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Peak-gain-bounded design of constrained controllable damping in vibrating structures

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***Abstract*— This paper concerns the design of mechanical vibration suppression systems with controllable mechanical damping. For such systems, the paper illustrates techniques for both linear and nonlinear damping design, which place an upper bound on the peak gain from the structural disturbances to response performance outputs. The technique is an extension of many Lyapunov-based damping techniques in the open literature on semiactive systems. Presently, such techniques admit performance measures which depend only on the system state. This paper discusses theoretical extensions to those methods to accommodate performance measures which are explicitly dependent on the external disturbance and control forces. The theory is illustrated via simulation of a base-excited structure equipped with viscous, semiactive, or regenerative damping systems, with the objective of minimizing the peak gain from the base acceleration amplitude to the vector of inter-story drifts and structural accelerations.**

***Index Terms*— Vibration, Mechatronics, Nonlinear Control**

I. Introduction

In many vibration suppression applications, constraints on available power result in restrictions on the manner in which control systems can be made to operate, and on the type of hardware used to effect the control forces. Sometimes these power constraints are motivated primarily by system efficiency concerns, such as in the case of automotive suspension control systems. However, they also frequently arise as a consequence of reliability issues, in applications for which the delivery of power for control is unpredictable. One such application pertains to earthquake response control in civil structures, for which the forces and power levels involved are immense, while power delivery from an external grid cannot be taken for granted [1], [2]. In many such applications, vibration suppression systems involving only passive components (i.e., tuned mass dampers, supplemental friction dampers, isolation bearings, etc.) can be designed for acceptable performance. However, it is often the case that performance can be improved beyond what these purely passive systems can achieve, by using feedback to vary parameters within a passive system. Such technologies are often called *semiactive* because, while they do require a power supply to operate, energy consumption is exclusively parasitic (such as to execute control intelligence, facilitate sensor feedback, open and close valves, etc). Examples of semiactive devices include variable-orifice dampers, magnetorheological fluid dampers, and electromechanical transducers with controllable resistive shunt networks.

A broad class of such systems can be characterized (to reasonable approximation) by systems with controllable damping matrices which relate the vector of device velocities v to the corresponding vector of device forces f; i.e.,

$$f(t) = -U(t)v(t) \tag{1}$$

in which $U(t)$, the controllable damping matrix, is subject to an associated algebraic constraint $U(t) \in \mathcal{U}$, $\forall t$. The device system must be instantaneously dissipative; i.e.,

$$f^T(t)v(t) < 0 \quad , \quad \forall t, \ \forall v(t) \neq 0 \tag{2}$$

which results in the restriction of $U(t)$ to matrices with positive-definite Hermitian part; i.e.,

$$\mathcal{U} = \left\{ U \ : \ U + U^T > 0 \right\} \tag{3}$$

However, there is often a maximum feasible damping matrix $U_m > 0$ for such problems, which further restricts $\mathcal{U}$ to

$$\mathcal{U} = \left\{ U \ : \ U + U^T - 2UU_m^{-1}U^T \geqslant 0 \right\} \tag{4}$$

For example, for variable-orifice dampers U_m is a diagonal matrix, with diagonal components u_{mi} equal to the viscous damping of device i with the orifice in the closed position. Likewise, for electromechanical transducers with controllable resistive shunts, U_m is also diagonal, with u_{mi} equal to the effective viscous damping of the transducer with its coils shorted [3]. Many controllable damping systems also typically have a nonzero minimum damping they can impose on a structure, but this damping can be absorbed into the mechanical model parameters of the structure.

In most multi-device semiactive systems, it is impractical to transmit power between devices, which effectively restricts $\mathcal{U}$ to include only diagonal matrices; i.e.,

$$\mathcal{U} = \{ \ U = \text{diag}\left\{...u_i...\right\} \ : \ u_i \in (0, u_{mi}], \ \forall i \ \} \tag{5}$$

To distinguish constraint (4) and the more restrictive (5), we hereafter refer to systems adhering to (4) as *regenerative* damping systems, while those additionally adhering to (5) as *semiactive*. In general, the theory we will discuss in this paper can be generalized to definitions of $\mathcal{U}$ as arbitrary convex subsets of (3), with (4) and (5) as special cases.

Irrespective of how $\mathcal{U}$ is defined, it will also be convenient to define the feasible force region for f, given v; i.e.,

$$\mathcal{F}(v) = \{ \ f \ : \ \exists U \in \mathcal{U} \ni f = -Uv \ \} \tag{6}$$

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In this paper we investigate the embedment of an n_f-device controllable-damping system into a linear structural system. The resultant system model is assumed to be

$$\dot{x} = Ax + B_f f + B_w w \tag{7a}$$
$$v = B_f^T x \tag{7b}$$
$$z = C_z x + D_{zf} f + D_{zw} w \tag{7c}$$

in which $w \in \mathbb{R}^{n_w}$ is a vector of exogenous disturbances, and $z \in \mathbb{R}^{n_z}$ is the vector of quantities by which performance will be judged. We make the following assumptions, which are always true for stable structural systems:

a) We assume A is asymptotically stable, and the mapping $f \mapsto v$ is positive-real and strictly proper.

b) If the above is true, there always exists a self-dual system realization [4], [5] in which B_f participates in both (7a) and (7b), and for which $A + A^T \leqslant 0$. We assume a self-dual realization in order to simplify some of the algebra. Beyond this, we assume $(A, A + A^T)$ is observable, as this is necessary (by Lasalle's Theorem) for asymptotic stability.

c) These two observations, together with the fact $\mathcal{U}$ is assumed to be a subset of (3), implies the closed-loop system is asymptotically stable for any time-varying $U(t) \in \mathcal{U}$, $\forall t$. This follows from the simple Lyapunov argument; i.e., for $w = 0$ and $x(0) \neq 0$,

$$\tfrac{d}{dt}\|x(t)\|_2^2 = 2x^T(t)\left[A - B_f U(t) B_f^T\right] x(t) \tag{8}$$
$$\leqslant x^T(t)\left[A + A^T\right] x(t) \leqslant 0 \tag{9}$$

with the equality holding instantaneously only (and not over an interval) if $(A, A + A^T)$ is observable.

In this context we consider two problems. The first problem is to design of a constant damping matrix $U(t) = U_0 \in \mathcal{U}$, which optimizes a bound on some measure of z. The second problem is to derive a criterion for full-state feedback laws $\phi : x(t) \to f(t) \in \mathcal{F}(v(t))$, which when satisfied, analytically guarantees to improve performance beyond the optimum attained with U_0.

Historically, such performance-bounded damping design techniques have been proposed in the contexts of several performance measures. For example, Tseng and Hedrick [6] (and, later, Scruggs et al [7]), investigated controllers which place a bound on

$$J_q = \mathcal{E}\, z^T z \tag{10}$$

where w is taken to be white noise and $D_{zw} = O$. In those papers, the optimal static $U_0 \in \mathcal{U}$ minimizing J_q was found precisely, as a constrained optimal static feedback problem. Then, a nonlinear state feedback law ϕ is synthesized from U_0, which is guaranteed to improve J_q beyond the optimally-attainable static performance. The resultant nonlinear controller has the form of a generalized saturation; i.e.,

$$f(t) = \underset{f' \in \mathcal{F}(v(t))}{\arg\min} \|D_{zf}(f' - Kx(t))\|_2 \tag{11}$$

where K is a feedback gain synthesized from U_0. In [8], Scruggs examined an extension of this approach, to control problems with multiple J_q objectives.

However, for some applications, it may be that the peak gain from w to z; i.e.,

$$J_p = \sup_{w \in \mathcal{L}_\infty} \|z\|_{\mathcal{L}_\infty} / \|w\|_{\mathcal{L}_\infty} \tag{12}$$

is the more meaningful performance measure.[1] For example, this can be the case in many vibration isolation applications for which the primary motivation for control is to protect against extreme events.

In this paper, we develop criteria analogous to J_q-bounded damping design, for the J_p measure. These techniques are an extension of several Lyapunov-based techniques in the open literature. The primary contributions of the paper, beyond these existing techniques, are *(i)* that this paper does not require that any restrictions be made on C_z, D_{zf}, or D_{zw}, and *(ii)* that it uses semidefinite programming to tighten the guaranteed bound on J_p. We concentrate primarily on techniques for optimizing U_0, while merely laying the analytical groundwork for synthesizing ϕ.

II. Lyapunov-Bounded Damping Design

Some of the more general techniques for linear and nonlinear structural damping control are in a Lyapunov context [9]. Such techniques ensure a bound on J_p, for the special case in which z is defined such that $D_{zf} = O$, $D_{zw} = O$, and $C_z^T C_z = I$; i.e.,

$$J_p = \sup_{w \in \mathcal{L}_\infty} \|x\|_{\mathcal{L}_\infty} / \|w\|_{\mathcal{L}_\infty} \tag{13}$$

The design procedure begins by determining, for some time-invariant $U_0 \in \mathcal{U}$, the associated Lyapunov matrix P_L as

$$0 = \left[A - B_f U_0 B_f^T\right]^T P_L + P_L\left[A - B_f U_0 B_f^T\right] + Q_L \tag{14}$$

where $Q_L > 0$ is a matrix of design parameters. We then have the following theorem [9].

THEOREM 1: Imposition of static damping $f = -U_0 v$ results in an upper bound on J_p, as defined in (13), equal to

$$J_p \leqslant \gamma_L = \frac{2\bar{\rho}\{P_L\}^{3/2}\bar{\sigma}\{B_w\}}{\underline{\rho}\{P_L\}^{1/2}\underline{\rho}\{Q_L\}} \tag{15}$$

where $\bar{\rho}\{\cdot\}$, $\underline{\rho}\{\cdot\}$, and $\bar{\sigma}\{\cdot\}$ denote maximum eigenvalue, minimum eigenvalue, and maximum singular value, respectively. Furthermore, imposition of any full-state nonlinear feedback law $\phi : x(t) \to f(t) \in \mathcal{F}(v(t))$ adhering to

$$x^T(t) P_L B_f \left(f(t) + U_0 B_f^T x(t)\right) \leqslant 0 \tag{16}$$

for all $x(t)$ guarantees to improve upon this bound. Furthermore, such a feedback law is guaranteed to exist.

Proof: Define $V(t) = x^T(t) P_L x(t)$. Then

$$\dot{V}(t) = -x^T(t) Q_L x(t) + 2x^T(t) P_L B_w w + 2x^T(t) P_L B_f\left(f(t) + U_0 B_f^T x(t)\right) \tag{17}$$

Let $f = -U_0 v$ and apply the Cauchy-Schwartz inequality:

$$\dot{V}(t) \leqslant -x^T(t) Q_L x(t) + 2V(t)^{1/2}\, \|P_L^{1/2} B_w w(t)\|_2 \tag{18}$$

[1] Here, we define $\|q\|_{\mathcal{L}_\infty} = \sup_t \|q(t)\|_2$.

which is conservatively bounded by

$$\dot{V}(t) \leqslant -V(t) \left(\underline{\rho}\{Q_L\}/\bar{\rho}\{P_L\}\right) + 2V(t)^{1/2} \, \bar{\rho}\{P_L\}^{1/2} \, \bar{\sigma}\{B_w\}\|w(t)\|_2 \quad (19)$$

which implies that for all t,

$$V(t)^{1/2} \leqslant 2 \left(\bar{\rho}\{P_L\}^{3/2}\bar{\sigma}\{B_w\}/\underline{\rho}\{Q_L\}\right) \|w\|_{\mathcal{L}_\infty} \quad (20)$$

Noting that $V(t)^{1/2} > \underline{\rho}\{P_L\}^{1/2}\|x(t)\|_2$, we obtain (15). It is immediate that any controller adhering to (16) will expand the region in x over which $\dot{V} < 0$. ■

Determination of the optimal constant $U_0 \in \mathcal{U}$ is accomplished by optimizing γ_L over the domain (U_0, Q_L). This optimization is nonconvex, and cumbersome due to the non-smooth dependency of γ_L. It is much more straight-forward to approach the design of U_0 as an inequality-constrained optimization, as will be shown in the next section.

To design a full-state feedback controller ϕ which outperforms the optimal bound attained over $U_0 \in \mathcal{U}$, Theorem 1 implies that the feedback law

$$f = \underset{f' \in \mathcal{F}(v(t))}{\arg\min} \left\{x^T(t)P_L B_f f'\right\} \quad (21)$$

thus guarantees to improve upon the bound $J_p \leqslant \gamma_L$ by minimizing $\dot{V}(t)$ at every time t, beyond the value with static damping. There are many extensions to this approach. For example, controller (21) exhibits switching surfaces, and therefore may result in sliding modes. However this situation may be remedied while still preserving the same bound on J_p. Indeed, it is straight-forward to show that Theorem 1 still holds if (16) is replaced with the more conservative

$$\|f(t) - Kx(t)\|_R^2 \leqslant x^T(t)P_L B_f R^{-1} B_f^T P_L x(t) \quad (22)$$

still adheres to the same bound, where the matrix R is positive definite but otherwise arbitrary, $\|q\|_R^2 = q^T R q$, and $K = -U_0 B_f^T - R^{-1}B_f^T P_L$. As such, the nonlinear controller

$$f = \underset{f' \in \mathcal{F}(v(t))}{\arg\min} \|f' - Kx\|_R \quad (23)$$

still guarantees $J_p < \gamma_L$. This feedback law is a continuous mapping from x to f for any $R > 0$, and therefore theoretically cannot exhibit switching surfaces or sliding modes. Qualitatively, the "smoothness" of the feedback law (and the degree to which $U(t)$ remains close to U_0) may be enhanced by increasing the eigenvalues of R, which may therefore be treated like tuning parameters. Also note that (23) and (21) are equivalent as $R \to O$, whereas the limit of (23) as $\underline{\rho}\{R\} \to \infty$ is optimal static damping; i.e., $f = -U_0 v$.

The Lyapunov-based methods discussed above are useful because the design of ϕ is straight-forward to apply, given U_0. Actually obtaining the $U_0 \in \mathcal{U}$ for minimal γ_L, however, can be challenging for multi-device systems. Even so, given any sub-optimal $U_0 \in \mathcal{U}$, the synthesis of ϕ as described above will still result in controllers adhering to a (sub-optimal) bound on J_p, and oftentimes in the application of Lyapunov-based variable-damping control, the optimization of U_0 is not carried out rigorously. Beyond the challenge of optimizing U_0, Lyapunov-based techniques have a few fundamental limitations. Most importantly, they place very specific restrictions on the definition of z; i.e., $C_z^T C_z$ must be nonsingular, $D_{zf} = O$, and $D_{zw} = O$. (Technically, in our discussion above we required that $C_z^T C_z = I$. However, if $C_z^T C_z \neq I$ but still nonsingular, then there exists an equivalent realization in which $C_z^T C_z = I$.) Secondarily, the bound on J_p is overly conservative. In the following two sections, we discuss a similar control design approach which places no restrictions on the definition of z, and which uses tighter (although still conservative) bounds on J_p.

III. Peak-Gain-Bounded Optimization of Linear Damping

We can approach the optimization of J_p over $U_0 \in \mathcal{U}$ as an application of the $\mathcal{S}$-procedure, and as an extension of the multiplier-based (i.e., LMI) methods of peak-gain-bounded linear control design originally proposed by Boyd et al in [10]. In the context of our problem, these concepts comprise the following observation [10], [11].

THEOREM 2: Define $V(t) = x^T(t)Px(t)$ for some time-invariant matrix $P = P^T > 0$, and let $U_0 \in \mathcal{U}$, and impose the feedback law $f = -U_0 v$. Define $Q(P, U_0) = [A - B_f U_0 B_f^T]^T P + P[A - B_f U_0 B_f^T]$. Then $\sup_t\{V(t)\} \leqslant \alpha\|w\|_{\mathcal{L}_\infty}^2$ for all $w \in \mathcal{L}_\infty$ if and only if there exist scalars $\lambda > 0$ and $\mu = \alpha\lambda$ such that

$$\begin{bmatrix} Q(P,U_0) & PB_w \\ B_w^T P & O \end{bmatrix} + \begin{bmatrix} \lambda P & O \\ O & -\mu I \end{bmatrix} < 0 \quad (24)$$

Furthermore, the additional inequality

$$\begin{bmatrix} \lambda P & & (sym) \\ O & (\gamma - \mu)I & \\ C_z - D_{zf}U_0 B_f^T & D_{zw} & \gamma I \end{bmatrix} \geqslant 0 \quad (25)$$

is sufficient to ensure that $\|z\|_{\mathcal{L}_\infty} \leqslant \gamma\|w\|_{\mathcal{L}_\infty}$.

Proof: Although this proof is a standard result, we provide a sketch of it here to provide context for the discussion of nonlinear damping controllers in the next section. Sufficiency for the bound on $V(t)$ follows from the fact that

$$\dot{V}(t) = \begin{bmatrix} x(t) \\ w(t) \end{bmatrix}^T \begin{bmatrix} Q(P,U_0) & PB_w \\ B_w^T P & O \end{bmatrix} \begin{bmatrix} x(t) \\ w(t) \end{bmatrix} \quad (26)$$

which, by (24), must be negative if $\lambda V(t) \geqslant \mu w^T(t)w(t)$. See [12] (Sec. B.2) for the necessity proof of this bound. The sufficiency of the bound on $\|z\|_{\mathcal{L}_\infty}$ follows from observing that if (25) holds then it follows through a Schur transformation that

$$\begin{bmatrix} \lambda P & O \\ O & (\gamma - \mu)I \end{bmatrix} \geqslant \frac{1}{\gamma} \begin{bmatrix} C_{zcl}^T \\ D_{zw}^T \end{bmatrix} \begin{bmatrix} C_{zcl} & D_{zw} \end{bmatrix} \quad (27)$$

where $C_{zcl} = C_z - D_{zf}U_0 B_f^T$. Taking quadratic forms of both sides of the above, on the vector $[x^T(t) \; w^T(t)]^T$, implies that

$$z^T(t)z(t) \leqslant \gamma\lambda V(t) + \gamma(\gamma - \mu)w^T(t)w(t) \quad (28)$$

$$\leqslant \sup_t \left\{\gamma\lambda V(t) + \gamma(\gamma - \mu)w^T(t)w(t)\right\} \quad (29)$$

$$\leqslant \gamma^2\|w\|_{\mathcal{L}_\infty}^2 \quad (30)$$

where the bound on $\lambda V(t)$ was used in the last line. ■

A. Optimization Approach

For fixed $U_0 \in \mathcal{U}$, inequalities $\lambda > 0$, (24), and (25), comprise a system of matrix inequalities which are linear in all variables except the product λP. Thus, as derived in [10], the minimal γ can be found, for fixed λ, as an LMI eigenvalue problem. By extension the minimal γ can be found over the set of all $\lambda > 0$ via a (not necessarily convex, but one-dimensional) line search over the compact domain $\lambda \in \left(0, -2\max_i \Re\{\rho_i(A - B_f U_0 B_f^T)\}\right)$. As such, determination of the optimal γ satisfying the above inequalities, while nonconvex, is straightforward and computationally efficient. Furthermore, the problem of optimal unconstrained (i.e., active) linear control for minimal γ, also turns out to an LMI problem for fixed λ, for controllers of the same order as the plant [11], and therefore may be solved via analogous line-search methods.

This is not true, however, for the optimization of $U_0 \in \mathcal{U}$, because even for fixed λ, this feasibility constraint cannot be incorporated into the optimization without losing convexity. Nonetheless, we can optimize U_0 for minimal γ, via an iterative process of convex redesign [13]. For iteration k of such a method, a given damping matrix $U_0^{(k)}$ and its corresponding Lyapunov matrix $P^{(k)}$ are redesigned via over-bounding of the bilinear terms in $Q(P, U_0)$. To see this, consider that

$$\begin{aligned} &Q(P, U_0) \\ &= Q(P, U_0^{(k)}) + Q(P^{(k)}, U_0) - Q(P^{(k)}, U_0^{(k)}) \\ &\quad - B_f \delta U_0^T B_f \delta P - \delta P B_f \delta U_0 B_f \end{aligned} \tag{31}$$

where we note that the first two terms are each linear in either U_0 or P, the third is constant. The last two terms are bilinear in $\delta U_0 = U_0 - U_0^{(k)}$ and $\delta P = P - P^{(k)}$. Now, we over-bound the above to remove the bilinear terms, by observing that for any $W = W^T > 0$,

$$\begin{aligned} &- B_f \delta U_0^T B_f \delta P - \delta P B_f \delta U_0 B_f \\ &\quad \leqslant B_f \delta U_0^T W \delta U_0^T B_f^T + \delta P B_f W^{-1} B_f^T \delta P \end{aligned} \tag{32}$$

As such, we have that (24) is conservatively ensured by

$$\begin{bmatrix} (\star) & PB_w & B_f(U_0 - U_0^{(k)})^T & (P - P^{(k)})B_f \\ & -\mu I & 0 & 0 \\ & & -W^{-1} & 0 \\ (sym) & & & -W \end{bmatrix} < 0 \tag{33}$$

where

$$(\star) = Q(P, U_0^{(k)}) + Q(P^{(k)}, U_0) - Q(P^{(k)}, U_0^{(k)}) + \lambda P \tag{34}$$

For fixed $\lambda > 0$ and $W > 0$, γ may be reduced by redesigning U_0, starting from a feasible pair $(U_0^{(k)}, P^{(k)})$, by minimizing γ over (U_0, P, γ, μ), and subject to (33), (25), and the feasibility constraint $U_0 \in \mathcal{U}$. For regenerative damping constraint (4), this constraint is

$$\begin{bmatrix} U_m & 2U_0 - U_m \\ 2U_0^T - U_m & U_m \end{bmatrix} \geqslant 0 \tag{35}$$

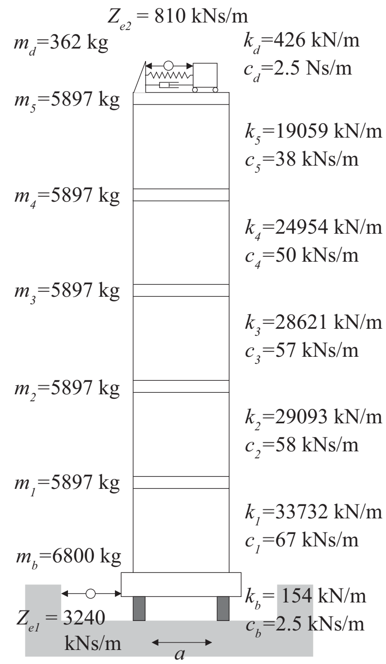

Fig. 1. Example structure with two damping devices

(For semiactive constraint (5), the only modification to the above is to restrict U_0 to diagonality.) The optimal redesign can be augmented to also redesign λ, by nesting the above convex optimization inside a line search for λ. The redesign process is thus repeated until convergence is reached.

B. Example

As an example, consider the five-story structure in Fig.1, which was modified from an example considered by Ramallo et al [14]. The device parameters are the same as those considered in several other papers [15], [8], and correspond to electromechanical transducers which may be operated as semiactive or regenerative, depending on whether their electronic systems are connected. The tuned mass damper (TMD) on the roof of the structure is tuned to the second structural natural frequency, and its primary purpose is for suppression of accelerations in the second natural mode. Note that each damping device consists of a linear-to-rotation conversion (facilitated, for example, through a planetary roller screw), with the rotary side connected to a three-phase permanent-magnet synchronous machine. Each machine has a non-negligible rotary inertia J_i, which is reflected into the linear dynamics as a linear inertia J_i/ℓ_i^2, where ℓ_i is the screw lead.

For this example, we define

$$z(t) = \begin{bmatrix} d_b(t)/\bar{d}_b & d_1(t)/\bar{d}_1 & a_b(t)/\bar{a}_b & a_5(t)/\bar{a}_5 \end{bmatrix}^T \tag{36}$$

where d_b is the base drift, d_1 is the drift of the first story relative to the base, a_b is the absolute acceleration of the base, and a_5 is the absolute acceleration of the top floor. The thresholds $\{\bar{d}_b, \bar{d}_1, \bar{a}_b, \bar{a}_5\}$ were chosen as $\{4\text{cm}, 1\text{mm}, 1\text{m/s}^2, 1\text{m/s}^2\}$. Note that for these performance variables, $C_z^T C_z$ is singular, $D_{zf} \neq 0$, and $D_{zw} \neq 0$.

For this system, the optimization approach from Subsection III-A was applied. The optimal values of U_0 for diagonal (i.e., viscous) and regenerative cases were

$$U_0 = \begin{bmatrix}135 & 0\\ 0 & 808\end{bmatrix} \text{kN-s/m}, \quad U_0 = \begin{bmatrix}233 & 77.8\\ 37.7 & 460\end{bmatrix} \text{kN-s/m}$$

respectively, and the corresponding γ values were 5.01 and 4.39. In both cases, it is interesting to note that under the J_p measure, the optimal U_0 exhibits extremely aggressive velocity-proportional damping forces for the TMD, in comparison to tuning techniques. In both cases, term $(2,2)$ of U_0 constitutes a viscous damping term which is on the same order as the critical damping of the TMD. This observation likely implies that under the J_p measure, at least insofar as the optimization of viscous damping is concerned, the use of a TMD is not necessarily effective. However, also note that for the regenerative case, significant coupling exists for the damping forces of the two devices, indicating that the optimal bound is only attained through significant energy transferral between the base to the roof.

We now assess the conservatism of the bound $J_p \leqslant \gamma$, by finding the true bound J_p at the optimum U_0. The worst-case w with $\|w\|_{\mathcal{L}_\infty} \leqslant 1$ can be found by solving a standard optimal control problem [16]; i.e., for T large,

$$\begin{array}{ll} \max & J_p = \|[C_z - D_{zf}U_0B_f^T]x(T) + D_{zw}w(T)\|_2^2 \\ \text{over:} & w(t) \;\;,\;\; t \in [0,T] \\ \text{s.t. :} & \dot{x}(t) = [A - B_fU_0B_f^T]x(t) + B_ww(t) \\ & \|w(t)\|_2 \leqslant 1 \;,\; t \in [0,T] \\ & x(0) = 0 \end{array}$$

(We note that the problem simplifies considerably if $n_w \geqslant n_z$, although this is not the case in the present example.) Solutions to problems such as the one above are standard, and in the interest of space we we suppress the details here. Results for worst-case w functions are shown in Fig.2, where T was taken to be 10s. The true values of J_p for the viscous and regenerative cases, respectively, were 4.60 and 4.32, respectively. This implies a degree of conservatism of 8.0% and 1.6% for the respective γ bounds. It is interesting that the presence of regenerative damping has fundamentally modified the worst-case scenario for w from a periodic oscillation at the natural frequency of the base, to a static load. This is because, due to the parameters chosen for the performance measure z, the U_0 imposes significant damping on the lower natural frequencies of the system for both the viscous and regenerative case. However, the regenerative case is able to bring two modes above critical damping, whereas the viscous case is only able to do this with one mode.

C. Relation to Lyapunov-bounded control

We now show that even when the problem data is restricted to the requirements of the Lyapunov-bounded controller in Sec. II, the bound derived by the process above is less conservative. This is important because the optimal γ found above only *bounds* J_p, but will not in general be equal to it.

COROLLARY 1: Let $C_z = I$, $D_{zf} = O$, and $D_{zw} = O$. For fixed $U_0 \in \mathcal{U}$, let γ^* be the minimal γ over (P,γ,μ,λ), subject to constraints (24), (25), and $\lambda > 0$. Meanwhile, let P_L be the solution to (14), and let γ_L be the bound in (15). Then $\gamma^* \leqslant \gamma_L$ for the same U_0, and irrespective of Q_L.

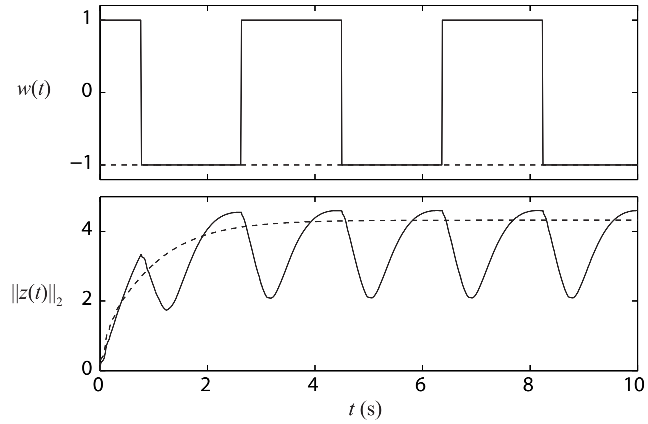

Fig. 2. Worst-case trajectories for w over $t \in [0, 10\text{s}]$, and $\|z(t)\|_2$ over that interval, for viscous (solid) and regenerative (dotted) cases

Proof: Condition (25), for the assumptions above, is equivalent to $\gamma\lambda P > I$, and also implies that at the optimum, $\mu = \gamma$. It is then straight-forward to verify that

$$\gamma = \mu = \gamma_L \;,\; \lambda = \frac{\underline{\rho}\{Q_L\}}{2\bar{\rho}\{P_L\}} \;,\; P = \frac{P_L}{\gamma\lambda\underline{\rho}\{P_L\}} \tag{37}$$

lies in the closure of (24) and (25). ■

IV. CRITERIA FOR PEAK-GAIN BOUNDED NONLINEAR DAMPING CONTROLLERS

Assuming U_0 has been optimized over $\mathcal{U}$ for minimal γ, we now wish to find a criterion for a nonlinear feedback law $\phi : x(t) \to f(t)$ which guarantees to improve on this bound, in a manner analogous to criterion (16) from Theorem 1 for Lyapunov-bounded control. The scope of this paper is merely to reinterpret the results of Theorem 1 in the broader class of performance measures considered here.

Specifically, we have the following result:

LEMMA 1: Let U_0 be the damping matrix which optimizes γ over the domain $(U_0, P, \gamma, \mu, \lambda)$, subject to constraints (24), (25), $\lambda > 0$, and $U_0 \in \mathcal{U}$. Let the optimal values of $(U_0, P, \gamma, \mu, \lambda)$ be denoted $(U_0^*, P^*, \gamma^*, \mu^*, \lambda^*)$. Then any full-state feedback $\phi : x(t) \to f(t) \in \mathcal{F}(v(t))$ satisfying

$$\begin{aligned} x^T(t)\left[A^TP^* + P^*A + \lambda^*P^*\right]x(t) & \\ + 2x^T(t)P^*B_ff(t) + 2x^T(t)P^*B_ww(t) & \\ - \mu^*w^T(t)w(t) < 0 & \end{aligned} \tag{38}$$

$$\begin{aligned} \lambda^*x^T(t)P^*x(t) + (\gamma^* - \mu^*)w^T(t)w(t) & \\ \geqslant \tfrac{1}{\gamma^*}\|C_zx(t) + D_{zf}f(t) + D_{zw}w(t)\|_2^2 & \end{aligned} \tag{39}$$

guarantees the bound $J_p < \gamma^*$. Furthermore, such a feedback law always exists.

Proof: It is known that there always exists an $f(t) \in \mathcal{F}(v(t))$ which simultaneously satisfies (38) and (39), because with $f(t) = -U_0^*v(t) \in \mathcal{F}(v(t))$, these expressions are just quadratic forms on $[x^T(t) \;\; w^T(t)]^T$, with weighting matrices in (24) and (27), respectively. For $V(t) = x^T(t)P^*x(t)$, satisfaction of (38) guarantees that $\dot{V}(t) < 0$ whenever $\lambda^*V(t) > \mu^*w^T(t)w(t)$, while (39) guarantees that $z^T(t)z(t) \leqslant \gamma^*\lambda V(t) + \gamma^*(\gamma^* - \mu^*)w^T(t)w(t)$, in the same way as in Theorem 2. ■

The above lemma suggests a strategy entirely analogous to the Lyapunov-based controllers discussed earlier. At every time, the feedback law $\phi : x \rightarrow f$ makes $\dot{V}(t)$ more negative than it would be with optimal static damping (i.e., $f(t) = -U_0^* v(t)$). This is accomplished, for example, with a control law such as (23), but synthesized with the optimal P^* and U_0^* as derived above. However, this minimization must constrained so as to also satisfy (39). Imposition of this constraint is inconvenient, as it requires knowledge of w. However, we can find more conservative condition which does not, as explained in the lemma below.

LEMMA 2: Condition (39) is conservatively satisfied by

$$\lambda^* x^T(t) P^* x(t) \geqslant \|C_z x(t) + D_{zf} f(t)\|_E^2 \tag{40}$$

where

$$E = \left[\gamma^* I - \tfrac{1}{\gamma^* - \mu^*} D_{zw} D_{zw}^T\right]^{-1} \tag{41}$$

Furthermore, inequalities (40) and (38) can always be satisfied by some $f(t) \in \mathcal{F}(v(t))$.

Proof: Let $C_{zcl}(t) = C_z - D_{zf} U(t) B_f^T$. Then, suppressing time-dependence and $(\cdot)^*$ superscripts, (39) is equivalent to

$$\begin{bmatrix} x \\ w \end{bmatrix}^T \begin{bmatrix} \gamma\lambda P - C_{zcl}^T C_{zcl} & -C_{zcl}^T D_{zw} \\ -D_{zw}^T C_{zcl} & \gamma(\gamma-\mu) I - D_{zw}^T D_{zw} \end{bmatrix} \begin{bmatrix} x \\ w \end{bmatrix} \geqslant 0 \tag{42}$$

which, through a Schur complement, and division by γ, is equivalent to the condition

$$x^T \left(\lambda P - C_{zcl}^T E C_{zcl}\right) x \geqslant \tilde{w}^T \left[D_{zw}^T D_{zw}/\gamma - (\gamma - \mu) I\right] \tilde{w} \tag{43}$$

where we have used the Matrix Inversion Lemma in the first quadratic form, and where $\tilde{w} = w - \left[\gamma(\gamma-\mu) I - D_{zw}^T D_{zw}\right]^{-1} D_{zw}^T C_{zcl} x$. Recognizing that the right-hand side must be negative-definite if $D_{zw} \neq 0$ and if (25) is feasible, we conclude that the condition is conservatively satisfied by enforcement of positivity of the left-hand side, which is equivalent to (40). ■

We thus have the following theorem, the proof of which follows immediately from the observations above, and which is the extension of Theorem 1

THEOREM 3: For any $R = R^T > 0$, and define $K = -U_0^* B_f^T - R^{-1} B_f^T P^*$. Then any full-state feedback controller $\phi : x \rightarrow f$ adhering to

$$\|f(t) - K x(t)\|_R^2 \leqslant x^T(t) P^* B_f R^{-1} B_f^T P^* x(t) \tag{44}$$

$$\|C_z x(t) + D_{zf} f(t)\|_E^2 \leqslant \lambda^* x^T(t) P^* x(t) \tag{45}$$

$$f(t) \in \mathcal{F}(v(t)) \tag{46}$$

for all $x(t)$ guarantees to improve upon the bound $J_p \leqslant \gamma^*$. Furthermore, such a feedback law is guaranteed to exist.

To summarize the ways in which Theorem 3 extends 1, we have that (i) it references performance to the optimized U_0^* as obtained in the previous section, and (ii) it requires an extra constraint be imposed; namely (45). This extra constraint, which requires the optimal solutions for $(U_0^*, P^*, \gamma^*, \mu^*, \lambda^*)$ to evaluate, is the additional restriction which results from nonzero D_{zf} and D_{zw} terms in the definition of z.

V. SUMMARY

The primary purpose of this paper has been to draw connections between the Lyapunov-bounded control design techniques which have been used extensively in the literature on semiactive structural control, and the LMI-based peak-bounded control design techniques which have emerged in the optimal control literature. We have shown that (*a*) Lyapunov-bounded feedback design is a special case of peak-gain-bounded design, which affords more flexibility in the definition of the performance measure, (*b*) For static damping design, LMI methods can be used to optimize tighter performance bounds than those traditionally used for Lyapuonv-based techniques, and (*c*) Criteria for peak-gain-bounded nonlinear state feedback controllers can be found, which ensure that the controller will always improve upon the best bound achievable with linear damping. These criteria are extensions of those which apply to Lyapunov-bounded controllers, but include an extra constraint.

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