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Dynamic Input Consensus using Integrators

Clark N. Taylor, Randal W. Beard and Jeffrey Humpherys

Abstract— **The consensus or agreement problem enables a team of agents to agree on certain information variables using a low-bandwidth, dynamic, and sparsely-connected graph. However, most prior work on agreement protocols has focused on converging to a single, static variable. In this paper, we propose a consensus filter that accepts dynamically changing inputs at each agent. We analyze several properties of this consensus filter, proving the outputs of the filter converge to a low-pass filtered version of the average of the inputs. Disagreement portions of the inputs can be significantly attenuated through judicious selection of filter parameters.**

I. INTRODUCTION

The consensus problem, also called the agreement problem, is primarily concerned with algorithms and strategies that enable a team of agents, usually moving robotic agents, to agree on certain information variables using a low-bandwidth, dynamic, sparsely connected, and potentially noisy communication channel. Recent work on agreement algorithms include [1] which considers flocking behavior, where agents communicate their heading to physically close neighbors. An important result shown in [1] is that if the (bi-directional) communication topology is not switching infinitely fast, and if the union of the communication graphs over repeated time intervals is connected, then all agents converge to the same heading. A similar result for directed graphs is given in [2]. It is important to note that while these works guarantee convergence, they do not specify the value to which the network converges. An important contribution along those lines is [3], where it is shown that a strongly connected directed graph converges to the average of the initial conditions of each agent if and only if the graph is balanced, or in other words, when every agent has the same number of incoming and outgoing communication links. Since undirected graphs are balanced, this result implies that average consensus is achieved in undirected graphs if the graph is connected.

The agreement protocols proposed in [1], [2], [3] are concerned with asymptotic agreement starting from initial conditions. This framework works well if the agreement problem is static, or rather, if the agents must agree on a static variable. However, the standard agreement protocol does not address the case where the agents must agree on a dynamically changing quantity. In the case where there is a single, globally known, or globally sensed reference input, the agreement protocol can be modified to track the reference input if the network is connected [4]. However, it is interesting to pose the agreement problem where each agent has a different time varying reference input. Is it possible to achieve consensus on the average of the inputs at each node?

This problem is addressed in [5], [6] where it is called the *dynamic consensus* problem. In [5], the standard first order agreement protocol is extended by adding an input where the input to the agreement protocol at each node is the time derivative of the reference input at each node. The analysis of the convergence properties of their paper is carried out in the frequency domain, where it is shown that if all of the reference input signals are decaying to constant values, and if the consensus protocol is initialized correctly (to zero), then the nodes converge to the average of the reference inputs. However, if there is persistent frequency content to the reference input signals, or if the consensus protocol is not initialized correctly, then there will be a bounded steady state error. In [6] a second order agreement protocol that uses an integrator is introduced to remove the need to initialize the protocol to zero, and is shown to have additional robustness properties. The analysis in [6] is conducted in the time domain and applies to switching networks. However, implementation requires each node to receive information not only from its immediate neighbors, but also from its neighbor's neighbors.

This paper builds on the work of [5], [6] by proposing an agreement protocol that uses a integrator in a way that is similar to [6] but using only neighbor information, and analyzes the convergence and $\mathcal{L}_2$-gain properties of the protocol in both the time and frequency domains.

Other work that is closely related to this paper is the controllability and observability of graphs as discussed in [7, Chapter 10], where some of the nodes are inputs that can inject signals into the network, while the remaining nodes execute the standard agreement protocol. Explicit conditions for controllability and observability are developed and it is shown that the system states converge. The protocol developed in [7] is similar in structure to that proposed in [5].

We believe that dynamic or input consensus is an extremely useful notion for cooperative control. The input consensus approach has a much more system theoretic feel, where the effects of initial conditions are attenuated, and the response to inputs is emphasized. We envision numerous applications including the case where each agent is making measurements of a dynamic world and sharing those measurements across the network. The input-output perspective allows us to talk of consensus filters, as opposed to consensus protocols.

This paper is organized as follows. In Section II we

C. N. Taylor and R. W. Beard are on the Faculty of Electrical and Computer Engineering, and Jeffrey Humpherys is on the Faculty of Mathematics, all at Brigham Young University, Provo, Utah 84602 taylor@ee.byu.edu, beard@byu.edu, jeffh@math.byu.edu

describe the basic input consensus algorithm. In Section III we analyze the properties of the algorithm, in particular its convergence to consensus for constant and asymptotically constant input. In Section IV we explore the filtering aspects of our algorithm. In Section V we give simulation results, and our conclusions are in Section VI.

II. Input Consensus Algorithm

Let L be the Laplacian matrix associated with a simple undirected graph $G = (V, E)$ where V is the node set and E is the (symmetric) edge set, and suppose that $|V| = n$. The standard continuous-time consensus algorithm is given by

$$\dot{z} = -Lz. \tag{1}$$

As shown in [3], if the graph is connected, then $z(t) \to \mathbf{1}\bar{z}$, where $\bar{z} = \frac{1}{n}\sum_{i=1}^{n} z_i(0)$ is the average of the initial conditions.

Suppose that there is a possibly time-varying input u_i at each node and suppose that the objective is to converge to the average of the inputs. In this case we propose using the agreement protocol given by

$$\dot{z} = -\kappa L z(t) - \alpha z(t) + \beta u(t) - \gamma \int_0^t L z(\tau) d\tau, \tag{2}$$

where α, β, γ, and κ are positive, design constants. We first note that Eq. (2) is a decentralized algorithm since each node implements the scheme

$$\dot{z}_i = \beta u_i - \alpha z_i - \sum_{j \in \mathcal{N}_i} \left[\kappa (z_i - z_j) + \gamma \int_0^t (z_i(\tau) - z_j(\tau)) d\tau \right],$$

where $\mathcal{N}_i \subset V$ is the set of neighbors of node i. The first term in Eq. (2) is the standard consensus term and tends to push the values at each node toward each other. The second term in Eq. (2) is a damping term that is inserted to remove the effect of the initial conditions of z. The third term in Eq. (2) is the input at node i. If the inputs at each node are unique and persistent, then there will be a steady state difference between the nodes. The final term in Eq. (2) integrates the error and removes the bias.

By defining

$$x(t) := \begin{pmatrix} z(t) \\ \int_0^t Lz(\tau)\, d\tau \end{pmatrix} \quad \text{where} \quad x(0) = \begin{pmatrix} z(0) \\ 0 \end{pmatrix},$$

we see that Eq. (2) can be written in state space form as

$$\dot{x} = Ax + Bu \tag{3a}$$

$$z = Cx, \tag{3b}$$

where

$$A = \begin{pmatrix} -(\kappa L + \alpha I) & -\gamma I \\ L & 0 \end{pmatrix},\ B = \begin{pmatrix} \beta I \\ 0 \end{pmatrix},\ C = \begin{pmatrix} I & 0 \end{pmatrix}.$$

Taking the Laplace Transform, we separate into zero-state and zero-input components

$$Z(s) = C(sI - A)^{-1} B U(s) + C(sI - A)^{-1} x(0),$$

where the transfer function $T(s) = C(sI - A)^{-1}B$ is given by

$$T(s) = \beta s \left[s^2 I + (\kappa L + \alpha I)s + \gamma L \right]^{-1}. \tag{4}$$

Hence, we can further simplify $Z(s)$ to get

$$Z(s) = s \left[s^2 I + (\kappa L + \alpha I)s + \gamma L \right]^{-1} (\beta U(s) + z(0)). \tag{5}$$

III. Properties

In this section we will derive some of the properties of the input consensus protocol (2). We begin by relating the eigenvalues and eigenvectors of A to the eigenvalues and eigenvectors of L.

Since G is undirected, L is a symmetric matrix with orthonormal eigenbasis $Q = [q_1, \ldots, q_n]$ corresponding to real eigenvalues $\Lambda = \mathrm{diag}(\lambda_1, \cdots, \lambda_n)$, thus satisfying $L = Q\Lambda Q^T$. Without loss of generality, we use the convention $0 = \lambda_1 \le \lambda_2 \le \cdots \le \lambda_n$.

Proposition III.1 *Assume $\gamma \neq \kappa\alpha$. Let λ be an eigenvalue of L with unit (pairwise orthogonal) eigenvector q. Then the eigenvalues μ of A take the form*

$$\mu = -\frac{\alpha + \kappa\lambda}{2} \pm \sqrt{\left(\frac{\alpha + \kappa\lambda}{2}\right)^2 - \gamma\lambda} \tag{6}$$

and have corresponding left and right eigenvectors, ℓ and r, respectively, of the form

$$\ell = \begin{pmatrix} -\mu q^T & \gamma q^T \end{pmatrix} \tag{7a}$$

$$r = \begin{pmatrix} (\kappa\mu + \gamma) q \\ -(\mu + \alpha) q \end{pmatrix}. \tag{7b}$$

These form a complete set of eigenvectors, and thus A is diagonalizable. Moreover each eigenvalue μ has a corresponding eigenprojection given by

$$P_\mu = \frac{r\ell}{\ell r} = \frac{\begin{pmatrix} \mu(\kappa\mu + \gamma)\Pi & -\gamma(\kappa\mu + \gamma)\Pi \\ -\mu(\mu + \alpha)\Pi & \gamma(\mu + \alpha)\Pi \end{pmatrix}}{\mu(\kappa\mu + \gamma) + \gamma(\mu + \alpha)} \tag{8}$$

where Π is the rank-one (orthogonal) eigenprojection of λ given by

$$\Pi = qq^T. \tag{9}$$

Thus, we can write A in terms of its spectral decomposition

$$A = \sum_{\mu \in \sigma(A)} \mu P_\mu. \tag{10}$$

Proof: Given our space constraints, we relegate most of the details to the reader. We remark that the characteristic polynomial for A is given by

$$\mu^2 + (\alpha + \kappa\lambda)\mu + \lambda\gamma.$$

From this one can easily verify Eq. (6) and the relations $Ar = \mu r$, $\ell A = \mu \ell$, and $P_\mu A = A P_\mu = \mu P_\mu$. Note the row-vector convention used for the left eigenvectors. ■

Remark III.2 *In the case that $\gamma = \kappa\alpha$, the matrix A will have a Jordan block if $\kappa\Lambda - \alpha I$ is singular, or rather $\lambda_i =$*

α/κ *for some* λ_i*. The results that follow can be extended to include this case with a little more work.*

Corollary III.3 *If the graph* G *is connected then there is exactly one null eigenvalue of* A*, and the remaining eigenvalues are in the open left half plane. Moreover,*

$$P_0 = \begin{pmatrix} 0 & -\frac{\gamma}{\alpha}\Pi_0 \\ 0 & \Pi_0 \end{pmatrix},$$

where

$$\Pi_0 = \frac{1}{n}\mathbf{1}^T\mathbf{1}$$

is the rank-one eigenprojection for $\lambda = 0$*.*

Proof: Since G is connected, there is exactly one null eigenvalue of L and the remaining eigenvalues of L are positive. By inspection of Eq. (6), we see that all the eigenvalues of A have negative real part except for the positive root of the $\lambda = 0$ case. Setting $\mu = 0$ in Eq. (8), we determine P_0. Finally, since $q_1 \in \text{span}(\mathbf{1})$ we have that $q_1 = \frac{1}{\sqrt{n}}\mathbf{1}$. This gives us the form of Π_0 above. ∎

Lemma III.4 *If* G *is connected, then the transfer function* (4) *can be written as*

$$T(s) = \frac{\beta}{s+\alpha}\Pi_0 + \sum_{\lambda>0} \frac{\beta s}{s^2 + (\kappa\lambda + \alpha)s + \gamma\lambda}\Pi_\lambda. \quad (11)$$

Moreover in the limit, we have

$$\lim_{s\to 0} T(s) = \frac{\beta}{\alpha}\Pi_0.$$

Proof:

$$\begin{aligned} T(s) &= \beta s\left[s^2 I + (\kappa L + \alpha I)s + \gamma L\right]^{-1} \\ &= \beta s\left[Q^T(s^2 I + (\kappa\Lambda + \alpha I)s + \gamma\Lambda)Q\right]^{-1} \\ &= \beta s Q^T\left[s^2 I + (\kappa\Lambda + \alpha I)s + \gamma\Lambda\right]^{-1} Q \\ &= \sum_{i=1}^{n} \frac{\beta s}{s^2 + (\kappa\lambda_i + \alpha)s + \gamma\lambda_i} q_i q_i^T. \end{aligned}$$

For $\lambda_1 = 0$ we have

$$s^2 + (\kappa\lambda_1 + \alpha)s + \gamma\lambda_1 = s(s+\alpha).$$

Therefore $T(s)$ is given by Eq. (11). The $s \to 0$ limit follows trivially. ∎

Lemma III.5 *Assume* G *is connected. If* $x(t_0)$ *is the state of Eq.* (3) *at time* t_0*, then for* $t \geq t_0$*, we have*

$$x(t) = e^{A(t-t_0)}P_- x(t_0) + P_0 x(t_0) + \int_{t_0}^{t} e^{A(t-s)}P_- Bu(s)\,ds,$$

where $P_- = I - P_0$ *is the projection onto the stable eigenspace of* A *along the null space of* A*.*

Proof: From the Variation of Constants Formula, we have that

$$x(t) = e^{A(t-t_0)}x(t_0) + \int_{t_0}^{t} e^{A(t-s)}Bu(s)\,ds.$$

We project $x(t)$ into the stable and center eigenspaces via P_- and P_0 by defining $x_-(t) := P_- x(t)$ and $x_c(t) := P_0 x(t)$. Since $I = P_- + P_0$, we have $x(t) = (P_- + P_0)x(t) = x_-(t) + x_c(t)$. Thus,

$$\begin{aligned} x_-(t) &= e^{A(t-t_0)}P_- x(t_0) + \int_{t_0}^{t} e^{A(t-s)}P_- Bu(s)\,ds \\ x_c(t) &= e^{A(t-t_0)}P_0 x(t_0) + \int_{t_0}^{t} e^{A(t-s)}P_0 Bu(s)\,ds \end{aligned}$$

One can verify that $P_0 B = 0$ and $e^{A(t-t_0)}P_0 = P_0$. Thus $x_c(t) = P_0 x(t_0)$. Adding $x_c(t)$ and $x_-(t)$ back together gives the result. ∎

Remark III.6 *When* $t_0 = 0$*, we see that* $P_0 x(0) = 0$ *since the integrator term is set to zero in* $x(0)$*; indeed this holds as long as the initial integrator state averages to zero.*

Lemma III.7 *Let* μ_- *be the stable eigenvalue with largest real part, that is, closest to the imaginary axis; define* $\eta = -\operatorname{Re}\mu_-$*. Then there exists a constant* $C > 0$ *such that* $\|e^{At}P_-\| \leq Ce^{-\eta t}$*, for all* $t \geq 0$*. The value* η *is called the spectral gap of* A*.*

Proof: Using Eq. (10), we can write

$$e^{At} = \sum_{\mu\in\sigma(A)} e^{\mu t}P_\mu.$$

Combining with P_- yields

$$\|e^{At}P_-\| \leq \sum_{\text{Re}(\mu)<0} \|e^{\mu t}P_\mu\| \leq C|e^{\mu_- t}| = Ce^{-\eta t}.$$

∎

Theorem III.8 (Constant input) *If* G *is connected, then constant input produces consensus, more precisely, if* $u(t) = c$ *for some* $c \in \mathbb{R}^n$*, then*

$$\lim_{t\to\infty} z(t) = \frac{\beta}{\alpha}\mathbf{1}\bar{c}, \quad (12)$$

where $\bar{c} = \frac{1}{n}\sum c_i$*, that is,* $\bar{c}$ *is the average value of* c_i*.*

Proof: From Lemma III.5, we have that

$$x(t) = e^{At}P_- x_0 + \left(\int_0^t e^{A(t-s)}P_-\,ds\right)Bc. \quad (13)$$

We claim that $x(t)$ converges as $t \to \infty$, and thus so does $z(t)$. Since $e^{At}P_- \to 0$ as $t \to \infty$, it suffices to show that the integral

$$\int_0^\infty e^{A(t-s)}P_-\,ds$$

exists. Note, however that

$$\int_0^t \|e^{A(t-s)}P_-\|\,ds \leq C\int_0^t e^{-\eta(t-s)}\,ds = \frac{C}{\eta}(1 - e^{-\eta t}),$$

which converges to C/η as $t \to \infty$. Thus $x(\infty)$ and $z(\infty)$ exist. The result then follows from the Final Value Theorem. From Lemma III.4, we have

$$\lim_{t\to\infty} z(t) = \lim_{s\to 0} sZ(s) = \lim_{s\to 0} sT(s)\left(\frac{c}{s} + \frac{z(0)}{\beta}\right)$$
$$= \frac{\beta}{\alpha}\Pi_0 c = \frac{\beta}{\alpha}\mathbf{1}\left(\frac{1}{n}\mathbf{1}^T c\right) = \frac{\beta}{\alpha}\mathbf{1}\bar{c}.$$

■

Corollary III.9 *The DC gain of the system is* $\frac{\beta}{\alpha}\Pi_0$.

Theorem III.10 (Asymptotically constant input) *Assume G is connected. If the input $u(t)$ converges exponentially to a constant value $c \in \mathbb{R}^n$, that is, there exists $K, \nu > 0$ so that $\|u(t) - c\| \le Ke^{-\nu t}$ for all $t \ge 0$, then Eq. (12) holds.*

Proof: Let $\bar{x}(t)$ denote the state function for the constant input satisfying Eq. (13). The state function for the asymptotically constant input takes the form

$$x(t) = e^{At}P_- x_0 + \int_0^t e^{A(t-s)}P_- Bu(s)\, ds.$$

It suffices to show that their difference converges to zero. Note that

$$\begin{aligned}\|x(t) - \bar{x}(t)\| &= \left\|\int_0^t e^{A(t-s)}P_- B(u(s) - c)\, ds\right\| \\ &\le \int_0^t \|e^{A(t-s)}P_-\|\|B\|\|u(s) - c\|\, ds \\ &\le CK\|B\|\int_0^t e^{-\eta(t-s)}e^{-\nu s}\, ds \\ &= CK\|B\|\begin{cases} te^{-\eta t} & \eta = \nu \\ \dfrac{e^{-\nu t} - e^{-\eta t}}{\eta - \nu} & \eta \ne \nu \end{cases}\end{aligned}$$

In either case, we have exponential convergence as $t \to \infty$.

■

IV. Input Consensus Filtering

In the prior section, some of the fundamental properties of our proposed consensus algorithm have been introduced. However, we would like to shift from discussing a consensus algorithm to analyzing the proposed system as a filter. To understand the input consensus filter, we will discuss the frequency-response and $\mathcal{L}_2$ gain of the system.

Define $\bar{u}(t) = \frac{1}{n}\sum_{i=1}^n u_i(t)$ to be the time varying average of the inputs at each node, and define $e_u(t) = u(t) - \bar{u}\mathbf{1}$ to be the input disagreement function. Note from Eq. (11) that the output of the system is

$$\begin{aligned}Z(s) &= \Bigg[\frac{\beta}{s+\alpha}\Pi_0 \\ &\quad + \sum_{\lambda>0}\frac{\beta s}{s^2 + (\kappa\lambda + \alpha)s + \gamma\lambda}\Pi_\lambda\Bigg](\bar{u}(s)\mathbf{1} + e_u(s)) \\ &= \mathbf{1}\frac{\beta}{s+\alpha}\bar{u}(s) + \sum_{\lambda<0}\frac{\beta s}{s^2 + (\kappa\lambda + \alpha)s + \gamma\lambda}\Pi_\lambda e_u(s).\end{aligned}$$

Therefore, the output of the system can be decomposed into a low pass filtered version of the average of the inputs, plus a filtered version of the input disagreement function. It is clear that the $\mathcal{L}_2$-gain on the average of the inputs is β/α.

Define the state disagreement function as

$$e_x(s) := x(s) - \mathbf{1}\frac{\beta}{s+\alpha}\bar{u}(s).$$

Proposition IV.1 *The $\mathcal{L}_2$-gain from e_u to e_x is*

$$\frac{\beta}{\kappa\lambda_2 + \alpha},$$

where λ_2 is second smallest eigenvalue of L, and is called the algebraic connectivity of the graph G.

Proof: It is straight forward to show that the $\mathcal{L}_2$-gain of the transfer function

$$T_i(s) := \frac{\beta s}{s^2 + (\kappa\lambda_i + \alpha)s + \gamma\lambda_i}$$

is

$$\frac{\beta}{\kappa\lambda_i + \alpha}, \tag{14}$$

and that the gain-maximizing frequency is $\omega^* = \sqrt{\gamma\lambda_i}$. Since $q_i^T q_j = 0$ for all $i \ne j$, the $\mathcal{L}_2$-gain is found by maximizing (14) over i. ■

Note that the algebraic connectivity λ_2 describes the degree of connectedness of a graph [8], [9], that is, λ_2 increases as L becomes more connected. Therefore, increased connectivity will reduce the effect of input disagreement on the state.

If we define $T_1(s) = \beta/(s+\alpha)$ and $T_2(s) = \beta s/(s^2 + (\kappa\lambda_2 + \alpha)s + \gamma\lambda_2)$, then Figures 1–4 show the Bode plots of T_1 and T_2 as a function of λ_2, γ, α, and κ respectively, where $\beta = \alpha$. Figure 1 clearly shows that the $\mathcal{L}_2$-gain of T_2 decreases as a function of λ_2.

Figure 2 shows that while increasing γ does not increase the $\mathcal{L}_2$-gain of T_2, it does cause the peaking frequency to increase in a way that causes the gain of T_2 to exceed the gain of T_1 over a certain frequency range. The next result ensures that the gain on the average of the inputs is always greater than the gain on the input disagreement.

Proposition IV.2 *If the graph G is connected, then $|T_1(j\omega)| > |T_i(j\omega)|$, for each $i = 2, \ldots, n$ and for every ω iff $\gamma \le \kappa\alpha + \kappa^2\lambda_i/2$.*

Proof: Note that

$$\begin{aligned} & |T_1(j\omega)|^2 > |T_i(j\omega)|^2 \\ \iff\quad & \frac{\beta^2}{\alpha^2 + \omega^2} > \frac{\beta^2\omega^2}{(\gamma\lambda_i - \omega^2)^2 + (\kappa\lambda_i + \alpha)^2\omega^2} \\ \iff\quad & (\gamma\lambda_i - \omega^2)^2 + (\kappa\lambda_i + \alpha)^2\omega^2 > \omega^2(\alpha^2 + \omega^2) \\ \iff\quad & (2\kappa\alpha + \kappa^2\lambda_i - 2\gamma)\omega^2 + \gamma^2\lambda_i > 0 \end{aligned} \tag{15}$$

Since G is connected, then Eq. (15) holds for all $\omega \in \mathbb{R}$ iff $\gamma \le \kappa\alpha + \kappa^2\lambda_i/2$. ■

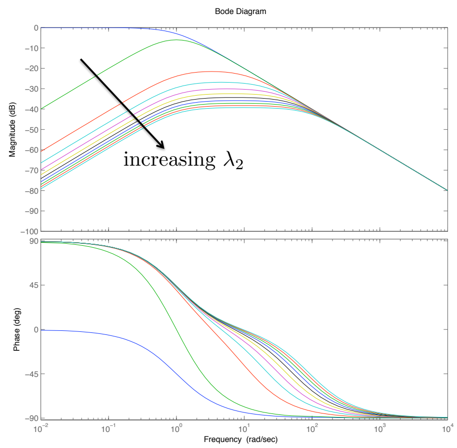

Fig. 1. Bode plot of T_1 and T_2 as a function of λ_2.

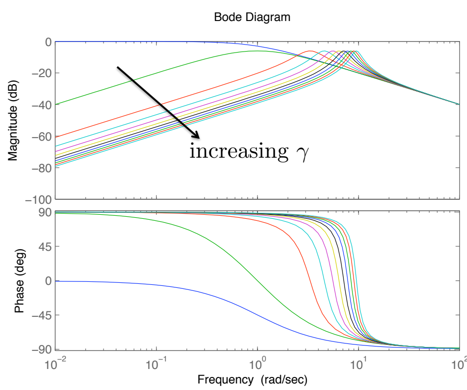

Fig. 2. Bode plot of T_1 and T_2 as a function of γ.

Figure 4 shows that increasing κ decreases the $\mathcal{L}_2$-gain of the input disagreement. We would like to be able to select κ to ensure a specified $\mathcal{L}_2$-gain, independent of the network topology.

Proposition IV.3 *If G is connected, a maximum input disagreement $\mathcal{L}_2$-gain of 10^{-a} can be achieved by setting $\kappa = \frac{10^a \beta}{2-2\cos(\pi/n)}$.*

Proof: From Equation (14), the $\mathcal{L}_2$-gain on the input disagreement is $\frac{\beta}{\kappa\lambda_2+\alpha}$. Noting that the algebraic connectivity λ_2 is smallest for a string topology, where, as shown in [10], $\lambda_2 = 2 - 2\cos(\pi/n)$, we obtain:

$$\frac{\beta}{\kappa\lambda_2 + \alpha} = 10^{-a}$$

$$\iff \quad \frac{\beta}{\kappa(2-2\cos(\pi/n)) + \alpha} \leq 10^{-a}$$

$$\iff \quad \kappa \geq \frac{10^a\beta - \alpha}{2 - 2\cos(\pi/n)}$$

Since $\alpha > 0$, we have that

$$\kappa \geq \frac{10^a \beta}{2 - 2\cos(\pi/n)}. \tag{16}$$

■

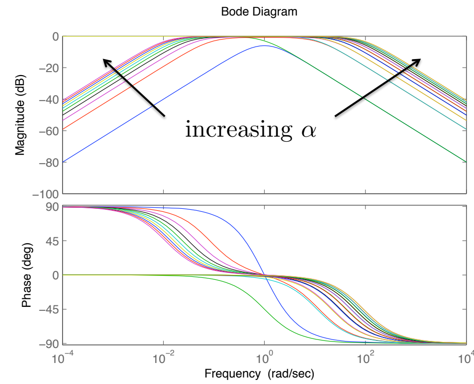

Fig. 3. Bode plot of T_1 and T_2 as a function of α.

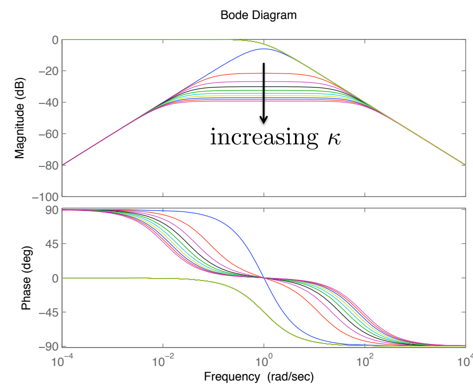

Fig. 4. Bode plot of T_1 and T_2 as a function of κ.

V. Simulation Results

To illustrate the properties described above, we simulated an 8-node system pictured in Figure 5. The parameters selected for the input consensus algorithm were $\alpha = 1$ and $\beta = \alpha$. κ was selected according to Eq. (16) to achieve an $\mathcal{L}_2$-gain on the disagreement input of $\frac{1}{5}$ (i.e., $a = 0.699$, $\kappa = 65.7$). The parameter γ was chosen as $\alpha\kappa/2$ to ensure it met the condition specified in Proposition IV.2. The inputs to the eight nodes were corrupted with noise (sine waves with random frequency and phase) as shown in Figure 6.

The outputs of the input consensus filter for the example system are shown in Figure 7. This graph demonstrates several important features of the input consensus filter, including:

- *Robustness to initial conditions:* To demonstrate the performance of the system with respect to initial conditions, the initial information states of the 8 nodes were set to $-4, -3, \ldots, 3$, respectively. Note that the outputs of the nodes quickly converge to the same value despite the widely varying input conditions.
- *Low-pass filtering of input average:* The outputs of the input consensus filter closely follow a low-pass filtered version of the input average. For comparison, note the input average denoted by the thick line in Figure 6.
- *Attenuation of input disagreement:* For comparison of the input and output disagreement, the axes in Figures 6

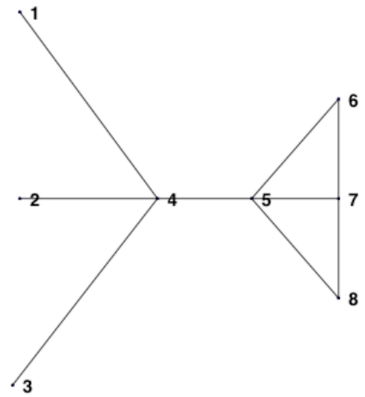

Fig. 5. Graph used to demonstrate the characteristics of input consensus.

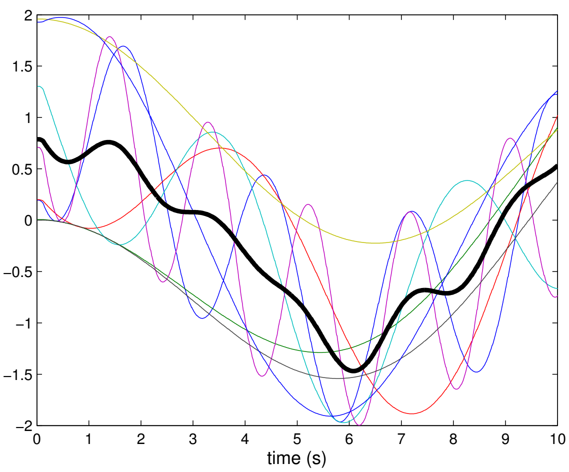

Fig. 6. The inputs to the eight node graph over time, with the true average denoted by the thick black line in the middle of the inputs.

and 7 are set to the same scale. Note that despite the large input disagreement, the output disagreement is barely visible after convergence from initial conditions.

In addition to illustrating these general properties of the input consensus filter, Figure 7 also demonstrates that the filter meets or exceeds the design constraints used to select the parameters for the filter. While the parameters were selected to have a maximum disagreement gain of $\frac{1}{5}$, the output disagreement is significantly more attenuated. This extra attenuation is due to two principle factors. First, κ was selected using the λ_2 for an 8-node string graph as described in Eq. (16) while the true λ_2 is approximately three times larger. Second, the $\mathcal{L}_2$ gain is computed using the q_2 portion of the input disagreement. For an n-element graph, however, this vector is orthogonal to $n-2$ other components of the input disagreement, each of which experience higher attenuation than the component associated with q_2. These two effects cause the realized attenuation to exceed the design constraint of $\frac{1}{5}$ for this system.

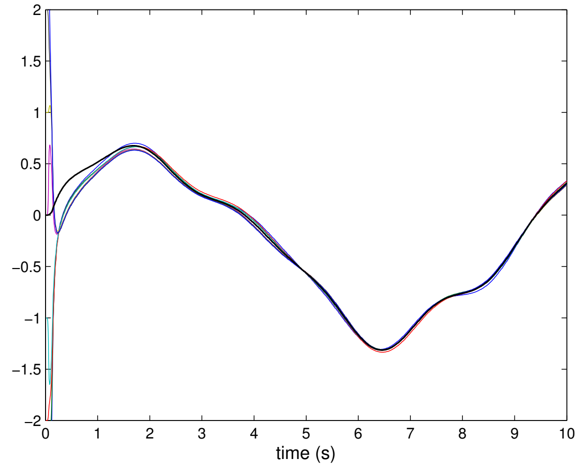

Fig. 7. The outputs of the input consensus algorithm on the eight node graph.

VI. Conclusions

This paper proposes a new network agreement protocol for systems with inputs at each node. The basic idea, similar to that proposed in [6], is to introduce an integrator that removes the steady-state error between nodes. We used both time and frequency domain techniques to analyze the properties of the protocol. The advantage of our consensus filtering approach is that initial conditions on the information state are forgotten, the average is over a time varying input at each node, and only neighbor information is required. In addition, the effect of the input disagreement can be significantly attenuated by judicious selection of the tuning parameters.

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