplified as follows:

(i) The cost function $c(\theta_i;\theta_{-i})$ is replaced by its mean-field approximation $\bar{c}(\theta,t)$ as in (6).

(ii) We assume $(p(\theta,t), h(\theta,t))$ is a traveling wave solution with wave speed 1. In this case,

$$\bar{c}(\theta,t) = \bar{c}(\theta - t),\, p(\theta,t) = p(\theta - t),\, h(\theta,t) = h(\theta - t).$$

From lemma 1, we have a relationship between the optimal control u_i and the density p: $u_i = \frac{\sigma^2}{2}\partial_\theta \ln(p)$.

Substituting this in (16), we obtain the approximation of $\eta_i^{(\mathrm{POP})}$ as

$$\eta_i(p;\bar{c}) = \lim_{T\to\infty}\frac{1}{T}\int_0^T [\bar{c}(\theta_i(s) - s) + \frac{R\sigma^4}{8}(\partial_\theta \ln(p))^2(\theta_i(s) - s)]\,\mathrm{d}s. \tag{17}$$

Since $\theta_i(s)$ is an ergodic process (See Prop. 3.1 [11]), the time average may be replaced by its expectation

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T \bar{c}(\theta_i(s) - s)\,\mathrm{d}s = \int_0^{2\pi} [p(\theta)\cdot \bar{c}(\theta)]\,\mathrm{d}\theta,$$
$$\lim_{T\to\infty}\frac{1}{T}\int_0^T (\partial_\theta \ln(p))^2(\theta_i(s) - s)\,\mathrm{d}s = \int_0^{2\pi} [p(\theta)\cdot(\partial_\theta \ln(p))^2(\theta)]\,\mathrm{d}\theta.$$

Substituting back in (17), we obtain

$$\eta_i(p;\bar{c}) = \int_0^{2\pi} p(\theta)[\bar{c}(\theta) + \frac{R\sigma^4}{8}(\partial_\theta \ln(p))^2(\theta)]\,\mathrm{d}\theta. \tag{18}$$

With a change of coordinate $v^2(\theta) = p(\theta)$, we arrive at the formula for $\eta^g(v;\bar{c})$ in (11). Since $p = v^2$ and p is a density function, we obtain the constraint (12). Finally, the constraint (15) is the consistency requirement of the mean field approximation. ■

The next result further clarifies the relationship and shows how an MFE of $\mathscr{G}_\infty$ may be constructed from a solution of $(\mathrm{V}^{\mathscr{G}}_\infty)$ and vice-versa.

Proposition 1: Suppose (v,λ) is a solution of $(\mathrm{V}^{\mathscr{G}}_\infty)$ satisfying (13)-(15). Then an MFE of $\mathscr{G}_\infty$, satisfying the coupled nonlinear PDEs (4)-(6), is given by

$$\begin{aligned} p(\theta,t) &= v^2(\theta - t), \\ h(\theta,t) &= -\sigma^2 R \ln v(\theta - t), \end{aligned} \tag{19}$$

with average cost $\eta^* = \lambda$. Conversely, suppose (p, h, η^*) is an MFE of $\mathscr{G}_\infty$ of the type (9) to (4)-(6). Then it is a solution of $(\mathrm{V}^{\mathscr{G}}_\infty)$ of the form (19) where (v, η^*) is a solution of (13)-(15).

Proof: Suppose (p, h, η^*) is a solution of the type (9). For such a solution, the left hand side of the FPK and the HJB PDEs,

$$\partial_t h + \partial_\theta h = 0, \quad \partial_t p + \partial_\theta p = 0. \tag{20}$$

We denote $u^* = -\frac{1}{R}\partial_\theta h$ and using (20), the FPK equation (5) is given by

$$\begin{aligned} \partial_\theta [pu^*] &= \frac{\sigma^2}{2}\partial^2_{\theta\theta} p, \\ \therefore,\ u^* &= \frac{\sigma^2}{2}\partial_\theta \ln(p) + \frac{K}{p}, \end{aligned} \tag{21}$$

where K is the constant of integration. Now, $u^* = -\frac{1}{R}\partial_\theta h$ where h is periodic. So, $\int_0^{2\pi} u^*\,\mathrm{d}\theta = 0$. Integrating both sides of (21) over $[0,2\pi]$, we have

$$0 = \int_0^{2\pi} u^*\,\mathrm{d}\theta = K\int_0^{2\pi}\frac{1}{p}\,\mathrm{d}\theta,$$

i.e., $K = 0$ and

$$u^* = \frac{\sigma^2}{2}\partial_\theta \ln(p). \tag{22}$$

Using (20), the HJB equation (4) is given by

$$-\frac{1}{2R}(\partial_\theta h)^2 + \frac{\sigma^2}{2}\partial^2_{\theta\theta} h = \eta^* - \bar{c}. \tag{23}$$

We introduce the Hopf-Cole transformation coordinate v as

$$v = \exp(-\frac{1}{\sigma^2 R}h), \tag{24}$$

to simplify the HJB equation (23) to

$$-\partial^2_{\theta\theta} v = \frac{2}{\sigma^4 R}(\eta^* - \bar{c})v.$$

Finally, using (22) and (24), we obtain

$$\frac{\sigma^2}{2}\partial_\theta \ln(p) = u^* = -\frac{1}{R}\partial_\theta h = \sigma^2 \partial_\theta \ln(v).$$

This gives $p = v^2$, where we have dropped the constant of integration because h is defined only up to a constant.

We thus obtain the eigenvalue problem expressed only in terms of v:

$$-\partial^2_{\theta\theta} v(\theta) = \frac{2}{\sigma^4 R}\left(\eta^* - \int_0^{2\pi} c^\bullet(\theta,\vartheta) v^2(\vartheta)\,\mathrm{d}\vartheta\right) v,$$

with the constraint that $\int v^2(\theta)\,\mathrm{d}\theta = 1$ because $p = v^2$ is a density. ■

In summary, solutions of $\mathscr{G}_\infty$ can be obtained by considering one of two problems:

1) The variational problem (11)-(12) with constraint (15);
2) The nonlinear eigenvalue problem (13)-(15).

B. Variational formulation of $\mathscr{W}_\infty$

As in the game-theoretic problem, we consider the following variational problem ($\mathrm{V}^{\mathscr{W}}_\infty$) as a means of obtaining a mean-field optimum (MFO) of $\mathscr{W}_\infty$, associated with (8). This problem requires a $v \in \mathbf{X}$ so as to minimize

$$\eta^{\mathrm{w}}(v) := \int_0^{2\pi} \left[\mathscr{C}(v)(\theta) v^2(\theta) + \frac{R\sigma^4}{2}(\partial_\theta v)^2(\theta)\right] \mathrm{d}\theta \quad (25)$$

$$\text{s.t.} \quad 1 = \int_0^{2\pi} v^2(\theta)\,\mathrm{d}\theta, \quad (26)$$

where $\mathscr{C}(v)$ is defined in (15).

Theorem 2: Consider an MFO of $\mathscr{W}_\infty$, i.e. a solution to $\mathscr{W}_\infty$ (8) of the type (9). Any MFO of $\mathscr{W}_\infty$ is a solution to the variational problem ($\mathrm{V}^{\mathscr{W}}_\infty$) given by (25)-(26).

Proof: As $N \to \infty$, $\eta^{(\mathrm{OPT})}(u)$ is given by

$$\begin{aligned}\eta^{\mathrm{w}}(p) &= \lim_{T\to\infty} \frac{1}{T}\int_0^T\int_0^{2\pi} p(\theta - s)\int_0^{2\pi} c^\bullet(\theta-\vartheta) p(\vartheta - s)\,\mathrm{d}\vartheta\,\mathrm{d}\theta\,\mathrm{d}s \\ &\quad + \lim_{T\to\infty}\frac{1}{T}\int_0^T\int_0^{2\pi} p(\theta)\tfrac{1}{2}Ru^2\,\mathrm{d}\theta\,\mathrm{d}s \\ &=: I_1 + I_2, \end{aligned} \quad (27)$$

using the mean-field approximation (6) and traveling wave ansatz (9) as in proof of Thm. 1. Then

$$\begin{aligned} I_1 &= \lim_{T\to\infty}\frac{1}{T}\int_0^T\int_0^{2\pi} p(\theta-s)\int_0^{2\pi} c^\bullet(\theta-\vartheta)p(\vartheta-s)\,\mathrm{d}\vartheta\,\mathrm{d}\theta\,\mathrm{d}s \\ &= \lim_{T\to\infty}\frac{1}{T}\int_0^T\int_0^{2\pi} p(\theta)\int_0^{2\pi} c^\bullet(\theta-\vartheta)p(\vartheta)\,\mathrm{d}\vartheta\,\mathrm{d}\theta\,\mathrm{d}s \\ &= \int_0^{2\pi} p(\theta)\int_0^{2\pi} c^\bullet(\theta-\vartheta)p(\vartheta)\,\mathrm{d}\vartheta\,\mathrm{d}\theta. \end{aligned} \quad (28)$$

The traveling wave ansatz (9) also gives $u = \frac{\sigma^2}{2}\partial_\theta \ln(p)$ as stated in lemma 1. Consequently, I_2 may be simplified as

$$\begin{aligned} I_2 &= \lim_{T\to\infty}\frac{1}{T}\int_0^T\int_0^{2\pi} p(\theta-s)\tfrac{1}{2}Ru^2\,\mathrm{d}\theta\,\mathrm{d}s \\ &= \lim_{T\to\infty}\frac{1}{T}\int_0^T\int_0^{2\pi} p(\theta-s)\frac{R\sigma^4}{8}\left(\partial_\theta \ln(p(\theta-s))\right)^2\mathrm{d}\theta\,\mathrm{d}s \\ &= \lim_{T\to\infty}\frac{1}{T}\int_0^T\int_0^{2\pi} p(\theta)\frac{R\sigma^4}{8}\left(\partial_\theta \ln(p(\theta))\right)^2\mathrm{d}\theta\,\mathrm{d}s \\ &= \int_0^{2\pi} p(\theta)\frac{R\sigma^4}{8}\left(\partial_\theta \ln(p(\theta))\right)^2\mathrm{d}\theta. \end{aligned} \quad (29)$$

Substituting (28)-(29) into (27), we obtain $\eta^{\mathrm{w}}(p)$ as

$$\int_0^{2\pi} p(\theta)\Big[\int_0^{2\pi} c^\bullet(\theta-\vartheta)p(\vartheta)\,\mathrm{d}\vartheta + \frac{R\sigma^4}{8}\left(\partial_\theta \ln(p(\theta))\right)^2\Big]\mathrm{d}\theta.$$

Making a change of coordinate $v^2(\theta) = p(\theta)$, we arrive at the formula for $\eta^{\mathrm{w}}(v)$ in (25). Because $p = v^2$ is a density function, we obtain the constraint (26). ■

Denote the average cost obtained this way as η^*_{w}, i.e., $\eta^*_{\mathrm{w}} := \min_{v\in V}\eta^{\mathrm{w}}(v)$. The Euler-Lagrange results are summarized as follows:

Lemma 4: Suppose v is a solution of ($\mathrm{V}^{\mathscr{W}}_\infty$), given by (25)-(26). Then (v,λ) satisfy the following:

$$\partial^2_{\theta\theta} v + \frac{2}{\sigma^4 R}(\lambda - 2\mathscr{C}(v))v = 0, \quad (30)$$

$$\int v^2(\theta)\,\mathrm{d}\theta - 1 = 0, \quad (31)$$

where λ is the Lagrange multiplier associated with (26). Furthermore,

$$\eta^*_{\mathrm{w}} = \lambda - \int_0^{2\pi}\mathscr{C}(v)(\theta)v^2(\theta)\,\mathrm{d}\theta. \quad (32)$$

Proof: The Euler-Lagrange equation (30) is obtained from considering the first variation of (25)-(26).

Multiply both sides of (30) by v and integrate from 0 to 2π to obtain

$$\lambda\int v^2 = \int \frac{\sigma^4 R}{2}(\partial_\theta v)^2 + 2\int \mathscr{C}(v)v^2 = \eta^*_{\mathrm{w}} + \int \mathscr{C}(v)v^2,$$

which gives $\eta^*_{\mathrm{w}} = \lambda - \int_0^{2\pi}\mathscr{C}(v)(\theta)v^2(\theta)\,\mathrm{d}\theta$. ■

In summary, solutions of $\mathscr{W}_\infty$ can be obtained by considering one of the following two problems:

1) The variational problem ($\mathrm{V}^{\mathscr{W}}_\infty$) (25)-(26), or
2) The nonlinear eigenvalue problem (30)-(31).

C. Efficiency loss

The efficiency loss is denoted by $\Delta_\eta(R;\sigma)$ and is given by $\eta^*_g(R;\sigma) - \eta^*_{\mathrm{w}}(R;\sigma)$. Our main result of this subsection is twofold in nature. First, we provide a precise relationship between an MFE and an MFO, in terms of v and λ. Second, using these these relationships, we then construct an expression for $\Delta_\eta(R)$.

Theorem 3: Let σ be fixed. For a given value of R, let $(\bar{c}^*, v^*_g(R))$ be the solution of ($\mathrm{V}^{\mathscr{G}}_\infty$), $\lambda^*_g(R)$ be the corresponding Lagrange multiplier, $v^*_{\mathrm{w}}(R)$ be the solution to the variational problem ($\mathrm{V}^{\mathscr{W}}_\infty$) and $\lambda^*_{\mathrm{w}}(R)$ be the corresponding Lagrange multiplier. Then we have

(i) $v_{\mathrm{w}}^*(R) = v_g^*(R/2)$, $\lambda_{\mathrm{w}}^*(R) = 2\lambda_g^*(R/2)$,

(ii) $$\Delta_\eta(R) = \lambda_g^*(R) - 2\lambda_g^*(\frac{R}{2}) + \int_0^{2\pi}\int_0^{2\pi} c^\bullet(\theta - \vartheta)(v_g^*)^2(\vartheta;\frac{R}{2})\,\mathrm{d}\vartheta(v_g^*)^2(\theta;\frac{R}{2})\,\mathrm{d}\theta.$$

Proof: Denote problem (13) as $G^g(v,\lambda,R) = 0$ and problem (30) as $G^{\mathrm{w}}(v,\lambda,R) = 0$. Consider the problem $G^{\mathrm{w}}(v^{\mathrm{w}},\lambda^{\mathrm{w}},R^{\mathrm{w}}) = 0$. Suppose $R^{\mathrm{w}} = 2R$ and $\lambda^{\mathrm{w}} = 2\lambda$. Then we obtain the relationship

$$\begin{aligned} G^{\mathrm{w}}(v^{\mathrm{w}},\lambda^{\mathrm{w}},R^{\mathrm{w}}) &= \partial^2_{\theta\theta}v^{\mathrm{w}} + \frac{2}{\sigma^4 2R}(2\lambda - 2\mathscr{C}(v^{\mathrm{w}})(\theta))v^{\mathrm{w}} \\ &= \partial^2_{\theta\theta}v^{\mathrm{w}} + \frac{2}{\sigma^4 R}(\lambda - \mathscr{C}(v^{\mathrm{w}})(\theta))v^{\mathrm{w}} \\ &= G^g(v^{\mathrm{w}},\lambda,R) = G^g(v^{\mathrm{w}},\lambda^{\mathrm{w}}/2,R^{\mathrm{w}}/2). \end{aligned}$$

That is to say, to solve the problem $G^{\mathrm{w}}(v^{\mathrm{w}},\lambda^{\mathrm{w}},R) = 0$, we could instead solve the equivalent problem $G^g(v^g,\lambda^g,R/2) = 0$. Then $v^{\mathrm{w}}(R) = v^g(R/2)$ and $\lambda^{\mathrm{w}}(R) = 2\lambda^g(R/2)$.

The formula for Δ_η is obtained from its definition using the relationship in (i) and lemma 4. ■

There are several insights that one can draw from the expression for $\Delta_\eta(R)$, particularly from the numerical study. We observe that as $R \to 0$, we have that η_{w}^* and η_g^* both tend to zero. In effect, the efficiency loss tends to zero, as $R \to 0$. Furthermore, when R is beyond a threshold, we again observe that the efficiency loss is zero. In fact, the efficiency loss is seen to be positive between these two regimes. In the next section, through a bifurcation analysis, we provide a locally valid upper bound on efficiency loss (Lemma 7).

IV. Bifurcation analysis

In this section, we investigate the solutions of the nonlinear eigenvalue problems (13)-(15) and (30)-(31) by using the methods of bifurcation theory and conclude with a bound on the efficiency loss.

We denote $\mathbf{Y} := C^0_{2\pi}([0,2\pi],\mathbb{R})$, the space of continuous nonnegative real-valued periodic functions on $[0,2\pi]$. Recall $\mathbf{X} := C^2_{2\pi}([0,2\pi],\mathbb{R}^+)$. The eigenvalue problem (Denoted as (EP$^\alpha$)) comprises of a nonlinear operator $G_\alpha : \mathbf{X}\times\mathbb{R}^+\times\mathbb{R}^+ \to \mathbf{Y}$, and a constraint $B : \mathbf{X} \to \mathbb{R}$:

$$G_\alpha(v,\lambda,R) := \partial^2_{\theta\theta}v + \frac{2}{\sigma^4 R}(\lambda - \alpha\mathscr{C}(v))\,v = 0, \tag{33}$$

$$B(v) := \int v^2(\theta)\,\mathrm{d}\theta - 1 = 0, \tag{34}$$

where $\mathscr{C}(v)$ is defined in (15) and $\alpha = 1,2$. When $\alpha = 1$, it is the problem (13)-(15), while when $\alpha = 2$, it is the problem (30)-(31).

For any fixed $R \in \mathbb{R}^+$, we are interested in obtaining *solutions* $(v,\lambda) \in \mathbf{X}\times\mathbb{R}^+$ such that $G_\alpha(v,\lambda,R) = 0$ and $B(v) = 0$, for $\alpha = 1,2$.

For the nonlinear eigenvalue problem, we define the incoherence solution

$$v = v_0 := \frac{1}{\sqrt{2\pi}}, \quad \lambda = \lambda_0 := \frac{\alpha}{2\pi}\int_0^{2\pi} c^\bullet(\theta)\,\mathrm{d}\theta.$$

About the incoherence solution, the linearization of (33) is given by

$$\mathscr{L}_R\tilde{v}(\theta) := \partial^2_{\theta\theta}\tilde{v} - \frac{2\alpha}{\sigma^4 R\pi}\int_0^{2\pi} c^\bullet(\theta-\vartheta)\tilde{v}(\vartheta)\,\mathrm{d}\vartheta$$

with $\tilde{v} \in \mathbf{X}$ and satisfies the integral constraint $\int_0^{2\pi}\tilde{v}(\theta)\,\mathrm{d}\theta = 0$.

The spectrum of the linear operator $\mathscr{L}_R : \mathbf{X} \to \mathbf{Y}$ is summarized in the following:

Lemma 5: Consider the linear eigenvalue problem $\mathscr{L}_R v = sv$. Suppose the Fourier expansion of the function $c^\bullet$ is

$$c^\bullet(\theta) = \sum_{k=-\infty}^{\infty} C_k^\bullet e^{ik\theta}. \tag{35}$$

Then the spectrum consists of eigenvalues $s = -k^2 - \frac{4\alpha}{\sigma^4 R}C_k^\bullet =: s_k$ for $k = 0,1,2,\ldots$ The eigenspace for the k^{th} eigenvalue $s = s_k$ is given by $\mathrm{span}\{\cos(k\theta),\sin(k\theta)\}$. ■

As the parameter R varies, the potential bifurcation points are where an eigenvalue crosses zero. The k^{th} such bifurcation point is given by

$$R = R_k := -\frac{4\alpha}{k^2\sigma^4}C_k^\bullet.$$

Example 1: Consider now the function $c^\bullet(\theta - \vartheta) = \frac{1}{2}\sin^2\left(\frac{\vartheta-\theta}{2}\right)$. In this case, $C_1^\bullet = -\frac{1}{8}$ and the first bifurcation point

$$R = \frac{\alpha}{2\sigma^4} =: R_c^\alpha$$

is the critical point at which the incoherence solution loses stability.

We state the bifurcation result for a specific cost function $c^\bullet$ next. The similar proof can be found in [11].

Theorem 4: Consider the nonlinear eigenvalue problem (33)-(34) with cost function $c^\bullet(\vartheta-\theta) = \frac{1}{2}\sin^2\left(\frac{\vartheta-\theta}{2}\right)$. Let (v_0,λ_0) denote the incoherence solution. Then from $R = R_c^\alpha = \frac{\alpha}{2\sigma^4}$ bifurcates a branch of non-constant solutions (v,λ) of (33)-(34). More precisely, there exists a neighborhood $J \subset \mathbb{R}$ of $x = 0$, functions $\hat{\lambda}(x), \hat{R}(x) \in C^1(J)$, and a family $v(x)$ of non-constant solutions of (33)-(34) in $\mathbf{X}$ such that

(i) $\lambda = \hat{\lambda}(x)$ and $\hat{\lambda}(x) \to \lambda_0$, $R = \hat{R}(x)$ and $\hat{R}(x) \to R_c^\alpha$ as $x \to 0$, and

(ii) the amplitude of $v(x) - v_0$ tends to zero as $x \to 0$.

■

A Lyapunov-Schmidt perturbation method is used to obtain an asymptotic formula for the non-constant bifurcating solution branch for the cost $c^\bullet(\vartheta-\theta) = \frac{1}{2}\sin^2\left(\frac{\vartheta-\theta}{2}\right)$. We substitute the expansion

$$\begin{aligned} R &= r_0 + \varepsilon r_1 + \varepsilon^2 r_2 + \ldots \\ v &= v_0 + \varepsilon v_1 + \varepsilon^2 v_2 + \ldots \\ \lambda &= \lambda_0 + \varepsilon\lambda_1 + \varepsilon^2\lambda_2 + \ldots \end{aligned} \tag{36}$$

into (33)-(34), and collect the terms according to different orders of ε. The results are summarized in the following lemma. The calculations can be found in the extended version on authors' website.

Lemma 6: Given the function $c^\bullet(\vartheta-\theta)=\frac{1}{2}\sin^2\left(\frac{\vartheta-\theta}{2}\right)$, the solution for (33)-(34) is given by an asymptotic formula in the small "amplitude" parameter x:

$$\begin{aligned} v(x) &= v_0 + 2x\cos(\theta+\theta_0) + \left(-\frac{1}{v_0} + v_0\pi\cos 2(\theta+\theta_0)\right)x^2 \\ \lambda &= \hat{\lambda}(x) = \lambda_0 - \alpha\pi x^2, \\ R &= \hat{R}(x) = r_0 - \frac{7\alpha\pi}{2\sigma^4}x^2 \end{aligned}$$

to the order of $o(x^2)$, where $r_0 = R_c^\alpha = \frac{\alpha}{2\sigma^4}$, $\lambda_0 = \frac{\alpha}{4}$ for $\alpha = 1,2$ and θ_0 is an arbitrary phase in $[0,2\pi]$. ■

Using the asymptotic formula from lemma 6 and the formula of $\Delta_\eta(R)$ in Thm. 3, we obtain an upper bound for the efficiency loss around the critical value of R_c^1.

Lemma 7: In a sufficiently small neighborhood of $R = R_c^1 = \frac{1}{2\sigma^4}$, the following bound holds for Δ_η:

$$\Delta_\eta(R) \le \frac{6}{49}(1-\sigma^4 R)^2 + O(x^3(1,R/2)), \tag{37}$$

where $x(\alpha,R) = \sqrt{\frac{2\sigma^4}{7\alpha\pi}(R_c^\alpha - R)}$.

V. NUMERICS

In this section we present the numerical results of the nonlinear eigenvalue problems (33)-(34) for $c^\bullet(\theta-\vartheta) = \frac{1}{2}\sin^2\left(\frac{\theta-\vartheta}{2}\right)$. We set the noise level at $\sigma^2 = 0.1$. The results of the solutions p and the average cost η^* from Lyapunov-Schmidt perturbation method as well as those from the AUTO software are depicted for comparison.

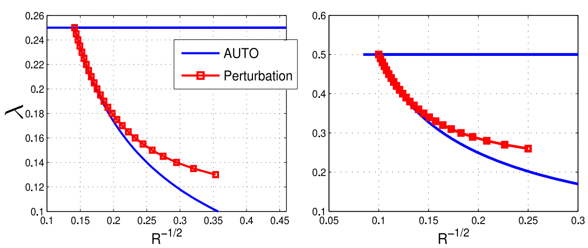

Fig. 1. Bifurcation diagram for the Lagrange multiplier λ as a function of parameter $1/\sqrt{R}$; (left) for (EP^1) of ($V_\infty^{\mathscr{G}}$) and (right) for (EP^2) of ($V_\infty^{\mathscr{W}}$). For (EP^1), λ also equals the average cost η_g^*.

a) Relationship of p and λ with R: Figure 1 depicts the bifurcation diagram for the Lagrange multiplier λ as a function of the bifurcation parameter R as well as the function of p for a particular value of R ($R = 10$ for (EP^1) and $R = 22.8$ for (EP^2)). For comparison, we also depict the corresponding numerical results of the problem that is obtained in AUTO using a continuation method [12]. The first row is the results for (EP^1) while the second is for (EP^2). This verifies the perturbation calculation results of Sec. IV.

b) Relationship of η with R: Next, we compare the average cost (η_g^* for (EP^1) and η_w^* for (EP^2)) obtained from solving the two nonlinear eigenvalue problems using AUTO. For (EP^1), we know $\eta_g^* = \lambda$ from lemma 3. For (EP^2), we know η_w^* from lemma 4. The results are depicted in Fig. 2 (Left). There are two critical points in the figure: One is R_c^1 for (EP^1) and the other is R_c^2 for (EP^2). When $R > R_c^2$ ($R^{-1/2} < (R_c^2)^{-1/2}$), we obtain the incoherence solution for both problems. When $R < R_c^1$ ($R^{-1/2} > (R_c^1)^{-1/2}$), we obtain the synchrony solution for both problems. The figure shows $\eta_g^* \ge \eta_w^*$. The equality holds when both are in incoherence solution, i.e., $R > R_c^2$.

c) Relationship of Δ_η with R: We calculate the difference of the average cost (efficiency loss) Δ_η for the case of $R < R_c^1$. The difference is calculated by two methods. One is the method stated in Thm. 3 (ii) and the other is the definition. The results are depicted in Fig. 2 (Right). It shows that the formula for $\Delta_\eta(R)$ in Thm. 3 is quite accurate, and the solution of ($V_\infty^{\mathscr{G}}$) is always inefficient except in the incoherent regime. From the numerics, we obtain that $\Delta_\eta/\eta_w^* < 20\%$.

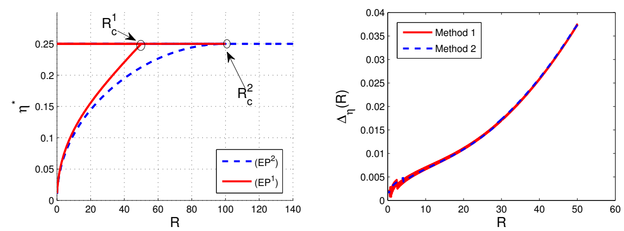

Fig. 2. (Left) The bifurcation diagram in terms of the average cost η^* for two eigenvalue problems (EP^α). The results are for $\sigma^2 = 0.1$. The critical value of R for (EP^1) of the game problem is $R_c^1 = 50\,((R_c^1)^{-1/2} = 0.1414)$ while for (EP^2) of the welfare optimization problem is $R_c^2 = 100\,((R_c^2)^{-1/2} = 0.1)$. (Right) Δ_η calculated using two methods: Method one is the method stated in Thm. 3 (ii); Method two is the definition $\Delta_\eta(R) = \eta_g^*(R) - \eta_w^*(R)$.

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