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Distributed Kalman Filtering Using the Internal Model Average Consensus Estimator

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Abstract— **We apply the internal model average consensus estimator in [1] to distributed Kalman filtering. The resulting distributed Kalman filter and the embedded average consensus estimator update at the same frequency. We show that if the internal model average consensus estimator is stable, the estimation error of the distributed Kalman filter is zero mean in steady state and has bounded covariance even when the dynamical system to be estimated is neutrally stable or unstable.**

I. Introduction

Recent years have witnessed increasing research efforts in distributed estimation, such as distributed Kalman filtering [2], distributed Kriged Kalman filtering [3], distributed H_∞ filtering [4], etc. In this paper, we consider the distributed Kalman filtering problem. One common approach to distributed Kalman filtering is to represent the centralized Kalman filter in the information form, where the correction step of the filter simply sums information from individual agents. To maintain an estimate of the dynamical system, each agent must calculate the necessary sums. When the communication is all-to-all or there exists a summing agent [5], this calculation is simple. If the network has a particular structure or is known to be acyclic with a known local topology, the necessary sum can be computed in a decentralized way. For example, the use of a channel filter in [6] allows the decentralized calculation of the necessary sum if the network is acyclic; if not, conservative channel filters could be implemented or a dynamic spanning tree may be found. However, in this paper, we assume that the communication structure does not have any specific structure, except that it is connected and undirected. To achieve decentralized data fusion under this assumption, we employ the decentralized average consensus estimator to estimate the sums needed to implement the Kalman filter.

When the average consensus estimator is updated sufficiently fast, the estimates of the necessary sums converge to their true values before the next Kalman filter update [2], [3]. In this case, a time-scale separation exists, leading to the decoupling of the average consensus estimator and the distributed Kalman filter. Although this approach can recover the optimality of the centralized Kalman filter at each agent, it requires high communication bandwidth and energy consumption for each agent, which may not be feasible for some applications. Reference [7] showed that the behavior of the distributed Kalman filter varies smoothly from a centralized Kalman filter to a local Kalman filter as the average consensus update rate goes from sufficiently large to zero.

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Instead of the approaches in [2], [3], [7], we pursue a distributed implementation of the centralized Kalman filter with a *single time-scale*, that is, the average consensus estimator and the Kalman filter are updated at the same frequency. In this scenario, [8] proves that if the dynamical system is stable, the expected values of the distributed Kalman filter estimates converge to the centralized state estimate. Reference [9] demonstrated in simulations that with the PI average consensus estimator in [10], the distributed Kalman filter is able to approximately track slowly-varying systems.

The previous work [8], [9] used average consensus estimators of different flavors in conjunction with local Kalman filtering to achieve a decentralized approximation to centralized Kalman filtering at each agent. If the agent already has information on the model dynamics for its Kalman filter, however, it should also use this information in its average consensus estimator. Failure to do so may result in unbounded covariance of the estimation error for unstable or neutrally stable systems.

In this paper, we take advantage of our newly developed class of internal model average consensus estimators [1] to match the dynamics of each agent's consensus estimator to the dynamics of its Kalman filter. This ensures that the estimation error between each agent's estimate of the state and the true state of the dynamical system has bounded covariance even when the dynamical system is neutrally stable or unstable. Our assumption on the dynamical system to be estimated is that it is observable only in a centralized sense, that is, the state of the dynamical system may not be observable to individual agents but is observable when the measurements from the agents are fused. Such an observability condition is weaker than [11]–[13], where the states of the dynamical system are observable to each agent. As we will demonstrate, the embedded average consensus estimation allows each agent to estimate the entire state of the dynamical system.

The subsequent sections are organized as follows. We review the main results in [1] in Section II. The distributed Kalman filter is studied in Section III: Section III-A discusses the centralized Kalman filter and its distributed implementation while Section III-B studies the convergence properties of the distributed implementation in steady state. Simulation results of the distributed Kalman filter are presented in Section IV. Conclusions and future work are discussed in Section V.

Notation: The vectors 1_N and 0_N represent the N by 1 vectors with all entries 1 and 0, respectively. The space consisting of all N by M real matrices is given by $\mathbb{R}^{N\times M}$ while the $p\times p$ identity matrix is denoted by I_p. The notation $\mathrm{diag}\{k_1, k_2, \cdots, k_n\}$ denotes the n by n diagonal matrix with

k_i's on the diagonal. $A \otimes B$ denotes the Kronecker product of matrices A and B. The Z-transform of a time signal $\nu(t)$ is represented by $\nu(z)$. We denote by $n_h(z)$ and $d_h(z)$ the numerator and denominator polynomials of a discrete time transfer function $h(z)$. For two real polynomials of z, $a(z)$ and $b(z)$, $a(z)|b(z)$ means that there exists a real polynomial $p(z)$, such that $b(z) = p(z)a(z)$. The notation $\mathbb{E}(x)$ is the expectation of the random variable x. The normal distribution is denoted $\mathcal{N}(\mu, \sigma^2)$, where μ is the mean and σ^2 is the variance.

II. Internal Model Average Consensus Estimator

We consider a group of N agents whose communication topology is modeled by a connected undirected graph $\mathcal{G}$. Agent i has a discrete time input $\phi_i(k) \in \mathbb{R}$, where $k = 1, 2, \cdots, \infty$, is the time index, and employs an average consensus estimator to estimate the average of $\phi_i(k)$, $i = 1, \cdots, N$. Let the Z-transform of $\phi_i(k)$ be

$$\phi_i(z) = \frac{c_i(z)}{d(z)}, \tag{1}$$

where $c_i(z)$ is the numerator polynomial of $\phi_i(z)$, $d(z)$ is a common monic denominator polynomial among all $\phi_i(z)$, and $c_i(z)$ and $d(z)$ are coprime. Without loss of generality, we assume that $d(z) = 0$ does not have roots inside the unit circle since any root inside the unit circle results in an exponentially vanishing component in $\phi_i(k)$.

In [1], we proposed a class of robust dynamic average consensus estimators that employ the $d(z)$ information and achieve the average consensus estimation

$$\lim_{k\to\infty} \left| \nu(k) - \frac{1}{N} 1_N 1_N^T \phi(k) \right| = 0 \tag{2}$$

where $\nu(k) = [\nu_1(k), \cdots, \nu_N(k)]^T$, $\nu_i(k)$ is agent i's estimate of the average of $\phi_i(k)$'s, and $\phi(k) = [\phi_1(k), \cdots, \phi_N(k)]^T$. The block diagram for the proposed average consensus estimators is shown in Fig. 1, where L is the graph Laplacian matrix of $\mathcal{G}$ and is defined as

$$\ell_{ij} := \begin{cases} \sum_{j=1}^{N} a_{ij} & \text{if } i = j \\ -a_{ij} & \text{otherwise} \end{cases} \tag{3}$$

in which $a_{ij} = a_{ji} > 0$ if agents i and j are neighbors, and otherwise, $a_{ij} = a_{ji} = 0$. The gains k_p and k_I satisfy $k_p \geq 0$ and $k_I > 0$. The transfer functions $h(z)$ and $g(z)$ represent local filtering processes performed by each agent. In fact, denoting by (A_h, B_h, C_h, D_h) and (A_g, B_g, C_g, D_g) the minimal time domain representations of $h(z)$ and $g(z)$, respectively, we note that Fig. 1 gives rise to

$$\begin{aligned} X_i^h(k+1) &= A_h X_i^h(k) + B_h \left(\phi_i(k) - k_p \textstyle\sum_{j\in\mathcal{N}_i} a_{ij}(\nu_i(k) - \nu_j(k)) \right) \\ &\quad - B_h \left(k_I \textstyle\sum_{j\in\mathcal{N}_i} a_{ij}(\eta_i(k) - \eta_j(k)) \right) \\ \nu_i(k) &= C_h X_i^h(k) + D_h \left(\phi_i(k) - k_p \textstyle\sum_{j\in\mathcal{N}_i} a_{ij}(\nu_i(k) - \nu_j(k)) \right) \\ &\quad - D_h \left(k_I \textstyle\sum_{j\in\mathcal{N}_i} a_{ij}(\eta_i(k) - \eta_j(k)) \right) \end{aligned} \tag{4}$$

and

$$\begin{aligned} X_i^g(k+1) &= A_g X_i^g(k) + B_g \left(k_I \textstyle\sum_{j\in\mathcal{N}_i} a_{ij}(\nu_i(k) - \nu_j(k)) \right) \\ \eta_i(k) &= C_g X_i^g(k) + D_g \left(k_I \textstyle\sum_{j\in\mathcal{N}_i} a_{ij}(\nu_i(k) - \nu_j(k)) \right), \end{aligned} \tag{5}$$

where $X_i^h(k)$ and $X_i^g(k)$ are agent i's internal states for $h(z)$ and $g(z)$, respectively. The exogenous inputs $\xi^1(z)$ and $\xi^2(z)$ in Fig. 1 represent the effects due to the initial conditions of $X_i^h(k)$ and $X_i^g(k)$, respectively.

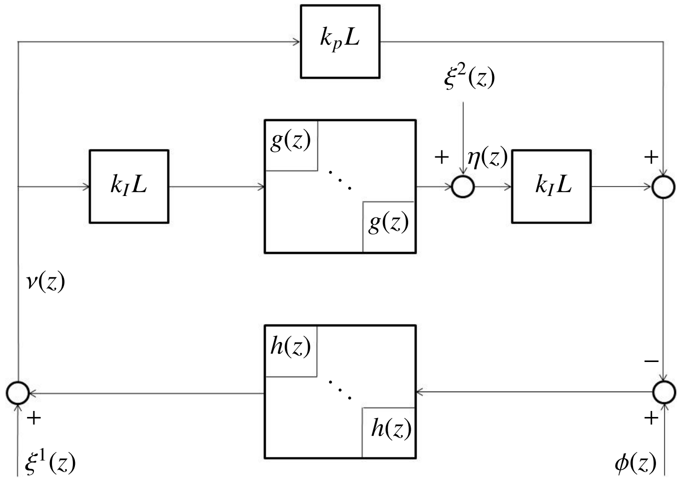

Fig. 1. The structure of the internal model average consensus estimators: $\xi^1(z)$ and $\xi^2(z)$ represent the exogenous inputs due to the initial conditions of the internal states of $h(z)$ and $g(z)$.

Theorem 1 below, proven in [1, Theorem 3], presents the conditions of $h(z)$ and $g(z)$ under which average consensus (2) is achieved regardless of $\xi^1(z)$ and $\xi^2(z)$.

***Theorem** 1:* Consider the closed-loop system in Fig. 1, where $\phi(z) = [\phi_1(z), \cdots, \phi_N(z)]^T$ and $\phi_i(z)$ is as in (1), in which $d(z) = 0$ does not contain any roots inside the unit circle. Assume that $\mathcal{G}$ is constant and that $n_h(z) = 0$ and $d_g(z) = 0$ have no common roots on or outside the unit circle. Then, the closed-loop stability of the system in Fig. 1 and the average consensus in (2) are guaranteed if and only if
A) $d(z)|(n_h(z) - d_h(z))$ and the roots of $d_h(z)$ are inside the unit circle;
B) $d(z)|d_g(z)$;
C) the roots of $d_g(z)d_h(z) + n_g(z)n_h(z)k_I^2\lambda_i^2 + d_g(z)n_h(z)k_p\lambda_i$ are inside the unit circle, $i = 2, \cdots, N$, where λ_i's are the eigenvalues of L such that $0 = \lambda_1 < \lambda_2 \leq \lambda_3 \cdots \leq \lambda_N$. □

***Remark** 1:* Define the estimation error $\epsilon(k) = \nu(k) - \frac{1}{N} 1_N 1_N^T \phi(k)$ and let $T(z)$ be the transfer matrix from $\phi(z)$ to $\epsilon(z)$. Then, condition *C)* guarantees the stability of $T(z)$ while conditions *A)* and *B)* ensure that the zeros of $T(z)$ contain those of $d(z) = 0$. □

***Example** 1:* When the inputs $\phi_i(k)$ are constant, $d(z) = z - 1$ in (1). In this case, Theorem 1 gives rise to the discrete PI estimator [14], where

$$h(z) = \frac{\gamma}{z - (1-\gamma)}, \quad 0 < \gamma < 2, \quad \text{and} \quad g(z) = \frac{1}{z-1}. \tag{6}$$

If the gains k_p and k_I are chosen such that *C)* is satisfied, the estimates from the discrete PI estimator converge to the average of the inputs regardless of the initial conditions of the internal states. □

III. Distributed Implementation of Centralized Kalman Filter

A. Distributed Implementation of Centralized Kalman filter

Consider a network of N agents estimating the state $x(k) \in \mathbb{R}^p$ from a linear dynamic system

$$x(k+1) = Fx(k) + w(k) \tag{7}$$

$$z_i(k) = H_i x(k) + v_i(k), \quad i = 1, \cdots, N \tag{8}$$

where the process noise $w(k) \sim \mathcal{N}(0, Q)$, $Q \geq 0$, the observation noise $v_i(k) \sim \mathcal{N}(0, R_i)$, $R_i > 0$, and they are independent. Without loss of generality, we suppose that both z_i and R_i are scalars. Because any stable eigenvalue of F in (7) results in a state whose expected value converges to zero and whose covariance remains bounded, there is no need to estimate this state using a distributed Kalman filter. Thus, we assume that the eigenvalues of F are on or outside of the unit circle, i.e., F is neutrally stable or unstable.

An optimal estimator for (7)-(8) is the Kalman Filter. Let $\hat{x}$ be the estimate of the state x and P the uncertainty associated with the estimate. Define the information vector $\hat{\tau} = P^{-1}\hat{x}$ and the information matrix $Y = P^{-1}$. The information form of the Kalman filter is given by

$$\text{Prediction:} \quad \tilde{Y}(k) = (FY^{-1}(k-1)F^T + Q)^{-1} \tag{9}$$

$$\tilde{\hat{\tau}}(k) = \tilde{Y}(k)(FY^{-1}(k-1)\hat{\tau}(k-1)) \tag{10}$$

$$\text{Update:} \quad Y(k) = \tilde{Y}(k) + NC \tag{11}$$

$$\hat{\tau}(k) = \tilde{\hat{\tau}}(k) + Ny(k) \tag{12}$$

where $z = [z_1^T, \cdots, z_N^T]^T$,

$$H = \left(H_1^T, \cdots, H_N^T\right)^T, \quad R = \text{diag}\{R_1, \cdots, R_N\}, \tag{13}$$

$$C = \frac{1}{N} H^T R^{-1} H = \frac{1}{N} \sum_{i=1}^{N} H_i^T R_i^{-1} H_i, \tag{14}$$

and

$$y(k) = \frac{1}{N} H^T R^{-1} z(k) = \frac{1}{N} \sum_{i=1}^{N} H_i^T R_i^{-1} z_i(k). \tag{15}$$

The matrix C consists of $p(p+1)/2$ unique scalar sums, as the matrix is symmetric, and the vector $y(k)$ consists of p scalar sums. The number of the agents, N, is assumed to be either known in advance, in the case of a fixed number of mobile sensors, or estimated by a separate decentralized estimation procedure [15], [16].

For each agent i to implement the Kalman filter (9)-(12), the information of NC and $Ny(k)$ must be available. To obtain this information, we make use of average consensus estimator to estimate C and $y(k)$. Let $\hat{C}_i(k)$ and $\hat{y}_i(k)$ be the ith agent's estimate of C and $y(k)$, respectively. Then, each agent implements (9)-(12) with C and $y(k)$ replaced by $\hat{C}_i(k)$ and $\hat{y}_i(k)$ and obtains its own estimate $\hat{x}_i(k)$.

We now present a single time-scale solution to distributed implementation of the Kalman filter, where the average consensus estimator and the Kalman filter are updated at the same frequency. Our solution consists of two steps, one for estimating C and the other for estimating $y(k)$:

Step 1. Follow Theorem 1 and design a stable internal model average consensus estimator with $d(z) = z - 1$ to estimate C in (14). Agent i's input to this estimator is a $p(p+1)/2$ by 1 vector consisting of the upper triangular and diagonal elements of $H_i^T R_i^{-1} H_i$.

Because $H_i^T R_i^{-1} H_i$ is constant, implementing an estimator designed from Step 1 ensures that $\hat{C}_i(k) \to C$ as $k \to \infty$. Note that this estimator can run at the same or a faster rate than the Kalman filter update since $H_i^T R_i^{-1} H_i$ is constant. One example of such an estimator is the discrete PI estimator in Example 1. During the transient, $\hat{C}_i(k)$ may not always be positive semidefinite. In this case, the agent can substitute a matrix $\tilde{\hat{C}}_i(k)$ obtained by projecting $\hat{C}_i(k)$ onto the convex set of positive semidefinite matrices.

Step 2. Follow Theorem 1 again to design a stable internal model average consensus estimator with $d(z) = \det(zI - F)$ to estimate $y(k)$ in (15). Agent i's input to this estimator is the $p \times 1$ vector $H_i^T R_i^{-1} z_i(k)$.

Step 2 matches the dynamics of an average consensus estimator to (7). Because $z_i(k)$ in (15) inherits the model information of F, this estimator guarantees $|\hat{y}_i(k) - y(k)| \to 0$ if the noise $w(k)$ and $v(k)$ in (7)-(8) is zero. When the noise is nonzero, this estimator ensures that the mean of $\hat{y}_i(k) - y(k)$ converges to zero and that the covariance of $\hat{y}_i(k) - y(k)$ remains bounded. Since this estimator contains the model of (7), it gets updated at the same rate as the Kalman filter.

B. Convergence properties of the distributed implementation

Given the two design steps in Section III-A, we next analyze the statistical properties of agent i's estimate $\hat{x}_i(k)$ in steady state. We make the following assumption:

Assumption 1: (F, H) in (7)-(8) is observable. □

This observability condition guarantees that $\tilde{Y}^{-1}(k)$ converges to a unique positive definite matrix $\tilde{P}$ that is the maximal solution to the following DARE (Discrete Algebraic Riccati Equation) [17, Theorem 17.5.3]

$$\tilde{P} = F(\tilde{P} - \tilde{P}H^T(H\tilde{P}H^T + R)^{-1}H\tilde{P})F^T + Q. \tag{16}$$

Note from (9)-(12) that the propagation of the information matrix Y is independent of the measurements. It follows that $\tilde{Y}^{-1}(k) \to \tilde{P}$ as $k \to \infty$, which means that in steady state, (9)-(12) reduces to the *steady state Kalman filter*

$$\hat{x}(k+1) = G\hat{x}(k) + NFPy(k) \tag{17}$$

where $y(k)$ is in (15) and the matrices P and G are given by

$$P = (\tilde{P}^{-1} + H^T R^{-1} H)^{-1} \tag{18}$$

and

$$G = F - KH \tag{19}$$

in which

$$K = FPH^T R^{-1} \tag{20}$$

is the steady state Kalman gain.

In practice, the agents can first run the consensus estimator from Step 1 for some time so that the convergence requirement of $\hat{C}_i(k) \to C$ is met. Next they can propagate (9) and (11) individually to obtain $\tilde{P}$. Then each agent uses the average consensus estimator from Step 2 to estimate $y(k)$ by $\hat{y}_i(k)$ and implements

$$\hat{x}_i(k+1) = G\hat{x}_i(k) + NFP\hat{y}_i(k), \tag{21}$$

where $\hat{x}_i(k)$ is the ith agent's estimate of $x(k)$.

Define the estimation error of agent i as

$$\hat{e}_i(k) = \hat{x}_i(k) - x(k) \tag{22}$$

and let $\hat{e}(k) = [\hat{e}_1(k)^T, \cdots, \hat{e}_N(k)^T]^T$. Theorem 2 below proves that our internal model average consensus estimator ensures that the expected value of $\hat{e}_i(k)$ converges to zero as $k \to \infty$ and that the covariance of $\hat{e}(k)$ remains bounded. Theorem 2 also indicates that the internal model average consensus estimator allows each agent to estimate the dynamical state even when some of the state is not observable to an individual agent.

***Theorem** 2:* Consider the centralized steady state Kalman filter (17) and its distributed implementation (21), where $\hat{y}_i(k)$ in (21) is agent i's estimate of $y(k)$ in (17). Suppose that $\hat{y}_i(k)$, $i = 1, \cdots, N$, are obtained from a stable average consensus estimator designed according to Step 2. If Q and R in (7)-(8) are upper bounded and Assumption 1 is satisfied, then

$$\mathbb{E}(\hat{e}_i(k)) \to 0 \quad \text{as} \quad k \to \infty \tag{23}$$

and $\mathbb{E}(\hat{e}(k)\hat{e}(k)^T)$, the covariance of $\hat{e}(k)$, is bounded. □

Proof: We define

$$\tilde{y}_i(k) = \hat{y}_i(k) - y(k) \tag{24}$$

and note from (7), (17) and (21) that

$$\hat{e}_i(k+1) = G\hat{e}_i(k) + NFP\tilde{y}_i(k) + Kv_i(k) - w(k) \tag{25}$$

or in a compact form

$$\hat{e}(k+1) = (I_N \otimes G)\hat{e}(k) + (I_N \otimes NFP)\tilde{y}(k) + (I_N \otimes K)v(k) - 1_N \otimes w(k) \tag{26}$$

where

$$v(k) = [v_1(k), \cdots, v_N(k)]^T \tag{27}$$

and

$$\tilde{y}(k) = [\tilde{y}_1(k)^T, \cdots, \tilde{y}_N(k)^T]^T. \tag{28}$$

To analyze the statistical properties of $\hat{e}(k)$ in (26), we derive the transfer matrix from $w(k)$ and $v(k)$ to $\hat{e}(k)$. First, we develop the transfer matrix from $w(k)$ and $v(k)$ to $\hat{y}(k)$, where

$$\hat{y}(k) = [\hat{y}_1(k)^T, \cdots, \hat{y}_N(k)^T]^T. \tag{29}$$

The block diagram in Fig. 2 shows the combination of the average consensus estimator with the dynamic plant in (7) and (8). Let

$$\phi_i(k) = H_i^T R_i^{-1} z_i(k) \tag{30}$$

and $\phi(k) = [\phi_1(k)^T, \cdots, \phi_N(k)^T]^T$. Suppose that the transfer matrix from the input $\phi(k)$ to the output $\hat{y}(k)$ is given by $T_{\phi\hat{y}}(z)$. Since the consensus estimator derived from Theorem 1 is robust to initial values of $h(z)$ and $g(z)$, we assume that the initial conditions of the average consensus estimator are zero. Denote by $T(z)[w(k)]$ the output from the system of which the initial conditions of the internal states are zero, the transfer function is $T(z)$ and the input is $w(k)$. We note that $z_i(k)$ in Fig. 2 is given by

$$z_i(k) = H_i(zI - F)^{-1}[w(k) + x(0)\delta(k+1)] + v_i(k) \tag{31}$$

which leads to

$$\begin{aligned}\hat{y}(k) = {} & T_{\phi\hat{y}}(z)\mathcal{H}_1(zI - F)^{-1}[w(k) + x(0)\delta(k+1)] \\ & + T_{\phi\hat{y}}(z)[\mathcal{H}_2 v(k)],\end{aligned} \tag{32}$$

where $\delta(\cdot)$ is the Kronecker delta function,

$$\mathcal{H}_1 = \begin{pmatrix} H_1^T R_1^{-1} H_1 \\ H_2^T R_2^{-1} H_2 \\ \vdots \\ H_N^T R_N^{-1} H_N \end{pmatrix} \quad \text{and} \quad \mathcal{H}_2 = \begin{pmatrix} H_1^T R_1^{-1} & & \\ & \ddots & \\ & & H_N^T R_N^{-1} \end{pmatrix}. \tag{33}$$

Next, we compute from (15) and (31) the transfer function from $w(k)$ and $v(k)$ to $\tilde{y}(k)$ as

$$\tilde{y} = \hat{y} - 1_N \otimes y \tag{34}$$

$$\begin{aligned} &= T_{\phi\tilde{y}}(z)\mathcal{H}_1(zI - F)^{-1}[w(k) + x(0)\delta(k+1)] \\ &\quad + T_{\phi\tilde{y}}(z)[\mathcal{H}_2 v(k)] \end{aligned} \tag{35}$$

where $T_{\phi\tilde{y}}(z) = T_{\phi\hat{y}}(z) - \frac{1}{N}1_N 1_N^T \otimes I_p$. Define

$$T_1(z) = (I_N \otimes NFP)T_{\phi\tilde{y}}(z)\mathcal{H}_1(zI - F)^{-1} \tag{36}$$

and

$$T_2(z) = (I_N \otimes NFP)T_{\phi\tilde{y}}(z)\mathcal{H}_2. \tag{37}$$

It then follows from (26) and (35) that

$$\begin{aligned}\hat{e}(k+1) = {} & (I_N \otimes G)\hat{e}(k) + (I_N \otimes K)v(k) - 1_N \otimes w(k) \\ & + T_1(z)[w(k) + x(0)\delta(k+1)] + T_2(z)[v(k)].\end{aligned} \tag{38}$$

Let the minimal state space representations of $T_1(z)$ and $T_2(z)$ be, respectively,

$$\mathcal{X}_1(k+1) = \mathcal{A}_1\mathcal{X}_1(k) + \mathcal{B}_1 u(k) \tag{39}$$

$$\mathcal{Y}_1(k) = C_1\mathcal{X}_1(k) + \mathcal{D}_1 u(k) \tag{40}$$

and

$$\mathcal{X}_2(k+1) = \mathcal{A}_2\mathcal{X}_2(k) + \mathcal{B}_2 u(k) \tag{41}$$

$$\mathcal{Y}_2(k) = C_2\mathcal{X}_2(k) + \mathcal{D}_2 u(k). \tag{42}$$

Using the state space representations of $T_1(z)$ and $T_2(z)$, we rewrite (38) as

$$\Xi(k+1) = \mathbf{A}\Xi(k) + \begin{pmatrix} \mathcal{D}_1 \\ \mathcal{B}_1 \\ 0 \end{pmatrix} x(0)\delta(k+1) + \mathbf{B}\begin{pmatrix} w(k) \\ v(k) \end{pmatrix} \tag{43}$$

$$\hat{e}(k) = \mathbf{C}\Xi(k) \tag{44}$$

where

$$\Xi(k) = \begin{pmatrix} \hat{e}(k) \\ \mathcal{X}_1(k) \\ \mathcal{X}_2(k) \end{pmatrix} \tag{45}$$

$$\mathbf{A} = \begin{pmatrix} I_N \otimes G & C_1 & C_2 \\ 0 & \mathcal{A}_1 & 0 \\ 0 & 0 & \mathcal{A}_2 \end{pmatrix} \tag{46}$$

$$\mathbf{B} = \begin{pmatrix} I_N \otimes K + \mathcal{D}_1 & -1_N \otimes I_p + \mathcal{D}_2 \\ \mathcal{B}_1 & 0 \\ 0 & \mathcal{B}_2 \end{pmatrix} \text{ and } \mathbf{C} = \begin{pmatrix} I_{Np} & 0 & 0 \end{pmatrix}. \tag{47}$$

Because $w(k)$ and $v(k)$ are zero mean, we conclude from (43) and (44) that $\mathbb{E}(\hat{e}(k)) \to 0$ as $k \to \infty$ if $\mathbf{A}$ is stable. The covariance of $\hat{e}(k)$ is given by

$$\mathcal{P}_{\hat{e}}(k) := \mathbb{E}(\hat{e}(k)\hat{e}(k)^T) = \mathbf{C}\mathcal{P}(k)\mathbf{C}^T \tag{48}$$

where $\mathcal{P}(k)$ is the solution to

$$\mathcal{P}(k) = \mathbf{A}\mathcal{P}(k-1)\mathbf{A}^T + \mathbf{B}\begin{pmatrix} Q & \\ & R \end{pmatrix}\mathbf{B}^T. \tag{49}$$

Because $w(k)$ and $v(k)$ are zero mean and Q and R are upper bounded, it follows from (38) that $\mathbb{E}(\hat{e}(k))$ converges to zero and $\mathbb{E}(\hat{e}(k)\hat{e}(k)^T)$ remains bounded if G, $T_1(z)$ and $T_2(z)$ are stable. Since (F, H) is observable, $R > 0$ and $Q \geq 0$, G is stable [17, Theorem 17.5.3]. Since the consensus estimator is designed to be stable, $T_{\phi\tilde{y}}(z)$ is also stable and thus, $T_2(z)$ in

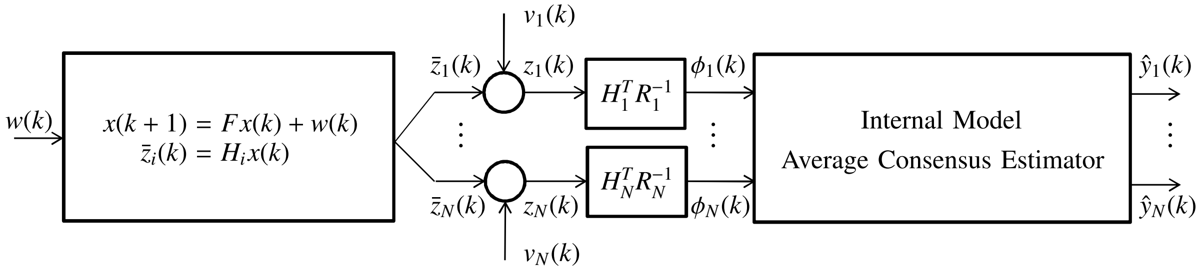

Fig. 2. The combination of the average consensus estimator with the dynamic plant in (7) and (8).

(37) is stable. Thanks to our internal model based design of the average consensus estimator, $T_1(z)$ in (36) is also stable because $T_{\phi\bar{y}}(z)$ contains the unstable or neutrally stable poles of $(zI - F)^{-1}$ as its zeros. Thus, $\hat{e}(k)$ has bounded variance and its expectation converges to zero as $k \to \infty$. ■

IV. Design Example

We consider a system with three states. Two states are the state variables of noisy sinusoidal dynamics, and the other state corresponds to a random walk. The sensor network consists of four agents, each of which makes a scalar measurement. Three agents sense one of the three system states, while the fourth agent measures the sum of the first two states. This system can be written as

$$x(k+1) = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 2\cos(\omega T) & 0 \\ 0 & 0 & 1 \end{pmatrix} x(k) + w(k) \tag{50}$$

$$z(k) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} x(k) + v(k) \tag{51}$$

The natural frequencies of (50) are $(1, \cos(\omega T) \pm \sin(\omega T) j)$. In the simulation, we choose $\omega = 5$, $T = 0.01$, $w(k) \sim \mathcal{N}(0_3, 10^{-4} I_3)$ and $v(k) \sim \mathcal{N}(0_4, 0.1 I_4)$. The initial estimates of the states for all the agents are set to zero and the covariance of each agent's initial estimate is chosen as I_3.

We compare the agents' estimation errors of the state for the following four kinds of Kalman filters: 1) Centralized Kalman filter (CKF); 2) Local Kalman filter (LKF); 3) Distributed Kalman filter with internal model average consensus estimator (DKF-IM); 4) Distributed Kalman filter with PI average consensus estimator (DKF-PI).

The CKF makes use of measurements from all the agents and generates an optimal estimate of the state. For the LKF, each agent implements a Kalman filter that employs measurements only from the agent itself, that is, there is no communication between agents.

In the distributed implementation, we assume that $\hat{C}_i(k)$ has converged to C and design average consensus estimators to estimate $y(k)$ in (15). We run average consensus estimators and the distributed Kalman filter at the same frequency. The communication graph $\mathcal{G}$ is a ring. The weight ℓ_{ij} in (3) is chosen as $\frac{1}{N}$ for neighboring agents i and j so that $\lambda_N \leq 1$.

For the DKF-IM, we design an internal model average consensus estimator from Step 2. The internal model of (50)

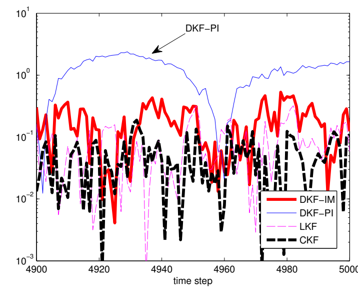

Fig. 3. A comparison of agent 2's estimation errors of the second state between four different Kalman filters. The DKF-PI has a larger estimation error than the DKF-IM since the PI estimator does not contain a model of the second state.

is given by

$$d(z) = \left| zI - \begin{pmatrix} 0 & 1 & 0 \\ -1 & 2\cos(\omega T) & 0 \\ 0 & 0 & 1 \end{pmatrix} \right|^{-1} = (z-1)(z^2 - 2\cos(\omega T) z + 1). \tag{52}$$

According to Theorem 1, we design $h(z) = 1$, $k_p = 0$, and

$$g(z) = \frac{(1 - 0.96 + 0.02j)(1 - 0.96 - 0.02j)}{d(z)}.$$

With this design, the inward and outward gain margins of $h(z)g(z)$ are given by 0 and approximately 2, respectively. Since $\lambda_N \leq 1$, choosing $k_I \leq \sqrt{2}$ ensures that condition *C)* in Theorem 1 is satisfied for arbitrary connected graphs. In the simulation, we choose $k_I = 1$.

For the DKF-PI, we use the discrete PI estimator in Example 1 to estimate $y(k)$. The discrete PI estimator yields the average of constant inputs asymptotically. When the inputs are time-varying, the estimate from the PI estimator is only approximate. One advantage of the PI estimator is that its robustness to estimator initialization errors. In the simulation, we take $\gamma = 0.95$, and $k_p = k_I = 1$, thereby satisfying *C)* in Theorem 1.

We simulate the four Kalman filters for 5000 time steps and show the agents' estimation errors of the last 100 time steps. Figs. 3-4 illustrate agent 2's estimation errors for the second and the third states, respectively. The estimation error for the first state is omitted since the first state is simply some white noise plus the second state delayed by one time step. In Fig. 3, the DKF-PI yields a larger estimation error than the DKF-IM because the PI average estimator does not contain an accurate model for the first two states while the internal model average estimator does. We note from Fig. 4 that the LKF has a larger estimation error for the third state. This is because the third state is not observable to agent 2

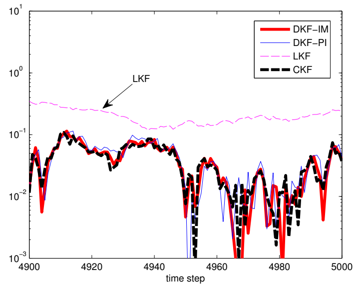

Fig. 4. A comparison of agent 2's estimation errors of the third state between four different Kalman filters. The LKF yields a larger estimation error since the third state is not observable to agent 2.

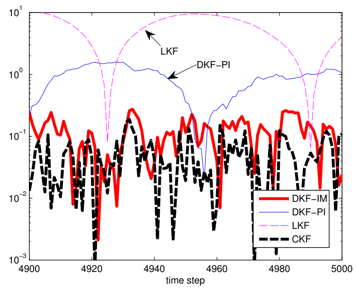

Fig. 5. A comparison of agent 3's estimation errors of the second state between four different Kalman filters. The LKF has the largest estimation error since the second state is not observable to agent 3. The DKF-PI outperforms the LKF due to the fusion of agents' information. The DKF-IM yields a smaller error than the DKF-PI since the DKF-IM contains the model information of the first two states.

and thus the LKF is not stable for the third state. Since the DKF-IM and the DKF-PI both contain an accurate model for the third state, they achieve smaller estimation error than the LKF. This illustrates an advantage of DKF that the average consensus process allows an individual agent to estimate the state that is not observable to that agent. In Figs. 5 and 6, we show the estimation errors of agent 3. Since the second state is not observable to agent 3, the LKF in Fig. 5 performs worst. The DKF-PI achieves smaller estimation error than the LKF because the PI estimator fuses information from the other agents, which observe the second state. However, because the PI estimator does not contain an accurate model of the second state, it is outperformed by the DKF-IM in Fig. 5. Because the DKF-IM and the DKF-PI both contain the model of the third state, they achieve estimation errors similar to those of the LKF and the CKF in Fig. 6.

V. Conclusions and Future work

We applied our internal model average consensus estimators in [1] to distributed Kalman filtering. We proved that the internal model in the consensus estimator guarantees bounded covariance of the estimation error even when the underlying dynamical system is neutrally stable or unstable. Our distributed Kalman filter updates at the same frequency as the average consensus estimator. Simulation results were presented to illustrate the effectiveness of embedding the internal model average consensus estimator in distributed Kalman filtering. Future work will consider applications of the internal model average consensus estimator in other distributed estimators, such as a distributed H_∞ filter.

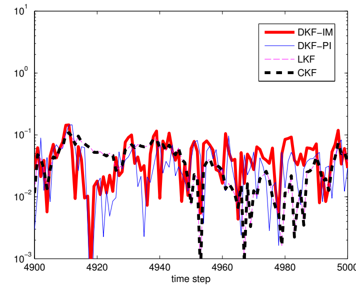

Fig. 6. A comparison of agent 3's estimation errors of the third state between four different Kalman filters. The LKF and the CKF have the same estimation error since the third state is observable only to agent 3. The DKF-PI and the DKF-IM perform similarly since they both contain an accurate model of the third state.

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