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Control Design with Limited Model Information

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Abstract— **We introduce the family of limited model information control design methods, which construct controllers by accessing the plant's model in a constrained way, according to a given design graph. This class generalizes the notion of communication-less control design methods recently introduced by one of the authors, which construct each sub-controller using only local plant model information. We study the trade-off between the amount of model information exploited by a control design method and the quality of controllers it can produce. In particular, we quantify the benefit (in terms of the competitive ratio and domination metrics) of giving the control designer access to the global interconnection structure of the plant-to-be-controlled, in addition to local model information.**

I. Introduction

Two challenges often face the control designer confronted with a large-scale plant composed of interconnected subsystems. The first challenge regards controller structure, and stems from the requirement that the control signal sent to a subsystem should depend only on the state of subsystems in its immediate neighborhood. This requirement is due to the high cost or impossibility of relaying measurements between physically remote subsystems, and leads to the traditional problem of decentralized or structured control [1]–[3].

The second control design challenge originates from the same concern for localization, but pertains to model information rather than plant measurements. Since one would like to *not* modify sub-controller K_i if the characteristics of a particular subsystem, which is not directly connected to subsystem i, vary, and/or a precise model of other subsystems in the plant may be unavailable when designing K_i in the first place, it is natural to try and design controllers without the full knowledge of a plant's model or, even more specifically, such that K_i depends solely on the description of subsystem i's model. When the latter situation holds, we say that control design method is "communication-less", to capture the fact that subsystem i and subsystem $j \neq i$ do not "communicate" plant information with each other (even though they might be dynamically coupled) during the control design phase.

The main goal of this paper is to study the trade-off between the amount of plant information exploited by a control design method, and the quality of controllers it can produce. To do so, we look at limited model information control design methods as elements of a particular class of maps between the plant and controller sets, and characterize their intrinsic limitations via the competitive ratio and domination metrics introduced in [4]. The class of plants and limitations on control design methods addressed in the present paper are significantly larger than in this reference, since we consider subsystems of arbitrary order, and we investigate the case where, even though detailed subsystems' model information is not available, the global interconnection structure of the plant (which we call the plant graph) is known at the time of control design.

This paper is organized as follows. In Section II, we formulate the problem of interest rigorously and define the performance metrics. In Section III, we characterize the best communication-less control design method according to both competitive ratio and domination metrics, for various possible plant graphs. In the case where the plant graph contains no sink, we generalize the fact proven in [4] that the deadbeat strategy is the best communication-less control design method. However the deadbeat strategy is dominated when the plant graph contains sinks, and we exhibit a better, undominated, communication-less control design method, which takes advantage of the knowledge of the sinks location to lower closed-loop performance for all plants. In Section IV, we show that achieving a strictly better competitive ratio than this control design method requires a complete design graph. Finally, we end with an illustrative example of limited model information control design in Section V and the conclusions in Section VI.

A. Notation

Sets will be denoted by calligraphic letters, such as $\mathcal{P}$ and $\mathcal{A}$. If $\mathcal{A}$ is a subset of $\mathcal{M}$ then $\mathcal{A}^c$ is the complement of $\mathcal{A}$ in $\mathcal{M}$, i.e., $\mathcal{M} \setminus \mathcal{A}$.

Matrices are denoted by capital roman letters such as A. A_j will denote the j^{th} row of the A. A_{ij} denotes a sub-matrix of matrix A, the dimension and the position of which will be defined in the text. The entry in the i^{th} row and the j^{th} column of the matrix A is a_{ij}.

Let S^n_{++} (S^n_{+}) be the set of symmetric positive definite (positive semidefinite) matrices in $\mathbb{R}^{n\times n}$. $A > (\geq) 0$ means symmetric matrix $A \in \mathbb{R}^{n\times n}$ is positive definite (positive semidefinite) and $A > (\geq) B$ means that $A - B > (\geq) 0$.

$\underline{\lambda}(Y)$ and $\bar{\lambda}(Y)$ denote the smallest and the largest eigenvalues of the matrix Y, respectively. Similarly, $\underline{\sigma}(Y)$ and $\bar{\sigma}(Y)$ will denote the smallest and the largest singular values of the matrix Y, respectively. Vector e_i will denote the

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column-vector with all entries zero except the i^{th} entry which is equal to one.

All graphs considered in this paper are directed, possibly with self-loops, with vertex set $\{1, ..., q\}$ for some positive integer q. If $G = (\{1, ..., q\}, E)$ is a directed graph, we say that i is a sink if there does not exist $j \neq i$ such that $(i, j) \in E$. A loop of length t in G is a set of distinct vertices $\{i_1, ..., i_t\}$ such that $(i_t, i_1) \in E$ and $(i_p, i_{p+1}) \in E$ for all $1 \leq p \leq t-1$. We will sometimes refer to this loop as $(i_1 \to i_2 \to ... \to i_t \to i_1)$. The adjacency matrix S of graph G is the $q \times q$ matrix whose entries satisfy

$$s_{ij} = \begin{cases} 1 & \text{if } (j, i) \in E \\ 0 & \text{otherwise.} \end{cases}$$

II. Control Design With Limited Model Information

A. Plant Model

Let a graph $G_{\mathcal{P}} = (\{1, ..., q\}, E_{\mathcal{P}})$ be given, with adjacency matrix $S_{\mathcal{P}} \in \{0, 1\}^{q \times q}$. We define the following set of matrices associated with $S_{\mathcal{P}}$:

$$\mathcal{A}(S_{\mathcal{P}}) = \{\tilde{A} \in \mathbb{R}^{n \times n} | \tilde{A}_{ij} = 0 \in \mathbb{R}^{n_i \times n_j} \text{ for all } 1 \leq i, j \leq q \text{ such that } (s_{\mathcal{P}})_{ij} = 0\}.$$

Also, for a given scalar $\epsilon > 0$, we let

$$\mathcal{B}(\epsilon) = \{\tilde{B} \in \mathbb{R}^{n \times n} \mid \underline{\sigma}(\tilde{B}) \geq \epsilon, \tilde{B}_{ij} = 0 \in \mathbb{R}^{n_i \times n_j} \text{ for all } 1 \leq i \neq j \leq q\}.$$

With these definitions, we can introduce the set $\mathcal{P}$ of plants of interest as the space of all discrete time, linear time invariant systems of the form

$$x(k+1) = Ax(k) + Bu(k) \; ; \; x(0) = x_0, \tag{1}$$

with $x_0 \in \mathbb{R}^n$, $A \in \mathcal{A}(S_{\mathcal{P}})$, $B \in \mathcal{B}(\epsilon)$. Clearly $\mathcal{P}$ is isomorph to $\mathcal{A}(S_{\mathcal{P}}) \times \mathcal{B}(\epsilon) \times \mathbb{R}^n$ and, slightly abusing notation, we will thus identify a plant $P \in \mathcal{P}$ with the corresponding triple (A, B, x_0).

A plant $P \in \mathcal{P}$ can be thought of as the interconnection of q subsystems, with the structure of the interconnection specified by graph $G_{\mathcal{P}}$, i.e., subsystem j's output feeds into subsystem i only if $(j, i) \in E_{\mathcal{P}}$. As a consequence, we refer to $G_{\mathcal{P}}$ as the "plant graph". For each $1 \leq i \leq q$, subsystem i is of dimension n_i. Implicit in these definitions is the fact that $\sum_{i=1}^{q} n_i = n$. We will denote the ordered set of state indices pertaining to subsystem i as $\mathcal{I}_i$, i.e., $\mathcal{I}_i := (1 + \sum_{j=1}^{i-1} n_j, \ldots, n_i + \sum_{j=1}^{i-1} n_j)$. For subsystem i, state vector and input vector are defined as $\underline{x}_i = [x_{\ell_1} \; \cdots \; x_{\ell_{n_i}}]^T$ and $\underline{u}_i = [u_{\ell_1} \; \cdots \; u_{\ell_{n_i}}]^T$ where the ordered set of indices $(\ell_1, \ldots, \ell_{n_i}) \equiv \mathcal{I}_i$, and dynamics specified by

$$\underline{x}_i(k+1) = \sum_{j=1}^{q} A_{ij} \underline{x}_j(k) + B_{ii} \underline{u}_i(k).$$

B. Controller Model

Let a control graph $G_{\mathcal{K}}$ be given, with adjacency matrix $S_{\mathcal{K}}$. The control laws of interest in this paper are linear static state-feedback control laws of the form

$$u(k) = Kx(k),$$

where

$$K \in \mathcal{K}(S_{\mathcal{K}}) = \{\tilde{K} \in \mathbb{R}^{n \times n} | \tilde{K}_{ij} = 0 \in \mathbb{R}^{n_i \times n_j} \text{ for all } 1 \leq i, j \leq q \text{ such that } (s_{\mathcal{K}})_{ij} = 0\}.$$

In particular, when $G_{\mathcal{K}}$ is the complete graph, $\mathcal{K}(S_{\mathcal{K}}) = \mathbb{R}^{n \times n}$ and controllers are unstructured while, if $G_{\mathcal{K}}$ is totally disconnected with self-loops, $\mathcal{K}(S_{\mathcal{K}})$ represents the set of fully decentralized controllers. When adjacency matrix $S_{\mathcal{K}}$ is not relevant or can be deduced from context, we refer to the set of controllers as $\mathcal{K}$.

C. Control Design Methods

A control design method Γ is a map from the set of plants $\mathcal{P}$ to a set of controllers $\mathcal{K}$. Just like plants and controllers, a control design method can exhibit structure which, in turn, can be captured by a design graph. Let a control design method be partitioned according to subsystems dimensions as

$$\Gamma = \begin{bmatrix} \Gamma_{11} & \cdots & \Gamma_{1q} \\ \vdots & \ddots & \vdots \\ \Gamma_{q1} & \cdots & \Gamma_{qq} \end{bmatrix} \tag{2}$$

and a graph $G_{\mathcal{C}} = (\{1, ..., q\}, E_{\mathcal{C}})$ be given, with adjacency matrix $S_{\mathcal{C}}$. In (2), each block Γ_{ij} represents a map $\mathcal{A}(S_{\mathcal{P}}) \times \mathcal{B}(\epsilon) \to \mathbb{R}^{n_i \times n_j}$.

We say that Γ has structure $G_{\mathcal{C}}$ if, for all i, the map $[\Gamma_{i1} \; \cdots \; \Gamma_{iq}]$ is only a function of $\{[A_{j1} \; \cdots \; A_{jq}], B_{jj} \mid (s_{\mathcal{C}})_{ij} \neq 0\}$. In words, a control design method has structure $G_{\mathcal{C}}$ if and only if, for all i, the subcontroller of subsystem i is constructed with knowledge of the plant model of only those subsystems j such that $(j, i) \in E_{\mathcal{C}}$. The set of all control design methods with structure $G_{\mathcal{C}}$ will be denoted by $\mathcal{C}$. In the particular case where $G_{\mathcal{C}}$ is the totally disconnected graph with self-loops (i.e., $S_{\mathcal{C}} = I_q$), we say that a control design method in $\mathcal{C}$ is "communication-less", so as to capture the fact that subsystem i's subcontroller is constructed with no information coming from (and, hence, no communication with) any other subsystem j, $j \neq i$. When $G_{\mathcal{C}}$ is not the complete graph, we refer to $\Gamma \in \mathcal{C}$ as being a "limited model information control design method".

Note that $\mathcal{C}$ can be considered as a subset of $(\mathcal{A}(S_{\mathcal{P}}) \times \mathcal{B}(\epsilon))^{\mathcal{K}}$, since a design method with structure $G_{\mathcal{C}}$ is not a function of initial state x_0. Hence, when $\Gamma \in \mathcal{C}$ we will write $\Gamma(A, B)$ instead of $\Gamma(P)$ for plant $P = (A, B, x_0) \in \mathcal{P}$.

D. Performance Metrics

The goal of this paper is to investigate the influence of the plant and design graph on the properties of controllers constructed by limited model information control design methods. To this end, we will use two performance metrics for control design methods. These performance metrics are adapted from the notions of competitive ratio and domination first introduced in [4], so as to take plant, controller, and control design structures into account. We start by introducing the (closed-loop) performance criterion.

To each plant $P = (A, B, x_0) \in \mathcal{P}$ and controller $K \in \mathcal{K}$, we associate the performance criterion

$$J_P(K) = \sum_{k=1}^{\infty} x(k)^T Q x(k) + \sum_{k=0}^{\infty} u(k)^T R u(k), \quad (3)$$

where $Q \in S^n_{++}$ and $R \in S^n_{++}$ are block diagonal matrices, with each diagonal block entry belonging to $S^{n_i}_{++}$. We make the following two standing assumptions:

Assumption 2.1: $Q = R = I$.

This is without loss of generality because the change of variables $(\bar{x}, \bar{u}) = (Q^{1/2}x, R^{1/2}u)$ transforms the performance criterion into

$$J_P(K) = \sum_{k=1}^{\infty} \bar{x}(k)^T \bar{x}(k) + \sum_{k=0}^{\infty} \bar{u}(k)^T \bar{u}(k), \quad (4)$$

without affecting the plant, controller, or design graph (due to the block diagonal structure of Q and R).

Assumption 2.2: We replace the set $\mathcal{B}(\epsilon)$ by its intersection with the set of diagonal matrices.

This assumption is without loss of generality. Indeed, consider a plant $P = (A, B, x_0) \in \mathcal{P}$. Every sub-system's B_{ii} matrix has a singular value decomposition $B_{ii} = U_{ii}\Sigma_{ii}V_{ii}^T$ with $\Sigma_{ii} \geq \epsilon I_{n_i \times n_i}$, as $\underline{\sigma}(B) \geq \epsilon$ for all $B \in \mathcal{B}(\epsilon)$ by definition. Combining these singular value decompositions together results in a singular value decomposition for matrix $B = U\Sigma V^T$ where $U = \mathrm{diag}(U_{11}, \cdots, U_{qq})$, $\Sigma = \mathrm{diag}(\Sigma_{11}, \cdots, \Sigma_{qq})$, and $V = \mathrm{diag}(V_{11}, \cdots, V_{qq})$. Using the change of variable $(\bar{x}, \bar{u}) = (U^T x, V^T u)$ results the performance criterion of the form (4), because both U and V are unitary matrices. Besides, because of the block diagonal structure of matrices U and V, this change of variable does not affect the plant, controller, or design graph.

We are now ready to define the performance metrics of interest in this paper.

Definition 2.3: (Competitive Ratio) Let a plant graph $G_{\mathcal{P}}$, controller graph $G_{\mathcal{K}}$ and constant $\epsilon > 0$ be given. Assume that, for every plant $P \in \mathcal{P}$, there exists a controller $K^*(P) \in \mathcal{K}$ such that

$$J_P(K^*(P)) \leq J_P(K), \ \forall K \in \mathcal{K}.$$

The competitive ratio (against $\mathcal{P}$) of a control design method Γ is defined as

$$r_{\mathcal{P}}(\Gamma) = \sup_{P=(A,B,x_0)\in\mathcal{P}} \frac{J_P(\Gamma(A,B))}{J_P(K^*(P))},$$

with the convention that "$\frac{0}{0}$" equals one.

Note that the mapping $K^* : P \to K^*(P)$ is not itself required to lie in the set $\mathcal{C}$, as every component of the optimal controller may depend on all entries of the model matrices A and B. Also note that the existence and ease of computation of K^* depends on the nature of set $\mathcal{K}$.

Definition 2.4: (Domination) A control design method Γ is said to dominate another control design method Γ' if

$$J_P(\Gamma(A, B)) \leq J_P(\Gamma'(A, B)), \ \forall\, P = (A, B, x_0) \in \mathcal{P}, \quad (5)$$

with strict inequality holding for at least one plant in $\mathcal{P}$. When $\Gamma' \in \mathcal{C}$ and no control design method $\Gamma \in \mathcal{C}$ exists that satisfies (5), we say that Γ' is undominated in $\mathcal{C}$ for plants in $\mathcal{P}$.

E. Problem Formulation

With the definitions of the previous subsections in hand, we can reformulate the main high-level question of this paper regarding the connection between closed-loop performance, plant structure, and limited model information control design as follows. For a given plant graph, control graph, and design graph, we would like to determine

$$\arg\min_{\Gamma\in\mathcal{C}} r_{\mathcal{P}}(\Gamma). \quad (6)$$

Since several design methods may achieve $\min_{\Gamma\in\mathcal{C}} r_{\mathcal{P}}(\Gamma)$, we are additionally interested in determining strategies in the set (6) that are *undominated*.

In [4], this problem was solved when the plant graph $G_{\mathcal{P}}$ and the control graph $G_{\mathcal{K}}$ are complete graphs, the design graph $G_{\mathcal{C}}$ is a totally disconnected graph with self-loops (i.e., $S_{\mathcal{C}} = I_q$), and $\mathcal{B}(\epsilon)$ is replaced with $\{I_n\}$. In this paper, we investigate the role of more general plant and design graphs. We also extend the results in [4] for scalar subsystems into subsystems of arbitrary order $n_i \geq 1$, $1 \leq i \leq q$.

III. Plant Graph Influence on Achievable Performance

In this section, we study the relationship between the plant graph and the achievable closed-loop performance in term of the competitive ratio and the domination.

Definition 3.1: The deadbeat control design method $\Gamma^{\Delta} : \mathcal{A}(S_{\mathcal{P}}) \times \mathcal{B}(\epsilon) \to \mathcal{K}$ is defined as

$$\Gamma^{\Delta}(A, B) = -B^{-1}A, \text{ for all } P = (A, B, x_0) \in \mathcal{P}.$$

This control design method is communication-less because subsystem i's controller gain $[\Gamma_{i1}(A, B) \ \cdots \ \Gamma_{iq}(A, B)]$ equals to $B_{ii}^{-1}[A_{i1} \ \cdots \ A_{iq}]$. The name "deadbeat" comes from the fact that the closed-loop system obtained by applying controller $\Gamma^{\Delta}(A, B)$ to plant $P = (A, B, x_0)$ reaches the origin just in one time-step [5].

Theorem 3.2: Let the plant graph $G_{\mathcal{P}}$ contain no isolated node and the control graph $G_{\mathcal{K}}$ be a complete graph. Then the competitive ratio of the deadbeat control design method is $r_{\mathcal{P}}(\Gamma^{\Delta}) = 1 + 1/\epsilon^2$.

Proof: For any plant $P = (A, B, x_0) \in \mathcal{P}$, the optimal controller $K^*(P)$ exists (because the plant is controllable since B is invertible by assumption) and can be computed using the unique positive definite solution to the algebraic Riccati equation

$$X = A^T XA - A^T XB(I + B^T XB)^{-1}B^T XA + I. \quad (7)$$

The corresponding cost is $J_P(K^*(A,B)) = x_0^T(X-I)x_0$. Inserting the product BB^{-1} before every matrix A and $B^{-T}B^T$ after every matrix A^T in Equation (7) results in

$$\begin{aligned} X-I &= A^TB^{-T}B^TXBB^{-1}A \\ &\quad - A^TB^{-T}B^TXB(I+B^TXB)^{-1}B^TXBB^{-1}A. \end{aligned} \tag{8}$$

Naming B^TXB as Y simplifies Equation (8) into

$$X-I = A^TB^{-T}[Y-Y(I+Y)^{-1}Y]B^{-1}A. \tag{9}$$

Note that Y is a positive definite matrix because X is positive definite and B is full rank. Let us denote the right-hand side of (9) by $A^TB^{-T}g(Y)B^{-1}A$. Then we can make the following two claims regarding the rational function $g(.)$.

Claim 1: The function $y \mapsto g(y) = y/(y+1)$ is a monotonically increasing over $\mathbb{R}^+$.

Claim 2: Let $Y \in S^n_{++}$ and D, T be diagonal and unitary matrices, respectively, such that $Y = T^TDT$. Then $g(Y) = T^T\mathrm{diag}(g(d_{11}),\ldots,g(d_{nn}))T$, where the d_{ii} are the diagonal elements of D (and the eigenvalues of Y).

Claim 1 is proved by computing the derivative of g over $\mathbb{R}^+$, while Claim 2 follows from the fact that all matrices involved in the computation of $g(Y)$ can be diagonalized in the same basis. Using these two claims, we find that, for all Y with eigenvalues denoted by $\lambda_1(Y),\ldots,\lambda_n(Y)$

$$\begin{aligned} X-I &= A^TB^{-T}g(Y)B^{-1}A \\ &= A^TB^{-T}T^T\mathrm{diag}(g(\lambda_1(Y)),\ldots,g(\lambda_n(Y)))TB^{-1}A \\ &\geq (g(\underline{\lambda}(Y)))A^TB^{-T}B^{-1}A, \end{aligned} \tag{10}$$

where $\underline{\lambda}(Y)$ is a positive number because matrix Y is a positive definite matrix. Now, according to [6],

$$\underline{\lambda}(X) \geq \underline{\lambda}(A^T(I+BB^T)^{-1}A+I) \geq \frac{\underline{\sigma}^2(A)}{1+\bar{\sigma}^2(B)}+1. \tag{11}$$

Using Equation (11) in inequality $\lambda(Y) \geq \underline{\sigma}^2(B)\underline{\lambda}(X)$ gives

$$\underline{\lambda}(Y) \geq \frac{\underline{\sigma}^2(B)\underline{\sigma}^2(A)}{1+\bar{\sigma}^2(B)} + \underline{\sigma}^2(B),$$

and

$$\begin{aligned} g(\underline{\lambda}(Y)) &\geq \frac{\underline{\sigma}^2(B)[\underline{\sigma}^2(A)+\bar{\sigma}^2(B)+1]}{1+\bar{\sigma}^2(B)+\underline{\sigma}^2(B)[\underline{\sigma}^2(A)+\bar{\sigma}^2(B)+1]} \\ &\geq \frac{\underline{\sigma}^2(B)}{\underline{\sigma}^2(B)+1}. \end{aligned} \tag{12}$$

Combining equations (10) and (12) results in

$$X-I \geq \frac{\underline{\sigma}^2(B)}{\underline{\sigma}^2(B)+1}A^TB^{-T}B^{-1}A,$$

and therefore

$$\frac{J_P(\Gamma^\Delta(A,B))}{J_P(K^*(A,B))} = \frac{x_0^T(A^TB^{-T}B^{-1}A)x_0}{x_0^T(X-I)x_0} \leq 1+\frac{1}{\epsilon^2}.$$

for all $P=(A,B,x_0)\in\mathcal{P}$.

To show that this upper-bound is attained, let us pick $i_1 \in \mathcal{I}_i$ and $j_1 \in \mathcal{I}_j$ where $1 \leq i \neq j \leq q$ and $(s_\mathcal{P})_{ij} \neq 0$ (such indices i and j exist because plant graph $G_\mathcal{P}$ has no isolated node by assumption). Consider then matrix A defined as $A = e_{i_1}e_{j_1}^T$ and matrix B defined as $B = \epsilon I$. The unique solution of the Riccati equation is $X = I+[1/(1+\epsilon^2)]e_{j_1}e_{j_1}^T$ and $J_{(A,B,e_{j_1})}(K^*(A,B)) = 1/(1+\epsilon^2)$. On the other hand $\Gamma^\Delta(A,B) = -[1/\epsilon]e_{i_1}e_{j_1}^T$ and $J_{(A,B,e_{j_1})}(\Gamma^\Delta(A,B)) = 1/\epsilon^2$. Therefore, $r_\mathcal{P}(\Gamma^\Delta) = 1+1/\epsilon^2$. ∎

There is no loss of generality in assuming that there is no isolated node in the plant graph $G_\mathcal{P}$, since it is always possible to design a controller for an isolated subsystem without any model information about the other subsystems and without impacting cost (3). In particular, this implies that there are $q \geq 2$ vertices in the graph.

With this characterization of Γ^Δ in hand, we are now ready to tackle problem (6).

A. First case: plant graph $G_\mathcal{P}$ with no sink

In this subsection, we show that, when the plant graph $G_\mathcal{P}$ contains no sink, the deadbeat control method is undominated by communication-less control design methods for plants in $\mathcal{P}$ *and* that it exhibits the smallest possible competitive ratio among such control design methods.

First, we state the following two lemmas, in which we assume that the plant graph $G_\mathcal{P}$ contains no isolated node, the control graph $G_\mathcal{K}$ is a complete graph, and the design graph $G_\mathcal{C}$ is a totally disconnected graph with self-loops only.

Lemma 3.3: A control design method $\Gamma \in \mathcal{C}$ has bounded competitive ratio only if the following implication holds for all $1 \leq \ell \leq q$ and all j:

$$a_{ij} = 0 \text{ for all } i \in \mathcal{I}_\ell \Rightarrow \gamma_{ij}(A,B) = 0 \text{ for all } i \in \mathcal{I}_\ell.$$

Proof: Assume that this claim is not correct, i.e., that there exists a matrix A and indices ℓ, j, $i_0 \in \mathcal{I}_\ell$ such that $a_{ij}=0$ for all $i \in \mathcal{I}_\ell$ but $\gamma_{i_0j}(A,B) \neq 0$. Consider matrix $\bar{A}$ such that $\bar{A}_i = A_i$ for all $i \in \mathcal{I}_\ell$ and $\bar{A}_z = 0$ for all $z \notin \mathcal{I}_\ell$. Based on the definition of limited-model-information control design methods, we know $\Gamma_i(A,B) = \Gamma_i(\bar{A},B)$ for all $i \in \mathcal{I}_\ell$ and $\Gamma_i(\bar{A},B) = 0$ for all $i \notin \mathcal{I}_\ell$ (because $\Gamma_i(A,B) = \Gamma_i(0,B)$ for all $i \notin \mathcal{I}_\ell$ and, as shown in [4], it is necessary that $\Gamma(0,B) = 0$ for Γ to have a finite competitive ratio). For $x = e_j$, we have

$$\begin{aligned} J_{(\bar{A},B,e_j)}(\Gamma(\bar{A},B)) &\geq \sum_{i\in\mathcal{I}_\ell}\gamma_{ij}(\bar{A},B)^2 = \sum_{i\in\mathcal{I}_\ell}\gamma_{ij}(A,B)^2 \\ &\geq \gamma_{i_0j}(A,B)^2 > 0. \end{aligned}$$

Now, note that because the j^{th} column of matrix $\bar{A}$ is entirely zero, the j^{th} column of the optimal controller $K^*(\bar{A},B) = -(I+B^TXB)^{-1}B^TX\bar{A}$ is also zero. Thus, $J_{(\bar{A},B,e_j)}(K^*(\bar{A},B)) = 0$ and, as result,

$$r_\mathcal{P}(\Gamma) \geq \frac{J_{(\bar{A},B,e_j)}(\Gamma(\bar{A},B))}{J_{(\bar{A},B,e_j)}(K^*(\bar{A},B))} = \infty.$$

This proves the claim by contrapositive. ∎

Lemma 3.4: Assume plant graph $G_\mathcal{P}$ has at least one loop. Then, $r_\mathcal{P}(\Gamma) \geq 1+1/\epsilon^2$ for all limited model information control design method Γ in $\mathcal{C}$.

Proof: Without loss of generality, let us assume that the nodes of graph $G_{\mathcal{P}}$ are numbered such that it admits the following loop of length ℓ: $1 \to 2 \to \cdots \to \ell \to 1$. Let us choose indices $i_1 \in \mathcal{I}_1, i_2 \in \mathcal{I}_2, \ldots, i_\ell \in \mathcal{I}_\ell$ and consider the one-parameter family of matrices $A(r)$ defined by $a_{i_2 i_1}(r) = r, a_{i_3 i_2}(r) = r, \ldots, a_{i_\ell i_{\ell-1}}(r) = r, a_{i_1 i_\ell}(r) = r$, and all other entries equal to zero, for all r. Let $B = \epsilon I$. Because of Lemma 3.3, the controller gain entries $\gamma_{j_2 i_1}(A(r), B)$ for all $j_2 \in \mathcal{I}_2$, $\gamma_{j_3 i_2}(A(r), B)$ for all $j_3 \in \mathcal{I}_3, \ldots, \gamma_{j_\ell i_{\ell-1}}(A(r), B)$ for all $j_\ell \in \mathcal{I}_\ell$, $\gamma_{j_1 i_\ell}(A(r), B)$ for all $j_1 \in \mathcal{I}_1$ can be non-zero, but all other entries of the controller gain $\Gamma(A(r), B)$ are zero for all r. As a result, the characteristic polynomial of matrix $A(r) + B\Gamma(A(r), B)$ can be computed as:

$$\begin{aligned}&\lambda^{n-\ell}[\lambda^\ell + (-1)^\ell (r + \epsilon\gamma_{i_2 i_1}(A(r), B)) \cdots \\ &\quad \times (r + \epsilon\gamma_{i_\ell i_{\ell-1}}(A(r), B))(r + \epsilon\gamma_{i_1 i_\ell}(A(r), B))].\end{aligned} \tag{13}$$

Now, note that because Γ has a bounded competitive ratio against $\mathcal{P}$ by assumption, this polynomial should be stable for all r. Indeed, Γ can have a finite competitive ratio only if $A + B\Gamma(A, B)$ is stable for all matrix A, for otherwise it would yield an infinite cost for some plants while the corresponding optimal cost remains bounded since the pair (A, B) is controllable for all plant in $\mathcal{P}$. As a result, we must have

$$\begin{aligned}&|(r + \epsilon\gamma_{i_2 i_1}(A(r), B)) \cdots (r + \epsilon\gamma_{i_1 i_\ell}(A(r), B))| \\ &= |r + \epsilon\gamma_{i_2 i_1}(A(r), B)| \cdots |r + \epsilon\gamma_{i_1 i_\ell}(A(r), B)| < 1\end{aligned} \tag{14}$$

for all r. Let $\{r_z\}_{z=1}^\infty$ be a sequence of real numbers with the property that r_z goes to infinity as z goes to infinity. From (14), we know that there exists an index $\bar{m}$ such that

$$\forall N,\ \exists z > N \text{ s.t. } |r_z + \epsilon\gamma_{i_{\bar{m}\oplus 1} i_{\bar{m}}}(A(r_z), B)| < 1, \tag{15}$$

where "$\oplus$" designated addition modulo ℓ. Indeed, if this not the case, it is true that

$$\forall m,\ \exists N_m \text{ s.t. } |r_z + \epsilon\gamma_{i_{\bar{m}\oplus 1} i_{\bar{m}}}(A(r_z), B)| \geq 1, \forall z > N_m.$$

Then, for all $z > \max_m N_m$ and all m, $|r_z + \epsilon\gamma_{i_{\bar{m}\oplus 1} i_{\bar{m}}}(A(r_z), B)| \geq 1$, which contradicts (14). Without loss of generality (since this just amounts to renumbering the nodes in the plant graph), we assume that $\bar{m} = 1$. Using (15), we can then construct a subsequence $\{r_{\phi(z)}\}$ of $\{r_z\}$ with the property that

$$|r_{\phi(z)} + \epsilon\gamma_{i_2 i_1}(A(r_{\phi(z)}), B)| < 1 \ \text{ for all } z.$$

Now introduce the sequence of matrices $\{\bar{A}(z)\}_{z=1}^\infty$ defined by $\bar{A}_{i_2 i_1}(z) = r_{\phi(z)}$ for all z and every other row equal to zero. For large enough z (and hence, large enough $r_{\phi(z)}$),

$$\begin{aligned}J_{(\bar{A}(z),B,e_{i_1})}(\Gamma(\bar{A}(z), B)) &\geq \gamma_{i_2 i_1}(\bar{A}(z), B)^2 \\ &= \gamma_{i_2 i_1}(A(r_{\phi(z)}), B)^2 \\ &\geq \frac{(|r_{\phi(z)}| - 1)^2}{\epsilon^2}\end{aligned}$$

and, thus,

$$\frac{J_{(\bar{A}(z),B,e_{i_1})}(\Gamma(\bar{A}(z), B))}{J_{(\bar{A}(z),B,e_{i_1})}(K^*(\bar{A}(z), B))} \geq \frac{(|r_{\phi(z)}| - 1)^2/\epsilon^2}{r_{\phi(z)}^2/(1+\epsilon^2)}.$$

This, in particular, implies that

$$r_{\mathcal{P}}(\Gamma) \geq \lim_{z\to\infty} \frac{J_{(\bar{A}(z),B,e_{i_1})}(\Gamma(\bar{A}(z), B))}{J_{(\bar{A}(z),B,e_{i_1})}(K^*(\bar{A}(z), B))} \geq 1 + 1/\epsilon^2,$$

which finishes the proof. ∎

Theorem 3.5: Let the plant graph $G_{\mathcal{P}}$ contain no isolated node and no sink, the control graph $G_{\mathcal{K}}$ be a complete graph, and the design graph $G_{\mathcal{C}}$ be a totally disconnected graph with self-loops. Then, the competitive ratio of any control design strategy $\Gamma \in \mathcal{C}$ satisfies $r_{\mathcal{P}}(\Gamma) \geq 1 + 1/\epsilon^2$.

Proof: From Lemma 1.4.23 in [7], we know that a directed graph with no sink must have at least one loop. Hence, if $G_{\mathcal{P}}$ satisfies the assumptions of the theorem, it must contain a loop. The result then follows from Lemma 3.4. ∎

Theorem 3.5 shows that the deadbeat control design method Γ^Δ is a minimizer of the competitive ratio function $r_{\mathcal{P}}$ over the set of communication-less design methods. The following theorem shows that it is also undominated by methods of this type if and only if $G_{\mathcal{P}}$ has no sink.

Theorem 3.6: Let the plant graph $G_{\mathcal{P}}$ contain no isolated node, the control graph $G_{\mathcal{K}}$ be a complete graph, and the design graph $G_{\mathcal{C}}$ be a totally disconnected graph with self-loops. The deadbeat control design method is undominated in $\mathcal{C}$ for plants in $\mathcal{P}$ if and only if there is no sink in the plant graph $G_{\mathcal{P}}$.

Proof: The "if" part of the proof is similar to that of Theorem 3 in [4], with additional attention paid to the fact that the plants chosen to establish undomination of Γ^Δ by any other design method $\Gamma \in \mathcal{C}$ has the structure of a sink-less graph. For the "only if" part, we show that the communication-less design method Γ^Θ introduced later dominates the deadbeat for plants in $\mathcal{P}$, when plant graph $G_{\mathcal{P}}$ has at least one sink. See [8] for the detailed proof. ∎

B. Second case: plant graph $G_{\mathcal{P}}$ with at least one sink

In this section, we consider the case where plant graph $G_{\mathcal{P}}$ has $c \geq 1$ distinct sinks. Accordingly, its adjacency matrix $S_{\mathcal{P}}$ is of the form

$$S_{\mathcal{P}} = \left[\begin{array}{c|c} (S_{\mathcal{P}})_{11} & 0_{(q-c)\times(c)} \\ \hline (S_{\mathcal{P}})_{21} & (S_{\mathcal{P}})_{22}\end{array}\right], \tag{16}$$

where

$$(S_{\mathcal{P}})_{11} = \begin{bmatrix} (s_{\mathcal{P}})_{11} & \cdots & (s_{\mathcal{P}})_{1,q-c} \\ \vdots & \ddots & \vdots \\ (s_{\mathcal{P}})_{q-c,1} & \cdots & (s_{\mathcal{P}})_{q-c,q-c}\end{bmatrix},$$

$$(S_{\mathcal{P}})_{21} = \begin{bmatrix} (s_{\mathcal{P}})_{q-c+1,1} & \cdots & (s_{\mathcal{P}})_{q-c+1,q-c} \\ \vdots & \ddots & \vdots \\ (s_{\mathcal{P}})_{q,1} & \cdots & (s_{\mathcal{P}})_{q,q-c}\end{bmatrix},$$

and

$$(S_{\mathcal{P}})_{22} = \begin{bmatrix} (s_{\mathcal{P}})_{q-c+1,q-c+1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & (s_{\mathcal{P}})_{qq}\end{bmatrix},$$

if we assume, without loss of generality, that the vertices are numbered such that the sinks are labeled $q-c+1, \cdots, q$. With these notations, let us now introduce the control design method Γ^{Θ} defined by

$$\Gamma^{\Theta}(A,B) = -\mathrm{diag}(B_{11}^{-1}, \ldots, B_{q-c,q-c}^{-1}, W_{q-c+1}(A,B), \ldots, W_q(A,B))A \tag{17}$$

for all $(A,B) \in \mathcal{A}(S_{\mathcal{P}}) \times \mathcal{B}(\epsilon)$, and

$$W_i(A,B) = (I + B_{ii}^T X_{ii} B_{ii})^{-1} B_{ii}^T X_{ii} \tag{18}$$

for all $i \in \{q-c+1, \cdots, q\}$ and X_{ii} is the unique positive definite solution of the following Riccati equation

$$A_{ii}^T X_{ii} A_{ii} - A_{ii}^T X_{ii} B_{ii} (I + B_{ii}^T X_{ii} B_{ii})^{-1} B_{ii}^T X_{ii} A_{ii} - X_{ii} + I = 0.$$

The control design strategy Γ^{Θ} applies the deadbeat to every subsystem that is not a sink and, for every sink, applies the same optimal control law as if the node were decoupled from the rest of the graph. We will show that when the plant graph contains sinks, control design method Γ^{Θ} has, in the worst case, the same competitive ratio as the deadbeat strategy, but also has the additional property of being undominated by communication-less methods for plants on $\mathcal{P}$.

We start with a lemma. In the following lemma, we assume that the plant graph $G_{\mathcal{P}}$ contains no isolated node, the control graph $G_{\mathcal{K}}$ is a complete graph, and the design graph $G_{\mathcal{C}}$ is a totally disconnected graph with self-loops.

Lemma 3.7: Let Γ be a communication-less control design method. Suppose that there exist i and $j \neq i$ such that $(s_{\mathcal{P}})_{ij} \neq 0$ and node i is not a sink; i.e., there exists $\ell \neq i$ such that $(s_{\mathcal{P}})_{\ell i} \neq 0$. The competitive ratio of Γ against $\mathcal{P}$ is bounded only if

$$a_{i_1 j_1} + b_{i_1 i_1} \gamma_{i_1 j_1}(A,B) = 0, \quad \text{for all } i_1 \in \mathcal{I}_i \text{ and } j_1 \in \mathcal{I}_j.$$

Proof: For ease of notation in this proof, we use $[A]_i = [A_{i1} \ \cdots \ A_{iq}]$. The proof is by contrapositive. Assume that there exist matrices A and B and indices $i_1 \in \mathcal{I}_i$ and $j_1 \in \mathcal{I}_j$ such that $a_{i_1 j_1} + b_{i_1 i_1} \gamma_{i_1 j_1}(A,B) \neq 0$. Choose an index $\ell_1 \in \mathcal{I}_\ell$. Consider the one-parameter family of matrices $\bar{A}(r)$ defined by $[\bar{A}(r)]_i = [A]_i$, $\bar{a}_{\ell_1 i_1} = r$, and all other entries of $\bar{A}(r)$ being equal to zero for all r. We know that $[\Gamma(\bar{A}(r),B)]_i = [\Gamma(A,B)]_i$ and $\Gamma_{\bar{\ell}}(\bar{A},B) = \gamma_{\bar{\ell} i_1}(r) e_{i_1}^T$ for all $\bar{\ell} \in \mathcal{I}_\ell$ (because of Lemma 3.3), $[\Gamma(\bar{A},B)]_z = 0$ for all $z \neq i, \ell$. For $x_0 = e_{j_1}$, we have

$$J_{(\bar{A}(r),B,x_0)}(\Gamma(\bar{A},B)) \geq (a_{i_1 j_1} + b_{i_1 i_1} \gamma_{i_1 j_1}(A,B))^2 \times [\gamma_{\ell_1 i_1}(r)^2 + (r + b_{\ell_1 \ell_1} \gamma_{\ell_1 i_1}(r))^2].$$

The minimum value of function $y \mapsto [y^2 + (r + b_{\ell_1 \ell_1} y)^2]$ is $r^2/(1 + b_{\ell_1 \ell_1}^2)$. Hence, irrespective of function $\gamma_{\ell_1 i_1}$,

$$J_{(\bar{A}(r),B,e_{j_1})}(\Gamma(\bar{A}(r),B)) \geq \frac{(a_{i_1 j_1} + b_{i_1 i_1} \gamma_{i_1 j_1}(A,B))^2 r^2}{1 + b_{\ell_1 \ell_1}^2}.$$

Note that the term $(a_{i_1 j_1} + b_{i_1 i_1} \gamma_{i_1 j_1}(A,B))^2$ is independent from r because Γ is communication-less. In addition,

$$J_{(\bar{A}(r),B,e_{j_1})}(\Gamma^{\Delta}(\bar{A}(r),B)) = \sum_{z \in \mathcal{I}_i} \frac{\bar{a}_{z j_1}^2}{b_{zz}^2} = \sum_{z \in \mathcal{I}_i} \frac{a_{z j_1}^2}{b_{zz}^2}$$

for all r and, thus, $J_{(\bar{A}(r),B,e_{j_1})}(\Gamma^{\Delta}(\bar{A}(r),B))$ is also independent from r. Then

$$\begin{aligned} r_{\mathcal{P}}(\Gamma) &= \sup_{P \in \mathcal{P}} \frac{J_P(\Gamma(A,B))}{J_P(K^*(A,B))} \\ &= \sup_{P \in \mathcal{P}} \left[\frac{J_P(\Gamma(A,B))}{J_P(\Gamma^{\Delta}(A,B))} \frac{J_P(\Gamma^{\Delta}(A,B))}{J_P(K^*(A,B))} \right] \\ &\geq \sup_{P \in \mathcal{P}} \frac{J_P(\Gamma(A,B))}{J_P(\Gamma^{\Delta}(A,B))}, \end{aligned}$$

and, as a result,

$$r_{\mathcal{P}}(\Gamma) \geq \frac{(a_{i_1 j_1} + b_{i_1 i_1} \gamma_{i_1 j_1}(\bar{A},B))^2}{(1 + b_{\ell_1 \ell_1}^2) J_{(\bar{A}(r),B,e_{j_1})}(\Gamma^{\Delta}(\bar{A}(r),B))} \lim_{r \to \infty} r^2.$$

Since $(a_{i_1 j_1} + b_{i_1 i_1} \gamma_{i_1 j_1}(\bar{A},B)) \neq 0$ by assumption, we then deduce that Γ has an unbounded competitive ratio, which proves the theorem by contrapositive. ■

Theorem 3.8: Let the plant graph $G_{\mathcal{P}}$ contain no isolated node and at least one sink, and the control graph $G_{\mathcal{K}}$ be a complete graph. Then the competitive ratio of the control design method Γ^{Θ} in (17) is

$$r_{\mathcal{P}}(\Gamma^{\Theta}) = \begin{cases} 1, & \text{if } (S_{\mathcal{P}})_{11} = 0 \text{ and } (S_{\mathcal{P}})_{22} = 0, \\ 1 + 1/\epsilon^2, & \text{otherwise.} \end{cases}$$

Proof: Based on Theorem 3.2, we know that, for every plant $P = (A,B,x_0) \in \mathcal{P}$

$$J_{(A,B,x_0)}(K^*(A,B)) \geq \frac{\epsilon^2}{1+\epsilon^2} x_0^T A^T B^{-T} B^{-1} A x_0, \tag{19}$$

In addition, proceeding as in the proof of the "only if" part of Theorem 3.6, we know that

$$J_{(A,B,x_0)}(\Gamma^{\Delta}(A,B)) \geq J_{(A,B,x_0)}(\Gamma^{\Theta}(A,B)). \tag{20}$$

Plugging Equation (20) into Equation (19) results in

$$J_{(A,B,x_0)}(K^*(A,B)) \geq \frac{\epsilon^2}{1+\epsilon^2} J_{(A,B,x_0)}(\Gamma^{\Theta}(A,B))$$

and, therefore, in

$$\frac{J_{(A,B,x_0)}(\Gamma^{\Theta}(A,B))}{J_{(A,B,x_0)}(K^*(A,B))} \leq 1 + \frac{1}{\epsilon^2} \text{ for all } P = (A,B,x_0) \in \mathcal{P}.$$

As a result, $r_{\mathcal{P}}(\Gamma^{\Theta}) \leq 1 + 1/\epsilon^2$. To show that this upper-bound is tight, we now exhibit plants for which it is attained. We use a different construction depending on matrices $(S_{\mathcal{P}})_{11}$ and $(S_{\mathcal{P}})_{22}$. If $(S_{\mathcal{P}})_{11} \neq 0$, two situations can occur.

- Case #1: $(S_{\mathcal{P}})_{11}$ has an off-diagonal entry; i.e., there exist $1 \leq i \neq j \leq q-c$ such that $(s_{\mathcal{P}})_{ij} \neq 0$. In this case, choose indices $i_1 \in \mathcal{I}_i$ and $j_1 \in \mathcal{I}_j$ and define $A = e_{i_1} e_{j_1}^T$ and $B = \epsilon I$. Then, for $x_0 = e_{j_1}$, we find that

$$\frac{J_{(A,B,x_0)}(\Gamma^{\Theta}(A,B))}{J_{(A,B,x_0)}(K^*(A,B))} = \frac{1/\epsilon^2}{1/(1+\epsilon^2)} = 1 + \frac{1}{\epsilon^2}$$

because the controller design Γ^{Θ} acts like the deadbeat control design method on this plant.

- Case #2: $(S_{\mathcal{P}})_{11}$ is diagonal and it has a nonzero diagonal entry; i.e., there exists $1 \leq i \leq q-c$ such that $(s_{\mathcal{P}})_{ii} \neq 0$. Choose an index i_1 in the set $\mathcal{I}_i$ and consider $A(r) =$

$re_{i_1}e_{i_1}^T$ and $B = \epsilon I$. For $x_0 = e_{i_1}$, the optimal cost $J_{(A(r),B,x_0)}(K^*(A(r),B))$ is equal to

$$\frac{\sqrt{r^4+2r^2\epsilon^2-2r^2+\epsilon^4+2\epsilon^2+1}+r^2-\epsilon^2-1}{2\epsilon^2},$$

which results in

$$\lim_{r\to 0}\frac{J_{(A,B,x_0)}(\Gamma^\Theta(A,B))}{J_{(A,B,x_0)}(K^*(A,B))} = 1+\frac{1}{\epsilon^2}.$$

Now suppose that $(S_\mathcal{P})_{11} = 0$. Again, two different situations can occur.

• Case #1: $(S_\mathcal{P})_{22}$ is nonzero; i.e., there exists $q-c+1 \le i \le q$ such that $(s_\mathcal{P})_{ii} \neq 0$. From the assumption that the plant graph contains no isolated node, we know that there must exist $1 \le j \le q-c$ such that $(s_\mathcal{P})_{ij} \neq 0$. Accordingly, let us pick $i_1 \in \mathcal{I}_i$ and $j_1 \in \mathcal{I}_j$ and consider the 2-parameter family of matrices $A(r,s)$ in $\mathcal{A}(S_\mathcal{P})$ with all entries equal to zero except $a_{i_1i_1}$, which is equal to r, and $a_{i_1j_1}$, which is equal to s. Let $B = \epsilon I$. For any initial condition x_0, the corresponding closed-loop performance is

$$J_{(A(r,s),B,x_0)}(\Gamma^\Theta(A(r,s),B)) = \beta_\Theta x_0^T a(r,s)^T a(r,s)x_0,$$

where we have let $a(r,s) = A(r,s)_{i_1}$ and β_Θ is

$$\beta_\Theta = \frac{\sqrt{r^4+2r^2\epsilon^2-2ar^2+\epsilon^4+2\epsilon^2+1}+r^2-\epsilon^2-1}{2\epsilon^2 r^2}.$$

Besides, the optimal closed-loop performance can be computed as

$$J_{(A(r,s),B,x_0)}(K^*(A(r,s),B)) = \beta_{K^*} x_0^T a(r,s)^T a(r,s)x_0,$$

where β_{K^*} is

$$\beta_{K^*} = \frac{\epsilon^2 s^2 + r^2(1+\epsilon^2) - (\epsilon^2+1)^2 + \sqrt{c_+c_-}}{2\epsilon^2(\epsilon^2+1)(s^2+r^2)},$$

$$c_\pm = (\epsilon^2 s^2 + (r^2 \pm 2r)(\epsilon^2+r) + (\epsilon^2+1)^2).$$

Then,

$$r_\mathcal{P}(\Gamma^\Theta) \ge \lim_{r\to\infty,\frac{s}{r}\to\infty}\frac{J_{(A(r,s),B,x_0)}(\Gamma^\Theta(A(r,s),B))}{J_{(A(r,s),B,x_0)}(K^*(A(r,s),B))} = 1+\frac{1}{\epsilon^2}$$

• Case #2: $(S_\mathcal{P})_{22} = 0$. Then, every matrix $A \in \mathcal{A}(S_\mathcal{P})$ has the form $\left[\begin{array}{c|c} 0 & 0 \\ \hline * & 0 \end{array}\right]$ and, in particular, is nilpotent of degree 2; i.e., $A^2 = 0$. In this case, the Riccati equation yielding the optimal control gain $K^*(A,B)$ can be readily solved, and we find that $K^*(A,B) = -(I+B^TB)^{-1}B^TA$ for all (A,B). As a result, $K^*(A,B) = \Gamma^\Theta(A,B)$ for all plant $P = (A,B,x_0) \in \mathcal{P}$ (since $W_i(A,B) = (I+B_{ii}^TB_{ii})^{-1}B_{ii}^T$ for all $q-c+1 \le i \le q$), which implies that the competitive ratio of Γ^Θ is equal to one. ■

Theorem 3.9: Let the plant graph $G_\mathcal{P}$ contain no isolated node and at least one sink, the control graph $G_\mathcal{K}$ be a complete graph, and the design graph $G_\mathcal{C}$ be a totally disconnected graph with self-loops. If $(S_\mathcal{P})_{11}$ is not diagonal or $(S_\mathcal{P})_{22} \neq 0$, then $r_\mathcal{P}(\Gamma) \ge 1+1/\epsilon^2$ for any control design method $\Gamma \in \mathcal{C}$.

Proof: First, suppose that $(S_\mathcal{P})_{11} \neq 0$ and $(S_\mathcal{P})_{11}$ is not a diagonal matrix. Then, there exist $1 \le i,j \le q-c$ and $i \neq j$ such that $(s_\mathcal{P})_{ij} \neq 0$. Choose indices $i_1 \in \mathcal{I}_i$ and $j_1 \in \mathcal{I}_j$ and consider the matrix A defined by $A = e_{i_1}e_{j_1}^T$ and $B = \epsilon I$. From Lemma 3.7, we know that a communication-less method Γ has a bounded competitive ratio only if $\Gamma(A,B) = -B^{-1}A$ (because node i is a part of $(S_\mathcal{P})_{11}$ and it is not a sink). Therefore

$$r_\mathcal{P}(\Gamma) \ge \frac{J_{(A,B,e_{j_1})}(\Gamma(A,B))}{J_{(A,B,e_{j_1})}(K^*(A,B))} = 1+\frac{1}{\epsilon^2}$$

for any such method. Second, suppose that $(S_\mathcal{P})_{22} \neq 0$. There thus exists $q-c+1 \le i \le q$ such that $(s_\mathcal{P})_{ii} \neq 0$. Note that, there exists $1 \le j \le q-c$ such that $(s_\mathcal{P})_{ij} \neq 0$, since there is no isolated node in the plant graph. Choose indices $i_1 \in \mathcal{I}_i$ and $j_1 \in \mathcal{I}_j$. Consider A defined as $A = re_{i_1}e_{j_1}^T + se_{i_1}e_{i_1}^T$ and $B = \epsilon I$. For this particular family of plants, Γ^Θ is optimal global controller with limited model information and based on the proof of Theorem 3.8, hence, we know that $r_\mathcal{P} \ge 1+1/\epsilon^2$. ■

Combining Theorem 3.8 and Theorem 3.9 implies that if either $(S_\mathcal{P})_{11}$ is not diagonal or $(S_\mathcal{P})_{22} \neq 0$, control design method Γ^Θ exhibits the same competitive ratio as the deadbeat control strategy, which is the smallest ratio achievable by communication-less control methods. However, the next theorem shows that Γ^Θ is a more desirable control design method than the deadbeat when plant graph $G_\mathcal{P}$ has sinks, since it is then undominated by communication-less design methods for plants in $\mathcal{P}$. The case where $(S_\mathcal{P})_{11}$ is diagonal and $(S_\mathcal{P})_{22} = 0$ is still open.

Theorem 3.10: Let the plant graph $G_\mathcal{P}$ contain no isolated node and at least one sink, the control graph $G_\mathcal{K}$ be a complete graph, and the design graph $G_\mathcal{C}$ be a totally disconnected graph with self-loops. The control design method Γ^Θ is undominated by any other control design method $\Gamma \in \mathcal{C}$ for plants in $\mathcal{P}$.

Proof: See [8] for detailed proof. ■

As a final remark, we point out that for general weight matrices Q and R appearing in the performance cost, the competitive ratio of both Γ^Δ and Γ^Θ is $1+\bar{\sigma}(R)/(\underline{\sigma}(Q)\epsilon^2)$. In particular, the competitive ratio has a limit equal to one as $\bar{\sigma}(R)/\underline{\sigma}(Q)$ goes to zero. We thus recover the well-known observation (e.g., [9]) that, for discrete-time linear time-invariant systems, the optimal linear quadratic regulator approaches the deadbeat controller in the limit of "cheap control".

IV. Design Graph Influence on Achievable Performance

In the previous section, we have shown that communication-less control design methods (i.e., $G_\mathcal{C}$ is totally disconnected with self-loops) have intrinsic performance limitations, and we have characterized minimal elements for both the competitive ratio and domination metrics. A natural question, then, is "Given plant graph $G_\mathcal{P}$, which design graph $G_\mathcal{C}$ is necessary to ensure the existence of $\Gamma \in \mathcal{C}$ with better competitive ratio than Γ^Δ and Γ^Θ ?". We tackle this question in this section.

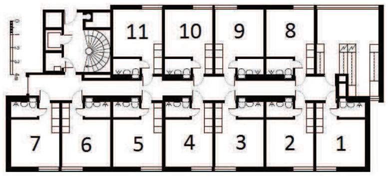

Fig. 1. The floor plan of a half block of a student house in Sweden.

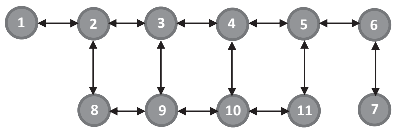

Fig. 2. The plant graph $G_{\mathcal{P}}$ of the system.

Theorem 4.1: Let the plant graph $G_{\mathcal{P}}$ and design graph $G_{\mathcal{C}}$ be given. Assume that $G_{\mathcal{P}}$ contains a path $k \to i \to j$, for distinct nodes i, j, and k. If $(j, i) \notin E_{\mathcal{C}}$, then $r_{\mathcal{P}}(\Gamma) \geq 1 + 1/\epsilon^2$ for all $\Gamma \in \mathcal{C}$.

Proof: See [8] for detailed proof. ■

Corollary 4.2: Let both the plant graph $G_{\mathcal{P}}$ and the control graph $G_{\mathcal{K}}$ be complete graphs. If the design graph $G_{\mathcal{C}}$ is not equal to the plant graph $G_{\mathcal{P}}$, then $r_{\mathcal{P}}(\Gamma) \geq 1 + 1/\epsilon^2$ for all $\Gamma \in \mathcal{C}$.

Proof: The proof is a direct application of Theorem 4.1. ■

Corollary 4.2 shows that, when $G_{\mathcal{P}}$ is a complete graph, achieving a better competitive ratio than the deadbeat design strategy requires each subsystem to have *full* knowledge of the plant model when constructing each subcontroller.

V. Illustrative Example

In this section, we illustrate limited model information control design through an example. Let us consider the problem of regulating the temperature in q different rooms. Let us suppose that each room can be warmed by a single heater. The goal is to maintain the temperature of each room at a prescribed value. Let us denote the average temperature of room i by $\bar{x}_i$. By applying Euler's constant step discretization scheme to the continuous-time model (both in time and space), we obtain

$$\bar{x}_i(k+1) = \sum_{j \neq i} a_{ij}(\bar{x}_j(k) - \bar{x}_i(k)) + b_i(\bar{x}_a - \bar{x}_i(k)) + u_i(k),$$

where $\bar{x}_a$ is the ambient temperature, which is assumed to be a known constant; b_i and a_{ij} are constants representing the average heat loss rates of room i to the ambient and to room j, respectively. Applying a change of variable $x(k) = \bar{x}(k) - x_d$ around the vector of desired temperature x_d results in a dynamical equation in the form of Equation (1) with $B = I$. In designing the controller, our aim is to minimize the cost function in Equation (3) with cost matrices $Q = R = I$. Note that when controlling the temperature in room i, the temperature of room $j \neq i$ may be measured, but the plant model parameters b_j and a_{jk} for all k could be unknown. Indeed, these parameters may depend on actions taking place in the room (such as opening doors and windows, cooking on a stove, etc...), which its owner may consider private and be unwilling to share with the thermostat of other rooms.

Figure 1 shows the floor plan of a student house in Sweden. The rooms are numbered from one to eleven. The corridors and stairways are supposed to have $\bar{x}_a$ as ambient temperature. If two rooms are not adjacent, their temperatures do not affect each other significantly, which we can use to generate the corresponding plant graph. In this particular problem, we have $q = 11$ rooms/subsystems and each room's dynamics is of dimension one. The plant graph $G_{\mathcal{P}}$ for this family of plants is shown in Figure 2. There is no sink in the plant graph. Using Theorem 3.5 and Theorem 3.6, we know that the deadbeat controller design strategy is undominated and has the best competitive ratio.

Now suppose that room number six is a refrigerated cold room that is perfectly isolated from all other spaces. This refrigerator warms up other places proportionally to the temperature difference, as it is cooling down room number six. In this case, node number seven becomes a sink in the new plant graph. Using Theorem 3.8 and Theorem 3.10, we now know that controller design strategy Γ^{Θ} in (17) is undominated and it achieves the best cost ratio for this problem.

VI. Conclusion

We presented a framework for the study of control design under limited model information, and investigated the connection between the control performance achieved by a design method and the amount of plant model information available to it. We showed that the best achievable performance by a limited model information control design method crucially depends on the structure of the plant graph. Possible future work will focus on extending the present framework to situations where the control graph is not complete and to plants with disturbances.

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