2011 American Control Conference
on O'Farrell Street, San Francisco, CA, USA
June 29 - July 01, 2011

Reconstruction of actuator fault for a class of nonlinear systems using sliding mode observer

Jian Zhang, Akshya Kumar Swain and Sing Kiong Nguang

Abstract— **The present paper proposes a sliding mode observer (SMO) for detection and isolation of actuator faults for a class of uncertain nonlinear systems (Lipschitz nonlinear systems). The sufficient condition of stability of the proposed SMO has been derived and expressed as Linear Matrix Inequalities (LMIs). The design parameters of the proposed SMO are determined by using LMI techniques. The constraint of the switching gain has been determined such that the proposed SMO satisfies the reachability condition. Then the equivalent output error injection is employed to reconstruct the actuator fault based on the structure of the uncertainty. The effectiveness of the proposed SMO in reconstructing actuator fault has been illustrated considering an example of a single-link flexible joint robot system and has been found to be satisfactory even with the presence of sensor noise.**

I. Introduction

Fault detection and isolation (FDI) has received considerable attention during the last two decades and the related literature can be found in [1]–[4] and the references there in. The approaches of FDI developed in the past can essentially be grouped into two main categories such as: signal-based FDI and model-based FDI. Signal-based FDI approaches employ statistical operations on the measurements or train some artificial network to extract the information regarding faults. The model-based FDI approaches generally compare the actual system's behavior with the predicted or estimated behavior based on its mathematical model [5]–[8]. The difference of these behaviors, referred to as residuals are very sensitive to any faults and therefore being used for fault detection. An alarm is triggered when the actual process behavior deviates from its expected behavior; more precisely, when the residuals exceed some predefined thresholds. However, due to the high dependence of the residual generation FDI to the corresponding mathematical models, any discrepancies between the actual process and its model can cause a misleading alarm, which often make the FDI ineffective.

One of the method to deal with the system uncertainty is to use the idea of sliding mode techniques. Sliding mode theory has been recognized as a promising robust control approach to confront uncertain or perturbed systems [9]–[11]. Several authors have reported sliding mode observer design methods. In [10], a discontinuous observer strategy has been used where the error between the estimated and measured outputs is forced to exhibit a sliding mode and measurement noise effects are reduced. Walcott and Zak used a Lyapunov-based approach to formulate an observer design where asymptotic stability can be obtained under certain assumptions in the presence of bounded nonlinearities or uncertainties [12]. Early work of applying the SMO for FDI was shown in [13] where a sliding mode observer approach is considered with the assumption that the states of the system are available. Hermans and Zarrop attempted to design an observer such that in the presence of a fault the sliding motion was destroyed [14]. However, the observer proposed in [15], which is similar to that of [12] can maintain the sliding mode even after the presence of faults. The actuator fault can therefore be reconstructed by the so-called equivalent output injection under certain conditions. Later it was extended to sensor fault reconstruction in [16]. Notice that the precise fault reconstruction shown in [15] and [16] was only for linear systems without uncertainties. When there are uncertainties, [17] provides a method to reconstruct faults for linear systems. It should be emphasized that the above work only consider linear systems. For nonlinear systems, the synthesis and computation of the switching gain of the SMO are much more difficult. In [18], an actuator fault detection and isolation scheme for a class of nonlinear systems with certain uncertainties was considered. [19] designed an adaptive method to update the sliding mode observer gain for counteracting uncertainty, so the upper bound of the uncertainty was not needed. In [20], a bank of observers were designed to isolate actuator faults for both linear and nonlinear systems. LMI techniques were used in [21] to design the SMO for a class of nonlinear systems with uncertainties.

In this paper, a different type of observer, based on principles of sliding mode has been proposed for reconstruction of actuator faults for nonlinear Lipschitz systems. The main contribution of the present work are the following: 1. The discontinuous switching component which induces a sliding motion and has been used for linear systems in [15] has been extended to nonlinear systems; 2. A new sufficient condition for the existence and stability of the SMO is derived and expressed in LMI form and 3. Actuator fault reconstruction has been carried out.

The paper is organized as follows: Section-II briefly describes the mathematical preliminaries required for designing SMO. Section-III describes the design procedure of the proposed SMO and derives the stability condition based on Lyapunov approach. The constraint of the switching gain is determined which satisfies the reachability condition. The procedure of reconstructing the actuator fault is presented in

This work was not supported by any organization

Jian Zhang, Akshya Kumar Swain and Sing Kiong Nguang are with Department of Electrical and Computer Engineering,University of Auckland,New Zealand jzha287@aucklanduni.ac.nz

section-IV. The results of simulation considering the example of single-link flexible joint robot system is shown in section-V with conclusions in section-VI.

II. Problem Formulation

Consider a nonlinear system described by

$$\begin{aligned} \dot{x}(t) &= Ax(t) + f(x,t) + Bu(t) + E\Delta\psi(x,t) + Df_a(t) \\ y(t) &= Cx(t) \end{aligned} \tag{1}$$

where $x \in \mathcal{R}^n$, $u \in \mathcal{R}^m$ and $y \in \mathcal{R}^p$ denote respectively the state variables, inputs and outputs ; $A \in \mathcal{R}^{n\times n}$, $B \in \mathcal{R}^{n\times m}$, $C \in \mathcal{R}^{p\times n}$, $D \in \mathcal{R}^{n\times q}$ and $E \in \mathcal{R}^{n\times r}(q \le p < n)$ are known constant matrices with C and D both being of full rank and the nonlinear term $f(x,t)$ is assumed to be known. The unknown nonlinear term $\Delta\psi(x,t)$ models lumped uncertainties and disturbances experienced by the system; the unknown function $f_a(t)$ represents actuator faults that is assumed to be bounded by a known function:

$$\|f_a(t)\| \le \rho(t) \tag{2}$$

Assumption 1. The matrix pair (A,C) is detectable.
It follows the assumption that there exists a matrix $L \in \mathcal{R}^{n\times p}$ such that $A - LC$ is stable, and thus for any $Q > 0$ the Lyapunov equation

$$(A - LC)^T P + P(A - LC) = -Q \tag{3}$$

has an unique solution $P > 0$ [18].

Assume that $P \in \mathcal{R}^{n\times n}, Q \in \mathcal{R}^{n\times n}$ are in the form:

$$P = \begin{bmatrix} P_1 & P_2 \\ P_2^T & P_3 \end{bmatrix}, Q = \begin{bmatrix} Q_1 & Q_2 \\ Q_2^T & Q_3 \end{bmatrix} \tag{4}$$

It follows from $P > 0$ and $Q > 0$ that $P_1 \in \mathcal{R}^{(n-p)\times(n-p)} > 0$, $P_3 \in \mathcal{R}^{p\times p} > 0$, $Q_1 \in \mathcal{R}^{(n-p)\times(n-p)} > 0$ and $Q_3 \in \mathcal{R}^{p\times p} > 0$.

Assumption 2. The nonlinear term $f(x,t)$ is assumed to be known and Lipschitz about x uniformly, i.e., $\forall x, \hat{x} \in \mathcal{X}$,

$$\|f(x,t) - f(\hat{x},t)\| \le \mathcal{L}_f \|x - \hat{x}\| \tag{5}$$

where $\mathcal{L}_f$ is the known Lipschitz constant.

Assumption 3. The function $\Delta\psi(x,t)$ representing the structured modeling uncertainty is unknown but bounded, and it satisfies

$$\|\Delta\psi(x,t)\| \le \xi(x,t) \tag{6}$$

where bounding function $\xi(x,t)$ is known and Lipschitz about x uniformly, i.e., $\|\xi(x,t) - \xi(\hat{x},t)\| \le \mathcal{L}_\xi$.

Assumption 4. There exist arbitrary matrices $F_1 \in \mathcal{R}^{r\times p}$ and $F_2 \in \mathcal{R}^{q\times p}$ such that:

$$\begin{bmatrix} E^T \\ D^T \end{bmatrix} P = \begin{bmatrix} F_1 \\ F_2 \end{bmatrix} C \tag{7}$$

Assumption 4 implies that $rank[D \quad E] \le p$. This property allows to decouple the dynamics of the observer error from the system uncertainty and fault.

Without loss of generality, it is assumed that the output matrix C has the form:

$$C = \begin{bmatrix} 0 & I_p \end{bmatrix} \tag{8}$$

However, if C does not have such a structure, there always exists a nonsingular transformation matrix T_c such that $CT_c^{-1} = [0 \quad I_p]$ since it has full row rank [22]. Assume that the triple (A,E,D) has the following structure:

$$A = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix}, E = \begin{bmatrix} E_1 \\ E_2 \end{bmatrix}, D = \begin{bmatrix} D_1 \\ D_2 \end{bmatrix} \tag{9}$$

where $A \in \mathcal{R}^{(n-p)\times(n-p)}$, $E_1 \in \mathcal{R}^{(n-p)\times r}$ and $D_1 \in \mathcal{R}^{(n-p)\times q}$. Then system (1) can be rewritten as:

$$\begin{aligned} \dot{x}_1 &= A_1x_1 + A_2x_2 + f_1(x,t) + B_1u(t) + E_1\Delta\psi + D_1f_a \\ \dot{x}_2 &= A_3x_1 + A_4x_2 + f_2(x,t) + B_2u(t) + E_2\Delta\psi + D_2f_a \\ y &= x_2 \end{aligned} \tag{10}$$

where $x = col(x_1, x_2)$ with $x_1 \in \mathcal{R}^{n-p}$, $f_1(x,t) \in \mathcal{R}^{n-p}$ is the first $n-p$ rows of $f(x,t)$ and $f_1(x,t) \in \mathcal{R}^{n-p}$ represents the remaining rows.

Lemma 1. If P and Q have been partitioned as in (4), then the following two conclusions are obvious :

1) $P_1^{-1}P_2E_2 + E_1 = 0$ and $P_1^{-1}P_2D_2 + D_1 = 0$ if (7) is satisfied;
2) The matrix $A_1 + P_1^{-1}P_2A_3$ is stable if Lyapunov equation (3) is satisfied.

Proof. See [18]

III. Sliding mode observer design

The design of sliding mode observer begins by introducing a new linear change of coordinates $z = Tx$ so as to impose specific structures on the uncertainty and fault distribution matrices, where

$$T := \begin{bmatrix} I_{n-p} & P_1^{-1}P_2 \\ 0 & I_p \end{bmatrix} \tag{11}$$

Using the conclusion (1) of Lemma 1, the system (10) can be transformed into the the new coordinate system z as :

$$\begin{aligned} \dot{z}_1 &= \tilde{A}_1z_1 + \tilde{A}_2z_2 + \tilde{B}_1u(t) \\ &\quad + f_1(T^{-1}z,t) + P_1^{-1}P_2f_2(T^{-1}z,t) \\ \dot{z}_2 &= \tilde{A}_3z_1 + \tilde{A}_4z_2 + \tilde{B}_2u(t) \\ &\quad + f_2(T^{-1}z,t) + E_2\Delta\psi(T^{-1}z,t) + D_2f_a \\ y &= z_2 \end{aligned} \tag{12}$$

where

$$TAT^{-1} = \begin{bmatrix} \tilde{A}_1 & \tilde{A}_2 \\ \tilde{A}_3 & \tilde{A}_4 \end{bmatrix}, TB = \begin{bmatrix} \tilde{B}_1 \\ \tilde{B}_2 \end{bmatrix}$$

$$TE = \begin{bmatrix} 0 \\ E_2 \end{bmatrix}, TD = \begin{bmatrix} 0 \\ D_2 \end{bmatrix}$$

$$\tilde{A}_1 = A_1 + P_1^{-1} P_2 A_3$$
$$\tilde{A}_2 = A_2 - A_1 P_1^{-1} P_2 + P_1^{-1} P_2 (A_4 - A_3 P_1^{-1} P_2)$$
$$\tilde{A}_3 = A_3$$
$$\tilde{A}_4 = A_4 - A_3 P_1^{-1} P_2$$

Based on the transformed system (12), the present study proposes the follwing sliding mode observer described as :

$$\begin{aligned}\dot{\hat{z}}_1 &= \tilde{A}_1 \hat{z}_1 + \tilde{A}_2 z_2 + \tilde{B}_1 u(t) \\ &\quad + f_1(T^{-1}\hat{z}, t) + P_1^{-1} P_2 f_2(T^{-1}\hat{z}, t) \\ \dot{\hat{z}}_2 &= \tilde{A}_3 \hat{z}_1 + \tilde{A}_4 \hat{z}_2 + \tilde{B}_2 u(t) \\ &\quad + f_2(T^{-1}\hat{z}, t) + (\tilde{A}_4 - A_0) e_y + \nu \\ \hat{y} &= \hat{z}_2\end{aligned} \tag{13}$$

where $\hat{z}_1$, $\hat{z}_2$ and $\hat{y}$ denote respectively the estimated states and output; $\hat{z} := col(\hat{z}_1, y)$; $A_0 \in \mathcal{R}^{p \times p}$ is a stable design matrix; $e_y = y - \hat{y}$ and the discontinuous vector ν is defined by

$$\nu = \begin{cases} k(t, y, u) \frac{P_0 e_y}{\|P_0 e_y\|} & \text{if } \|e_y\| > 0.0001 \\ 0 & \text{otherwise} \end{cases} \tag{14}$$

where $P_0 \in \mathcal{R}^{p \times p}$ is the symmetric definite Lyapunov matrix for A_0. 0.0001 in (14) is the threshold on the norm of e_y that can be chosen arbitrarily small. Note that the similar form of the discontinuous vector ν has been used in linear systems [15]. In the present study it has been extended to nonlinear systems. The positive scalar function $k(\cdot) : \mathcal{R}_+ \times \mathcal{R}^p \times \mathcal{R}^m \to \mathcal{R}_+$ satisfies:

$$\begin{aligned}k &\geq \|E_2\| \xi(T^{-1}\hat{z}, t) + \|E_2\| \mathcal{L}_\xi \|e_1\| \\ &\quad + \|D_2\| \rho + \mathcal{L}_{f_2} \|e_1\| + \eta\end{aligned} \tag{15}$$

where η is a positive constant.

If the state estimation errors are defined as $e_1 = z_1 - \hat{z}_1$ and $e_2 = z_2 - \hat{z}_2$, then the state estimation error dynamical system can be obtained as:

$$\dot{e}_1 = \tilde{A}_1 e_1 + \begin{bmatrix} I_{n-p} & P_1^{-1} P_2 \end{bmatrix} \left(f(T^{-1}z, t) - f(T^{-1}\hat{z}, t) \right) \tag{16}$$

$$\begin{aligned}\dot{e}_y &= \tilde{A}_3 e_1 + A_0 e_y + f_2(T^{-1}z, t) - f_2(T^{-1}\hat{z}, t) \\ &\quad + E_2 \Delta\psi(T^{-1}z, t) + D_2 f_a - \nu\end{aligned} \tag{17}$$

For error system (16)-(17), consider a sliding surface

$$\mathcal{S} = \{(e_1, e_y) | e_y = 0\} \tag{18}$$

The objective of the study is to derive the sufficient condition for the stability of the SMO of (13). This requires the analysis of the dynamical behavior of the state estimation error $e_1(t)$.

Lemma 2. Consider the system descried in (12) and the observer described in (13). Let a_0 and c_0 be positive constants such that $\|e^{\tilde{A}_1 t}\| \leq c_0 e^{-a_0 t}$. If $a_0 \geq c_0 \mathcal{L}_f \|[\ I_{n-p} \quad P_1^{-1} P_2\ ]\|$, where $\mathcal{L}_f$ is the Lipschitz constant given in (5), then the bound of the state estimation error $e_1(t)$ is independent of the system input and output and satisfies:

$$\|e_1(t)\| \leq c_0 \|e_1(0)\| exp\{(c_0 \mathcal{L}_f \|[\ I_{n-p} \quad P_1^{-1} P_2\ ]\| - a_0) t\} \tag{19}$$

Proof. From (16). we can obtain:

$$\begin{aligned}e_1(t) &= e^{\tilde{A}_1 t} e_1(0) + \int_0^t e^{\tilde{A}_1 (t-\tau)} \begin{bmatrix} I_{n-p} & P_1^{-1} P_2 \end{bmatrix} \\ &\quad \cdot \left(f\left(T^{-1}z, \tau\right) - f\left(T^{-1}\hat{z}, \tau\right) \right) d\tau\end{aligned} \tag{20}$$

From the fact that $\hat{z} := col(\hat{z}_1, y)$, we have:

$$\left\|T^{-1}z - T^{-1}\hat{z}\right\| = \left\|T^{-1} \begin{bmatrix} e_1 \\ 0 \end{bmatrix}\right\| = \|e_1\| \tag{21}$$

Therefore

$$\left\|f\left(T^{-1}z, t\right) - f\left(T^{-1}\hat{z}, t\right)\right\| \leq \mathcal{L}_f \|e_1\| \tag{22}$$

Using (5) and the triangle inequality, we can obtain that for any $t > 0$

$$\begin{aligned}\|e_1(t)\| &\leq c_0 e^{-a_0 t} \|e_1(0)\| + c_0 e^{-a_0 t} L_f \cdot \\ &\quad \left\| \begin{bmatrix} I_{n-p} & P_1^{-1} P_2 \end{bmatrix} \right\| \int_0^t e^{a_0 \tau} \|e_1(\tau)\| d\tau\end{aligned} \tag{23}$$

Applying Gronwall-Bellman inequality [23] to (23) with $\alpha = c_0 \|e_1(0)\|$, $u(t) = e^{a_0 t} \|e_1(t)\|$ and $\beta(t) = c_0 \mathcal{L}_f \left\| \begin{bmatrix} I_{n-p} & P_1^{-1} P_2 \end{bmatrix} \right\|$, we can obtain

$$e^{a_0 t} \|e_1(t)\| \leq c_0 \|e_1(0)\| \exp\left\{ c_0 L_f \left\| \begin{bmatrix} I & P_1^{-1} P_2 \end{bmatrix} \right\| t \right\} \tag{24}$$

Multiplying both sides of the above inequality by $e^{-a_0 t}$, (19) can be obtained.

Remark 1. a_0 and c_0 are positive constants chosen to ensure that $\|e^{\tilde{A}_1 t}\| \leq c_0 e^{-a_0 t}$. According to conclusion (2) of Lemma 1, $\tilde{A}_1$ is stable. Therefore such constants a_0 and c_0 always exist [23]. Since e_1 is bounded, we can assume that

$$\|e_1\| \leq \gamma \tag{25}$$

Proposition 1. Under the Assumption 1-4, the system (16)-(17) is asymptotically stable if there exist matrices $P_0 > 0$, $P_1 > 0$, A_0, P_2 and a positive scalar α satisfying $M :=$

$$\begin{bmatrix} \tilde{A}_1^T P_1 + P_1 \tilde{A}_1 + \frac{1}{\alpha} \bar{P}_1 \bar{P}_1^T + \alpha (L_f)^2 I & A_3^T P_0 \\ P_0 A_3 & A_0^T P_0 + P_0 A_0 \end{bmatrix} < 0 \tag{26}$$

where $\tilde{A}_1 = A_1 + P_1^{-1} P_2 A_3$, $\bar{P}_1 = P_1 \begin{bmatrix} I_{n-p} & P_1^{-1} P_2 \end{bmatrix}$.

Proof. Assume $V_1(e_1) = e_1^T P_1 e_1$ and $V_2(e_y) = e_y^T P_0 e_y$. Consider $V(e_1, e_y) = V_1(e_1) + V_2(e_y)$ as a Lyapunov candidate. The time derivative of V_1, V_2 along the trajectories of system (16)-(17) can be shown to be equal to:

$$\begin{aligned}\dot{V}_1 &= \dot{e}_1^T P_1 e_1 + e_1^T P_1 \dot{e}_1 \\ &= e_1^T(\tilde{A}_1^T P_1 + P_1 \tilde{A}_1)e_1 + 2e_1^T \bar{P}_1 \left(f(T^{-1}z,t) - f(T^{-1}\hat{z},t)\right)\end{aligned} \tag{27}$$

Since the inequality $2X^TY \leq \frac{1}{\alpha}X^TX + \alpha Y^TY$ is true for any scalar $\alpha > 0$, then

$$\begin{aligned}\dot{V}_1 &\leq e_1^T(\tilde{A}_1^T P_1 + P_1 \tilde{A}_1)e_1 + \frac{1}{\alpha}e_1^T \bar{P}_1 \bar{P}_1^T e_1 \\ &\quad + \alpha\left(f(T^{-1}z,t) - f(T^{-1}\hat{z},t)\right)^T \\ &\quad \cdot \left(f(T^{-1}z,t) - f(T^{-1}\hat{z},t)\right) \\ &\leq e_1^T(\tilde{A}_1^T P_1 + P_1 \tilde{A}_1)e_1 + \frac{1}{\alpha}e_1^T \bar{P}_1 \bar{P}_1^T e_1 \\ &\quad + \alpha(\mathcal{L}_f)^2\|e_1\|^2 \\ &= e_1^T\left(\tilde{A}_1^T P_1 + P_1 \tilde{A}_1 + \frac{1}{\alpha}\bar{P}_1 \bar{P}_1^T + \alpha(\mathcal{L}_f)^2 I_{n-p}\right)e_1\end{aligned} \tag{28}$$

$$\begin{aligned}\dot{V}_2 &= e_1^T A_3^T P_0 e_y + e_y^T P_0 A_3 e_1 + e_y^T(A_0^T P_0 + P_0 A_0)e_y \\ &\quad + 2e_y^T P_0\left(f_2(T^{-1}z,t) - f_2(T^{-1}\hat{z},t)\right) \\ &\quad + 2e_y^T P_0 E_2 \Delta\psi(T^{-1}z,t) + 2e_y^T P_0 D_2 f_a - 2e_y^T P_0 \nu\end{aligned} \tag{29}$$

From the Cauchy-Schwartz inequality and (15), we can impose a bound on the last four terms of (29).

$$\begin{aligned}&2e_y^T P_0\left(f_2(T^{-1}z,t) - f_2(T^{-1}\hat{z},t)\right) \\ &\quad + 2e_y^T P_0 E_2 \Delta\psi(T^{-1}z,t) + 2e_y^T P_0 D_2 f_a - 2e_y^T P_0 \nu \\ &\leq 2\|P_0 e_y\|\left(\mathcal{L}_{f_2}\|e_1\| + \|E_2\|\xi(T^{-1}z,t) + \|D_2\|\rho - k\right) \\ &\leq -2\eta\|P_0 e_y\|\end{aligned} \tag{30}$$

Therefore we can derive that

$$\begin{aligned}\dot{V} &= \dot{V}_1 + \dot{V}_2 \\ &\leq e_1^T\left(\tilde{A}_1^T P_1 + P_1 \tilde{A}_1 + \frac{1}{\alpha}\bar{P}_1 \bar{P}_1^T + \alpha(\mathcal{L}_f)^2 I_{n-p}\right)e_1 \\ &\quad + e_1^T A_3^T P_0 e_y + e_y^T P_0 A_3 e_1 + e_y^T(A_0^T P_0 + P_0 A_0)e_y \\ &\quad - 2k\|P_0 e_y\| \\ &\leq e_1^T\left(\tilde{A}_1^T P_1 + P_1 \tilde{A}_1 + \frac{1}{\alpha}\bar{P}_1 \bar{P}_1^T + \alpha(\mathcal{L}_f)^2 I_{n-p}\right)e_1 \\ &\quad + e_1^T A_3^T P_0 e_y + e_y^T P_0 A_3 e_1 + e_y^T(A_0^T P_0 + P_0 A_0)e_y \\ &= e^T M e \\ &< 0\end{aligned} \tag{31}$$

It follows that $e \to 0$ exponentially, namely, the error dynamical system (16)-(17) is asymptotically stable.

Remark 2. The inequality (26) can be transformed into the following LMI problem: find matrices P_0, P_1, P_2, Y and a scalar α such that:

$$\begin{bmatrix} \Theta + \alpha(\mathcal{L}_f)^2 I_{n-p} & P_1 & P_2 & A_3^T P_0 \\ P_1^T & -\alpha I_{n-p} & 0 & 0 \\ P_2^T & 0 & -\alpha I_p & 0 \\ P_0 A_3 & 0 & 0 & Y + Y^T \end{bmatrix} < 0 \tag{32}$$

where $\Theta := A_1^T P_1 + P_1 A_1 + A_3^T P_2^T + P_2 A_3$ and $Y = P_0 A_0$. If $\mathcal{L}_f$ is known, then the problem of finding P_0, P_1, P_2, Y to satisfy (32) is a standard LMI feasibility problem.

After getting the sufficient condition for the error system (16)-(17) to be asymptotically stable, the next objective is to determine the scalar gain function $k(\cdot)$ in (14) such that the system can be driven to the sliding surface $\mathcal{S}$ in finite time and a sliding motion can be maintained.

Proposition 2. Under the Assumption 1-4, the error system (16)-(17) is driven to the sliding surface (18) in finite time if the gain $k(\cdot)$ is chosen to satisfy

$$\begin{aligned}k &\geq (\|A_3\| + \|E_2\|\mathcal{L}_\xi + \mathcal{L}_{f_2})\gamma + \|D_2\|\rho \\ &\quad + \|E_2\|\xi(T^{-1}\hat{z},t) + \gamma_0\end{aligned} \tag{33}$$

where $\|e_1\| \leq \gamma$ (25), γ_0 is a positive scalar.

Proof. Consider a Lyapunov candidate function $V_2(e_y) = e_y^T P_0 e_y$.

$$\begin{aligned}\dot{V}_2 &= e_1^T A_3^T P_0 e_y + e_y^T P_0 A_3 e_1 + e_y^T(A_0^T P_0 + P_0 A_0)e_y \\ &\quad + 2e_y^T P_0\left(f_2(T^{-1}z,t) - f_2(T^{-1}\hat{z},t)\right) \\ &\quad + 2e_y^T P_0 E_2 \Delta\psi(T^{-1}z,t) + 2e_y^T P_0 D_2 f_a - 2e_y^T P_0 \nu \\ &\leq e_1^T A_3^T P_0 e_y + e_y^T P_0 A_3 e_1 \\ &\quad + 2e_y^T P_0\left(f_2(T^{-1}z,t) - f_2(T^{-1}\hat{z},t)\right) \\ &\quad + 2e_y^T P_0 E_2 \Delta\psi(T^{-1}z,t) + 2e_y^T P_0 D_2 f_a - 2e_y^T P_0 \nu \\ &\leq 2\|P_0 e_y\|(\|A_3\|\|e_1\| + \mathcal{L}_{f_2}\|e_1\| + \|E_2\|\xi(T^{-1}z,t) \\ &\quad + \|D_2\|\rho - k)\end{aligned} \tag{34}$$

From (33) and (34) it follows that

$$\dot{V}_2 \leq -2\gamma_0\|P_0 e_y\| \leq -2\gamma_0\sqrt{\lambda_{min}(P_0)}V_2^{1/2} \tag{35}$$

where $\lambda_{min}(P_0)$ is the smallest eigenvalue of P_0. This shows that the reachability condition [10] is satisfied. As a consequence, an ideal sliding motion will take place on the surface $\mathcal{S}$ and after some finite time t_s,

$$e_1 = \dot{e}_1 = 0, \quad \forall t > t_s \tag{36}$$

IV. Reconstruction of Actuator Fault

Given a sliding mode observer which satisfies (26) and (33), the task in this section is to reconstruct the actuator fault using the so-called equivalent output injection [15].

Assumption 5. There exists a nonsingular matrix $G \in \mathcal{R}^{p\times p}$ such that

$$G\begin{bmatrix} E_2 & D_2 \end{bmatrix} = \begin{bmatrix} H_1 & H_2 \\ 0 & H_3 \end{bmatrix} \tag{37}$$

where $H_1 \in \mathcal{R}^{(p-q)\times r}$ and $H_3 \in \mathcal{R}^{q\times q}$ is nonsingular.

Remark 3. The matrix structure G in Assumption 5 guarantees that the actuator can be distinguished from the system uncertainty, which makes actuator fault reconstruction possible.

Multiplying both sides of (17) by G, yields

$$G\dot{e}_y = GA_3e_1 + GA_0e_y + G\left(f_2(T^{-1}z,t) - f_2(T^{-1}\hat{z},t)\right) + \begin{bmatrix} H_1 & H_2 \\ 0 & H_3 \end{bmatrix}\begin{bmatrix} \Delta\psi(T^{-1}z,t) \\ f_a \end{bmatrix} - G\nu \tag{38}$$

After reaching the sliding surface, the sliding motion will be maintained thereafter, i.e., $e_y = 0$ and $\dot{e}_y = 0$, therefore (38) becomes

$$0 = GA_3e_1 + G\left(f_2(T^{-1}z,t) - f_2(T^{-1}\hat{z},t)\right) + \begin{bmatrix} H_1 & H_2 \\ 0 & H_3 \end{bmatrix}\begin{bmatrix} \Delta\psi(T^{-1}z,t) \\ f_a \end{bmatrix} - G\nu_{eq} \tag{39}$$

where ν_{eq} is the equivalent output error injection signal representing the average behavior of the discontinuous function ν. Since $\lim_{t\to\infty} e_1 = 0$, $f_2(T^{-1}z,t)-f_2(T^{-1}\hat{z},t)$ will also tends to zero. This implies (from (39)) that

$$f_a \to H_3^{-1}G_2v_{eq} \qquad \text{as } t \to \infty \tag{40}$$

where G_2 represents the last q rows of G.

The equivalent output error injection signal can be approximated as:

$$\nu_{eq} = k(t,y,u)\frac{P_0e_y}{\|P_0e_y\| + \delta} \tag{41}$$

where δ is a small positive scalar to reduce the chattering effect. It can be shown that ν_{eq} can be approximated to any degree of accuracy by (41) for a small enough choice of δ. The actuator fault can accordingly be approximated by

$$\hat{f}_a \approx k(t,y,u)H_3^{-1}G_2\frac{P_0e_y}{\|P_0e_y\| + \delta} \tag{42}$$

V. Simulation Results

The example of a single-link flexible joint robot system has been considered to demonstrate the effectiveness of the proposed SMO in reconstructing actuator faults. A dynamical model for the robot can be described by ([21], [24])

$$\begin{aligned}
\dot{\theta}_1 &= \omega_1 \\
\dot{\omega}_1 &= \frac{1}{J_1}(k_1(\theta_2-\theta_1) + k2(\theta_2-\theta_1)^3) - \frac{B_v}{J_1}\omega_1 + \frac{K_\tau}{J_1}\omega_1 + \frac{K_\tau}{J_1}u \\
\dot{\theta}_2 &= \omega_2 \\
\dot{\omega}_2 &= \frac{1}{J_2}(k_1(\theta_2-\theta_1) + k2(\theta_2-\theta_1)^3) - \frac{mgh}{J_2}sin\theta_2 \\
&\quad + \psi(\theta_1,\omega_1,\theta_2,\omega_2,t)
\end{aligned} \tag{43}$$

where θ_1 and ω_1 are the motor position and velocity, respectively; θ_2 and ω_2 are the link position and velocity; J_1 is the inertia of the DC motor, J_2 is the inertia of the link, $2h$ is the length of the link while m_l represents its mass, B_v is the viscous friction, k_1 and k_2 are positive constants and K_τ is the amplifier gain. It is assumed that the motor position, motor velocity and the sum of link velocity and link position can be measured. The values of the parameters used in this simulation are: $J_1 = 3.7 \times 10^{-3}kg \cdot m^2$, $J_2 = 9.3 \times 10^{-3}kg \cdot m^2$, $h = 1.5 \times 10^{-1}m$, $m = 0.21kg$, $B_v = 4.6 \times 10^{-2}m$, $k_1 = k_2 = 1.8 \times 10^{-1}Nm/rad$ and $K_\tau = 8 \times 10^{-2}Nm/V$.

To illustrate the effectiveness of the prosed SMO and to reconstruct the actuator faults, a nonlinear uncertainty is added to the system which satisfies the bound $\|\psi\| \leq 0.023(sin\theta_2)^2$. For the illustration purpose, a linear state feedback controller $u = [-14.1 - 25.6 - 16.2 - 12.1]z$ has been utilized to stabilize the system. Suppose that a fault f_a occurs in the input channel, where $f_a = 0.05t$ $(t < 2)$ and $f_a = 0.5sin(2\pi t)$ $(2 \leq t)$. Therefore the fault distribution matrix D will be equal to the input matrix. Reorder the system variables and let $x = col(x_1,x_2,x_3,x_4) := col(\theta_2,\omega_2,\theta_1,\omega_1)$, then the output distribution matrix C becomes:

$$C = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Notice that C does not have the form in (8). A nonsingular transformation matrix $T_c = [1\,0\,0\,0; 1\,1\,0\,0; 0\,0\,1\,0; 0\,0\,0\,1]$ is therefore introduced to obtain $CT_c^{-1} = [0 \quad I_p]$ and accordingly

$$A = \begin{bmatrix} -1.0000 & 1.0000 & 0 & 0 \\ -20.3548 & 1.0000 & 19.3548 & 0 \\ 0 & 0 & 0 & 1.0000 \\ 48.6486 & 0 & -48.6486 & -12.4324 \end{bmatrix}$$

$$f(x) = \begin{bmatrix} 0 \\ -19.3548(x_1-x_3)^3 - 33.1935sinx_1 \\ 0 \\ 48.6486(x_1-x_3)^3 \end{bmatrix}$$

$$C = \begin{bmatrix} 0 & I_3 \end{bmatrix}, E = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, D = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 21.6216 \end{bmatrix}$$

Imposing the stability constraint described in (26), and formulating the problem in a LMI framework gives the following solutions:

$$\begin{aligned}
\alpha &= 4.7220 \\
P_1 &= 3.2719 \\
P_2 &= \begin{bmatrix} -0.0018 & 0 & -0.0001 \end{bmatrix}
\end{aligned}$$

$$P_0 = \begin{bmatrix} 0.1210 & 0 & 0 \\ 0 & 4.6859 & 0 \\ 0 & 0 & 0.0519 \end{bmatrix}$$

$$A_0 = \begin{bmatrix} -28.4926 & 0 & 0 \\ 0 & -0.5000 & 0 \\ 0 & 0 & -63.2322 \end{bmatrix}$$

It is verified that the conclusion of Lemma 1 and Proposition 1 are all satisfied. The transformation matrix T is determined and the system is transformed into a new coordinate z and all the parameters of the proposed SMO (13) are obtained . The simulation results are shown in Fig-1, Fig-2 and Fig-

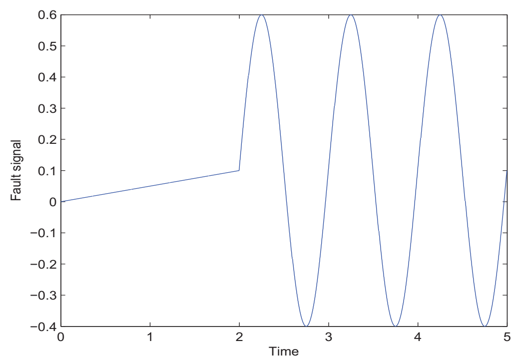

Fig. 1. Fault signal

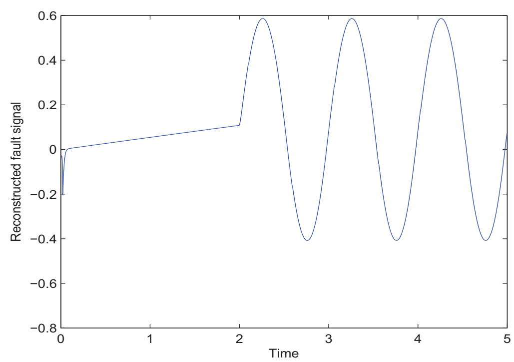

Fig. 2. Reconstructed fault signal

3. From the figures, it can be seen that the fault signal can accurately be reconstructed using the proposed sliding mode observer even in the presence of sensor noise.

VI. Conclusions

A new scheme for robust fault estimation for a class of nonlinear Lipschitz system using a sliding mode observer has been proposed in this work. The stability and reachability condition of the proposed sliding mode observer has been studied. The design parameters of the observer are obtained by LMI techniques. Under certain conditions, the actuator fault can be reconstructed to any degree of accuracy even in the presence of nonlinear uncertainties. The effectiveness of the proposed SMO has been demonstrated considering the example of a single-link flexible joint robot system.

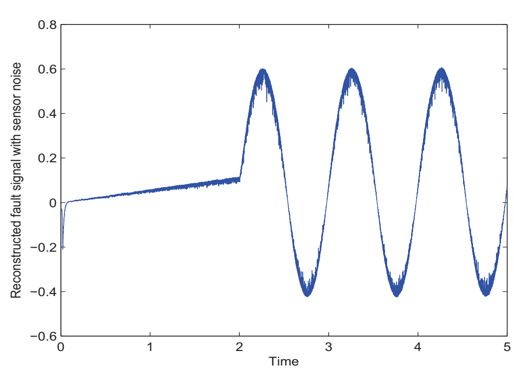

Fig. 3. Reconstructed fault signal with sensor noise of 30dB

References

[1] P. M. Frank, “Fault diagnosis in dynamic systems using analytical and knowledge-based redundancy-a survey and some new results,” *Automatica*, vol. 26, pp. 459–474, 1990.

[2] P. M. F. R. J. Patton and R. N. Clark, *Fault Diagnosis in Dynamic Systems: Theory and Applications*. Upper Saddle River, NJ: Prentice-Hall, 1989.

[3] R. Isermann, “Process fault detection on modeling and estimation methods-a survey,” *Automatica*, vol. 20, pp. 387–404, 1984.

[4] ——, *Fault diagnosis of technical process-applications*. Springer, Heidelberg, 2006.

[5] J. Chen and R. Patton, *Robust model-based fault diagnosis for dynamic systems*. Kluwer, Boston, 1999.

[6] S. Simani, C. Fantuzzi, and R. Patton, *Model-based fault diagnosis in dynamic systems using identification techniques*. Springer, London, 2003.

[7] T. Floquet, J. P. Barbot, W. Perruquetti, and M. Djemai, “On the robust fault detection via sliding mode disturbance observer,” *International Journal of Control*, vol. 77, pp. 622–629, 2004.

[8] D. N. Shields, “Observer-based residual generation for fault diagnosis for non-affine non-linear polynomial systems,” *International Journal of Control*, vol. 78, pp. 363–384, 2005.

[9] V. I. Utkin, *Sliding Modes and Their Application in Variable Structure Systems*. Mir, Moscow, 1978.

[10] ——, *Sliding Modes in Control Optimization*. Berlin: SpringerVerlag, 1992.

[11] C. Edwards and S. K. Spurgeon, *Sliding Mode Contol: Theory and Applications*. Taylor and Francis, 1998.

[12] B. L. Walcott and S. H. Zak, “State observation of nonlinear uncertain dynamical systems,” *IEEE Transaction on Automatic Control*, vol. 32, pp. 166–170, 1987.

[13] R. Sreedhar, B. Fernandez, and G. Masada, “Robust fault detection in nonlinear systems using sliding mode observers,” in *Proc. IEEE Conference on Control Application*, 1993.

[14] F. Hermans and M. Zarrop, “Sliding mode observers for robust sensor monitoring,” in *Proceedings of the 13th IFAC World Congress*, 1996.

[15] C. Edwards, S. K. Spurgeon, and R. J. Patton, “Sliding mode observers for fault detection and isolation,” *Automatica*, vol. 36, pp. 541–553, 2000.

[16] C. P. Tan and C. Edwards, “Sliding mode observers for detection and reconstruction of sensor faults,” *Automatica*, vol. 38, pp. 1815–1821, 2002.

[17] S. Dhahri, F. B. Hmida, A. Sellami, and M. Gossa, “Actuator fault reconstruction for linear uncertain systems using sliding mode observer,” in *International Conference on Signals, Circuits and Systems*, 2009.

[18] X. G. Yan and C. Edwards, “Robust sliding mode observer-based actuator fault detection and isolation for a class of nonlinear systems,” *International Journal of Systems Science*, vol. 39, pp. 349–359, 2008.

[19] W. Chen and M. Saif, “Novel sliding mode observers for a class of uncertain systems,” in *2006 American Control Conference*, Minneapolis, Minnesota, 2006, pp. 2622–2627.

[20] ——, “An actuator fault isolation strategy for linear and nonlinear systems,” in *2005 American Control Conference*, Potland, OR, 2005, pp. 3321–3326.

[21] X. G. Yan and C. Edwards, “Nonlinear robust fault reconstruction and estimation using a sliding mode observer,” *Automatica*, vol. 43, pp. 1605–1614, 2007.

[22] C. Edwards and S. K. Spurgeon, “On the development of discontinuous observers,” *International journal of control*, vol. 59, pp. 1211–1229, 1994.

[23] P. A. Ioannou and J. Sun, *Robust adaptive control*. Englewood Cliffs. NJ: Prentice Hall, 1996.

[24] X. Fan and M. Arcak, “Observer design for systems with multivariable monotone nonlinearities,” *Systems & Conrol Letters*, vol. 50, pp. 319–330, 2003.