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$\mathcal{L}_1$ Adaptive Control for Optical Soliton Propagation

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***Abstract*— This paper presents the application of an $\mathcal{L}_1$ adaptive control architecture to the problem of active dispersion management for propagation of Gaussian-shaped solitons along uncertain optical fibers. The derivations in the paper are based on a reduced variational model for optical soliton propagation. Simulation results demonstrate the benefits of the proposed closed-loop adaptive approach over existing passive dispersion management techniques.**

I. Introduction

Optical solitons are spatially localized, pulse-like, nonlinear waves that almost retain their shapes while propagating in ideal lossless fibers. This absence of dispersion stems from an exact balance between nonlinear and dispersion terms in the conservative form of the Nonlinear Schrödinger Equation which describes ideal fibers [1]. The fact that solitons are spatially localized and propagate with (little to) no deformation makes them the carriers of choice for modern optical communication, as they can be used to encode a bit in a small amount of space through the presence or absence of the pulse in a designated temporal window [2], [3]. However, the balance between nonlinearity and dispersion is not exact in real optical fibers, which may cause the pulse to broaden and "spill over" the designated window. This can result in transmission errors or, if the window size is increased to avoid spill-over, in a reduction of the transmission rate [3].

While error-correcting codes can be used at the receiving end to compensate for dispersion-induced errors, *all-optical* approaches are typically preferred because they do not affect the speed at which information is processed. A typical all-optical dispersion-management technique consists of periodically alternating lengths of fibers with positive and negative group-velocity dispersions, so as to compensate for pulse broadening *on average*. While traveling through the fiber, the pulse experiences broadening and recompression so that, at the end of each compensation element, the width and frequency chirp of the pulse are restored to the initial desired values. This method is effective if both fiber nonlinearity and residual dispersion only slightly affect the evolution of the pulse over one compensation period [4]. If this is not the case, these dispersion maps may fail to work appropriately, especially if the characteristics of the fiber are uncertain.

From a control theory perspective, this (passive) dispersion technique can be seen as open-loop control, where the group-velocity dispersion is the input, pulse width is the output, and the goal is to regulate the output to its desired value, whatever it may be, at a fixed propagation length. With this analogy in mind, it seems tempting to try and use feedback control to improve dispersion management of optical solitons, provided dispersion can be precisely tuned and pulse deformation can be precisely measured in real-time along the fiber. In this sense, several new technologies have been developed recently that allow for some degree of continuous dispersion tuning, from microfluidics-based tunable dispersion materials to fiber-based approaches ranging from fiber gratings to higher-order mode fibers, to name but a few, (see [5] for a comprehensive overview of these fiber-based technologies).

While a number of challenges still need to be overcome to arrive at spatially continuous, fine resolution, sensing and actuation, the time seems ripe for the investigation of advanced control techniques for active dispersion management. A first step in this direction can be found in [6], where a nonlinear state-feedback control law was derived based on controlled Hopf bifurcation. This approach, however, requires *a priori* precise knowledge of the characteristics of the fiber in order to achieve dispersion correction. To overcome this limitation, in this paper, we present the application of an $\mathcal{L}_1$ adaptive control scheme to the problem of active dispersion management for propagation of solitons along *uncertain* fibers. In particular, the developed active dispersion management scheme is able to compensate for the uncertainties encountered along the fiber, and regulates the pulse shape to ensure an error-free transmission. The derivations are based on a reduced model for soliton propagation obtained from the variational approach described in [7].

The paper is organized as follows. Section II describes the soliton propagation problem and gives a brief overview of passive dispersion management. Sections III and IV introduces an adaptive active dispersion management scheme for soliton propagation along uncertain fibers. Simulation results and conclusions are presented in Sections V and VI.

II. Dispersion Managed Transmission Systems

A. Optical Soliton Propagation

For pulses wider than $5\,\mathrm{ps}$, pulse propagation along optical fibers can be described by a generalization of the cubic nonlinear Schrödinger equation [2]:

$$i\frac{\partial\psi}{\partial z} - \frac{1}{2}\beta_2(z)\frac{\partial^2\psi}{\partial t^2} = -\gamma(z)|\psi|^2\psi - \frac{i}{2}\alpha(z)\psi\,,$$

where $\psi = \psi(z,t)$ is the complex-valued envelope of the electric field in the fiber, z is the physical length along the longitudinal axis of the fiber, and t is time initialized to the pulse center, while $\beta_2(z)$, $\gamma(z)$, and $\alpha(z)$ model the spatially varying group-velocity dispersion, Kerr nonlinearity, and effective losses along the optical fiber.

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The envelope ψ is usually conveniently transformed into a new function A defined as:

$$\psi(z,t) = A(z,t)\exp\left(-\frac{1}{2}\int_0^z \alpha(\xi)d\xi\right),$$

which leads to the nonlinear Schrödinger equation:

$$i\frac{\partial A}{\partial z} - \frac{1}{2}\beta_2(z)\frac{\partial^2 A}{\partial t^2} + \kappa(z)|A|^2 A = 0\,, \tag{1}$$

where the parameter $\kappa(z)$ is defined as:

$$\kappa(z) \triangleq \gamma(z)\exp\left(-\int_0^z \alpha(\xi)d\xi\right).$$

If the envelope A is assumed to have the Gaussian form

$$A(z,t) = M(z)\exp\left(-\frac{1}{2}\frac{t^2}{a^2(z)} + ib(z)t^2\right),$$

where $M(z)$, $a(z)$, and $b(z)$ characterize the (complex) amplitude, the width, and the frequency chirp of the pulse, then the variational method described in [7] yields the following set of coupled ordinary differential equations describing the evolution of the parameters $a(z)$ and $b(z)$:

$$\frac{\mathrm{d}a}{\mathrm{d}z} = -2\beta_2(z)ab, \qquad a(0) = a_0, \tag{2a}$$

$$\frac{\mathrm{d}b}{\mathrm{d}z} = 2\beta_2(z)b^2 - \frac{1}{2}\frac{\beta_2(z)}{a^4} - \frac{E_0}{2\sqrt{2}}\frac{\kappa(z)}{a^3}, \qquad b(0) = b_0, \tag{2b}$$

where $E_0 = a(z)|M(z)|^2$ is the energy of the pulse. The use in this paper of Gaussian pulses is motivated by results obtained in theoretical and experimental investigations [8]–[11], which indicate that dispersion managed solitons have nearly Gaussian shape for certain dispersion maps.

B. Passive Dispersion Management

Current dispersion management techniques consist of alternating sections of constant-dispersion fibers whose lengths and dispersions are selected to periodically reproduce the desired pulse at the output of each element [2], [4]. This dispersion management technique offers several advantages relative to constant-dispersion or even dispersion decreasing fibers, and it is seen as a promising approach to increase the transmission capacity in communication systems. In particular, using dispersion management allows to enhance the energy of the transmitted soliton, which increases the signal-to-noise ratio and reduces timing jitter problems. Moreover, dispersion management is an all-optical approach, and it is therefore preferred over non-optical error-correcting codes at the receiving end. Dispersion compensation arrangements for lossless fibers are described in [4]. The effectiveness of passive dispersion management has been demonstrated in several experiments (see [9] and references therein).

However, this approach relies on the assumption that both fiber nonlinearity and residual dispersion only slightly affect the evolution of the pulse over one compensation period. In fact, if the effect of the parameter $\kappa(z)$ related to the Kerr nonlinearity and the loss compensation by optical amplification cannot be neglected, then passive periodic dispersion maps may fail to work appropriately. To overcome this limitation, in the subsequent sections, we propose an active dispersion management scheme that, for pulse durations, power levels, and distances of interest, has the potential to ensure error-free soliton transmission in *uncertain* fibers.

III. Nonadaptive Active Dispersion Management

Next we describe a state-feedback control law for active dispersion management. The approach relies on the assumption that dispersion can be tuned and pulse deformation can be measured in real time along the fiber. In this section, we present a nonadaptive feedback-linearization control law that ensures transmission of a pulse of desired width with frequency chirp below a given pre-specified tolerance for the case in which the parameter $\kappa(z)$ is known and constant.

A. Transformation of State Variables

First, we show that there exists a nonlinear change of variables that transforms the nonlinear system in (2) into a nonlinear state equation with the following structure:

$$\frac{\mathrm{d}\zeta}{\mathrm{d}z} = A(z)\zeta + b_m\lambda(a,b)\big(\beta_2 - \mu(a,b,z)\big)\,, \qquad \zeta(0) = \zeta_0\,, \tag{3}$$

where $\zeta(z) \in \mathbb{R}^2$ is the new system state, $A(z)$ is a 2×2 matrix, $b_m \in \mathbb{R}^2$ is constant, and $\lambda : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ and $\mu : \mathbb{R}\times\mathbb{R}\times\mathbb{R} \to \mathbb{R}$ are nonlinear functions.

To this end, consider the change of variables:

$$\zeta_1(z) \triangleq a^2(z)b^2(z) + \frac{1}{4a^2(z)}\,, \quad \zeta_2(z) \triangleq -\frac{E_0}{2\sqrt{2}}\frac{b(z)}{a(z)}\,. \tag{4}$$

Then, the state equation can be rewritten as:

$$\frac{\mathrm{d}\zeta_1}{\mathrm{d}z} = \kappa(z)\zeta_2, \qquad \zeta_1(0) = a_0^2 b_0^2 + \frac{1}{4a_0^2},$$

$$\frac{\mathrm{d}\zeta_2}{\mathrm{d}z} = \frac{E_0}{\sqrt{2}}\lambda(a,b)\big(\beta_2 - \mu(a,b,z)\big), \quad \zeta_2(0) = -\frac{E_0}{2\sqrt{2}}\frac{b_0}{a_0}, \tag{5}$$

where the nonlinear functions $\lambda(\cdot)$ and $\mu(\cdot)$ are given by:

$$\lambda(a,b) \triangleq \frac{1}{a}\left(\frac{1}{2a^4} - 4b^2\right),$$

$$\mu(a,b,z) \triangleq \left(\frac{1}{2a^4} - 4b^2\right)^{-1}\left(\frac{\kappa_z(z)}{\kappa(z)}b - \frac{E_0}{2\sqrt{2}}\frac{\kappa(z)}{a^3}\right),$$

with $\kappa_z(z)$ being defined as $\kappa_z(z) = \frac{\mathrm{d}\kappa(z)}{\mathrm{d}z}$. Letting $\zeta(z) = [\zeta_1(z),\ \zeta_2(z)]^\top$ and defining $A(z)$ and b_m as:

$$A(z) \triangleq \begin{bmatrix} 0 & \kappa(z) \\ 0 & 0 \end{bmatrix}, \qquad b_m \triangleq \begin{bmatrix} 0 \\ \frac{E_0}{\sqrt{2}} \end{bmatrix},$$

the dynamics in (5) can be rewritten in the compact form in (3) with the initial condition $\zeta_0 = [\zeta_1(0),\ \zeta_2(0)]^\top$.

The state transformation in (4) is a local diffeomorphism (see [12, Lemma 6.2]) in the domain $\mathcal{D}$ defined as:

$$\mathcal{D} \triangleq \left\{(a,b) \in \mathbb{R}^2 \mid a_{\min} \le a \le a_{\max},\ |b| \le \frac{1}{2\sqrt{2}a^2} - \epsilon\right\},$$

where $a_{\min} > 0$ and $a_{\max} > 0$ characterize the range of pulse widths of interest, and ϵ is a small positive constant ($\epsilon \ll \frac{1}{2\sqrt{2}a_{\max}^2}$). We also note that $\lambda(a,b)$ is positive and bounded away from zero for all $(a,b) \in \mathcal{D}$. This is important from a control perspective. Injectivity of the change of variables allows to reformulate the problem in terms of the new system state, while the fact that $\lambda(a,b)$ is positive and bounded away from zero implies that the system in (5) is controllable.

B. Nonadaptive Feedback Linearization Control

For the design of the nonadaptive feedback linearization control law, we assume a nominal constant value κ_0 for the parameter $\kappa(z)$, which leads to the nominal soliton dynamics:

$$\frac{\mathrm{d}\zeta}{\mathrm{d}z} = A_0\zeta + b_m\lambda(a,b)\big(\beta_2 - \mu_0(a,b,z)\big)\,, \quad \zeta(0) = \zeta_0\,, \tag{7}$$

where A_0 and $\mu_0(\cdot)$ are defined as:

$$A_0 \triangleq \begin{bmatrix} 0 & \kappa_0 \\ 0 & 0 \end{bmatrix}, \quad \mu_0(a,b) \triangleq -\left(\frac{1}{2a^4} - 4b^2\right)^{-1} \frac{E_0}{2\sqrt{2}} \frac{\kappa_0}{a^3}.$$

For this nominal system, the control objective is to design an active dispersion control law $\beta_2(z)$ to ensure that $\zeta_1(z)$ tracks the setpoint $\zeta_{1\mathrm{cmd}}$ given by:

$$\zeta_{1\mathrm{cmd}} \triangleq 1/(4a_{\mathrm{cmd}}^2)\,,$$

where a_{cmd}, satisfying $a_{\min} < a_{\mathrm{cmd}} < a_{\max}$, is the desired pulse width. The point $\zeta_r \triangleq (\zeta_{1\mathrm{cmd}},\ 0)$ in $\mathcal{D}^*$ corresponds uniquely to $(a_{\mathrm{cmd}},\ 0)$ in $\mathcal{D}$, which implies that, if $\zeta_1(z)$ tracks $\zeta_{1\mathrm{cmd}}$, then the transmitted pulse has desired width.

For this purpose, we consider the following state-feedback control law for the group-velocity dispersion:

$$\beta_2(z) = \mu_0(a(z), b(z)) + \lambda^{-1}(a(z), b(z))u(z)\,, \tag{8}$$

where $u(z)$ is a control signal, yet to be defined. With this control law, the nominal system in (7) becomes:

$$\frac{\mathrm{d}\zeta(z)}{\mathrm{d}z} = A_m\zeta(z) + b_m u(z)\,, \qquad \zeta(0) = \zeta_0\,.$$

Next, we define the nominal control signal $u_0(z)$ as:

$$u_0(z) = k^\top\zeta(z) + k_g\zeta_{1\mathrm{cmd}}\,, \tag{9}$$

where the feedback gain k is chosen to ensure that, for a given desired rate of convergence λ_c, there exists a positive definite matrix X, $X = X^\top > 0$, such that:

$$(A_m + \lambda_c\mathbb{I}_n)^\top X + X(A_m + \lambda_c\mathbb{I}_n) < 0\,,$$

with $A_m \triangleq A_0 + b_m k^\top$. The gain k_g is given by:

$$k_g \triangleq -\left(c^\top A_m^{-1} b_m\right)^{-1}\,,$$

where c is the constant vector $c \triangleq [1,\ 0]^\top$.

The dispersion law in (8) with the nominal control signal in (9) leads to the nominal closed-loop system dynamics:

$$\frac{\mathrm{d}\zeta(z)}{\mathrm{d}z} = A_m\zeta(z) + b_m k_g\zeta_{1\mathrm{cmd}}\,, \qquad \zeta(0) = \zeta_0\,.$$

The point ζ_r is a locally exponentially stable equilibrium of this nominal closed-loop soliton propagation dynamics. An estimate of the region of attraction of this point is given by

$$\Omega^*_\gamma \triangleq \left\{x \in \mathbb{R}^2 \mid (x - \zeta_r)^\top X(x - \zeta_r) \le \gamma\right\}\,,$$

where γ is any positive constant such that $\Omega^*_\gamma \subset \mathcal{D}^*$.

IV. $\mathcal{L}_1$ Adaptive Control Augmentation

In the previous section, we have designed a nonadaptive feedback law which ensures asymptotic tracking of a desired pulse width when the parameter $\kappa(z)$ is known and constant. However, in a real communication channel, this parameter might not be known or it might experience large variations along the fiber. Under these circumstances, the active dispersion law in (8) –where κ_0 is now the best available guess for $\kappa(z)$– leads to the partially closed-loop dynamics:

$$\frac{\mathrm{d}\zeta}{\mathrm{d}z} = A_0\zeta + b_m\big(u + \eta_1(a,b,z)\big) + b_{um}\eta_2(a,b,z), \ \ \zeta(0) = \zeta_0, \tag{10}$$

where $b_{um} \triangleq [1,\ 0]^\top$, and $\eta_i(\cdot)$ are nonlinear uncertainties:

$$\eta_1(a,b,z) \triangleq \frac{1}{a}\left(\frac{E_0}{2\sqrt{2}}\frac{\kappa(z) - \kappa_0}{a^3} - \frac{\kappa_z(z)}{\kappa(z)}b\right)\,,$$

$$\eta_2(a,b,z) \triangleq -\frac{E_0}{2\sqrt{2}}\frac{b}{a}\left(\kappa(z) - \kappa_0\right)\,.$$

In the partially closed-loop dynamics in (10), $\eta_1(\cdot)$ and $\eta_2(\cdot)$ represent, respectively, the matched and unmatched components of the uncertainties. We note that if the parameter $\kappa(z)$ and its first and second derivatives with respect to the physical length z are assumed to satisfy the bounds

$$0 < \kappa_{\min} \le \kappa(z) \le \kappa_{\max} < \infty\,,$$

$$|\mathrm{d}\kappa(z)/\mathrm{d}z| \le d_{\kappa_z} < \infty\,, \qquad \left|\mathrm{d}^2\kappa(z)/\mathrm{d}z^2\right| \le d_{\kappa_{zz}} < \infty\,,$$

then, for any given set $\Omega^* \subset \mathcal{D}^*$, there exist positive constants $K^{\Omega^*}_{1\zeta}$, $K^{\Omega^*}_{1z}$, $K_{2\zeta}$, $K^{\Omega^*}_{2z}$, and B_1 such that:

$$\left\|\frac{\partial\eta_1}{\partial\zeta}\right\|_1 \le K^{\Omega^*}_{1\zeta}\,, \quad \left|\frac{\partial\eta_1}{\partial z}\right| \le K^{\Omega^*}_{1z}\,, \quad |\eta_1(a_{\mathrm{cmd}}, 0, z)| \le B_1\,,$$

$$\left\|\frac{\partial\eta_2}{\partial\zeta}\right\|_1 \le K_{2\zeta}\,, \quad \left|\frac{\partial\eta_2}{\partial z}\right| \le K^{\Omega^*}_{2z}\,.$$

while $\eta_2(a_{\mathrm{cmd}}, 0, z) = 0$.

For the system (10), we define the control signal $u(z)$ as:

$$u(z) = u_0(z) + u_{\mathrm{ad}}(z)\,,$$

where $u_0(z)$ was defined in (9), while $u_{\mathrm{ad}}(z)$ is the adaptive augmentation signal. This control signal yields the dynamics:

$$\begin{aligned}\frac{\mathrm{d}\zeta}{\mathrm{d}z} &= A_m\zeta \ + \ b_m k_g\zeta_{1\mathrm{cmd}} \ + \ b_m\big(u_{\mathrm{ad}} + \eta_1(a,b,z)\big) \\ &\quad + \ b_{um}\eta_2(a,b,z)\,, \qquad \zeta(0) = \zeta_0\,.\end{aligned} \tag{11}$$

The purpose of the adaptive augmentation loop is thus to compensate for the uncertainties $\eta_1(\cdot)$ and $\eta_2(\cdot)$. For this purpose, we consider the implementation of an $\mathcal{L}_1$ adaptive controller. The main benefit of $\mathcal{L}_1$ adaptive architectures is the decoupling of identification from control, which enables fast adaptation without sacrificing robustness. Fast adaptation allows for compensation of rapidly varying uncertainties and significant changes in system dynamics, and it is critical to achieve predictable performance without enforcing persistency of excitation or resorting to high-gain feedback [13].

The $\mathcal{L}_1$ architecture developed next is based on the theory presented in [13, Section 3.2], which can be easily extended to systems in which nonlinearities verify only *local* assumptions on the uniform boundedness of their partial derivatives.

A. Control Architecture

The $\mathcal{L}_1$ adaptive controller is defined for the tracking error dynamics by considering the error state $e_\zeta \triangleq \zeta - \zeta_r$. In terms of this error state, the dynamics in (11) take the form:

$$\frac{\mathrm{d}e_\zeta}{\mathrm{d}z} = A_m e_\zeta \;+\; b_m\big(u_{\mathrm{ad}} + \eta_1(a,b,z)\big) \\ +\; b_{um}\eta_2(a,b,z)\,, \qquad e_\zeta(0) = \zeta_0 - \zeta_r\,. \tag{12}$$

The $\mathcal{L}_1$ adaptive control architecture consists of a *fast adaptation scheme* and a *control law*. The fast adaptation scheme includes a state predictor and appropriately designed adaptation laws, which are used to generate estimates of the system uncertainties. Based on these estimates, the control law generates a control signal as the output of low-pass filter. The elements of the $\mathcal{L}_1$ adaptive controller are detailed next:

1) State predictor: Consider the following state predictor:

$$\frac{\mathrm{d}\hat{e}_\zeta}{\mathrm{d}z} = A_m\hat{e}_\zeta \;+\; b_m\big(u_{\mathrm{ad}} + \hat{\eta}_1\big) \;+\; b_{um}\hat{\eta}_2 \\ +\; K_{\mathrm{sp}}\tilde{e}_\zeta\,, \qquad \hat{e}_\zeta(0) = \zeta_0 - \zeta_r\,, \tag{13}$$

where $K_{\mathrm{sp}} \in \mathbb{R}^{n\times n}$ is such that the matrix A_s defined as $A_s \triangleq A_m + K_{\mathrm{sp}}$ is Hurwitz, $\tilde{e}_\zeta(z) \triangleq \hat{e}_\zeta(z) - e_\zeta(z)$ is the prediction error, while $\hat{\eta}_1(z)$ and $\hat{\eta}_2(z)$ represent respectively the estimates of $\eta_1(\cdot)$ and $\eta_2(\cdot)$ and are given by:

$$\hat{\eta}_1(z) \triangleq \hat{\theta}_1(z)\varphi(a(z),b(z)) + \hat{\sigma}_1(z), \quad \hat{\eta}_2(z) \triangleq \hat{\theta}_2(z)\zeta_2(z),$$

with $\hat{\theta}_1(z)$, $\hat{\sigma}_1(z)$, and $\hat{\theta}_2(z)$ being the adaptive estimates, and $\varphi(\cdot)$ being the regressor:

$$\varphi(a(z),b(z)) \triangleq \left\| \left[\begin{smallmatrix} a(z) \\ b(z) \end{smallmatrix}\right] \right\|_\infty .$$

2) Adaptation laws: The adaptation laws are given by:

$$\begin{aligned} \frac{\mathrm{d}\hat{\theta}_1}{\mathrm{d}z} &= \Gamma\,\mathrm{Proj}_{\Theta_1}(\hat{\theta}_1, -\tilde{e}_\zeta^\top P b_m \varphi(a,b)), & \hat{\theta}_1(0) &= \hat{\theta}_{10}, \\ \frac{\mathrm{d}\hat{\sigma}_1}{\mathrm{d}z} &= \Gamma\,\mathrm{Proj}_{\Sigma_1}(\hat{\sigma}_1, -\tilde{e}_\zeta^\top P b_m), & \hat{\sigma}_1(0) &= \hat{\sigma}_{10}\,, \\ \frac{\mathrm{d}\hat{\theta}_2}{\mathrm{d}z} &= \Gamma\,\mathrm{Proj}_{\Theta_2}(\hat{\theta}_2, -\tilde{e}_\zeta^\top P b_{um}\zeta_2), & \hat{\theta}_2(0) &= \hat{\theta}_{20}, \end{aligned} \tag{14}$$

where $\Gamma > 0$ is the adaptation gain, $\mathrm{Proj}(\cdot,\cdot)$ denotes the projection operator [14], and $P = P^\top > 0$ is the solution to the algebraic Lyapunov equation $A_s^\top P + PA_s = -Q$, $Q = Q^\top > 0$. The sets Θ_1, Σ_1, and Θ_2 are the compact convex sets used in the definition of the projection operators.

3) Control law: The $\mathcal{L}_1$ control signal is generated as:

$$u_{\mathrm{ad}}(s) = -C(s)\hat{\eta}_1(s) - C(s)H_m^{-1}(s)H_{um}(s)\hat{\eta}_2(s)\,, \tag{15}$$

where $C(s)$ is a strictly proper and stable low-pass filter with unit DC gain, $H_m(s)$ and $H_{um}(s)$ are defined as:

$$H_m(s) \triangleq c^\top (s\mathbb{I} - A_m)^{-1} b_m\,, \\ H_{um}(s) \triangleq c^\top (s\mathbb{I} - A_m)^{-1} b_{um}\,,$$

while $\hat{\eta}_1(s)$ and $\hat{\eta}_2(s)$ are the Laplace transforms of $\hat{\eta}_1(z)$ and $\hat{\eta}_2(z)$. The structure of $C(s)$ needs to ensure that $C(s)H_m^{-1}(s)H_{um}(s)$ is proper. Also, from the definitions of c, A_m, and b_m, it follows that $C(s)H_m^{-1}(s)H_{um}(s)$ is stable. The first term of the control law compensates for matched uncertainties, while the second term compensates for the effect of unmatched uncertainties at the system output.

B. Closed-Loop Stability and Performance

For the system in (12) with the $\mathcal{L}_1$ controller in (13)-(15), we can derive sufficient stability conditions similar to the ones in [13, Section 3.2]. In particular, the design of $C(s)$ is subject to an $\mathcal{L}_1$-norm condition similar to [13, Equation (3.71)], while the adaptation gain and the projection bounds need to be chosen sufficiently large (see [13, Equations (3.94) and (3.95)]).

If the $\mathcal{L}_1$-norm condition and the design constraints for the adaptation scheme are satisfied, then the closed-loop adaptive system is stable and, moreover, one can derive computable uniform performance bounds for the error signals between the input and state signals of the actual closed-loop system –$u_{\mathrm{ad}}(z)$ and $\zeta(z)$– and the input and state signals of the *closed-loop reference system* –$u_{\mathrm{ref}}(t)$ and $\zeta_{\mathrm{ref}}(t)$–, which is defined in terms of the nonadaptive version of the adaptive controller and characterizes the best achievable performance of the control law in (15) by assuming perfect knowledge of the uncertainties. The next theorem summarizes this result.

Theorem 1: If the design of the $\mathcal{L}_1$ adaptive controller satisfies the (sufficient) stability conditions, then:

$$\zeta \in \mathcal{D}^*\,, \qquad \|u_{\mathrm{ad}}\|_{\mathcal{L}_\infty} \le \rho_u\,, \\ \|\zeta - \zeta_{\mathrm{ref}}\|_{\mathcal{L}_\infty} \le \gamma_\zeta\,, \qquad \|u_{\mathrm{ad}} - u_{\mathrm{ref}}\|_{\mathcal{L}_\infty} \le \gamma_u\,,$$

where ρ_u is a positive constant, while γ_ζ and γ_u are uniform performance bounds inverse proportional to $\sqrt{\Gamma}$.

Proof: This result can be proven along the same lines as the proof of [13, Theorem 3.2.1]. □

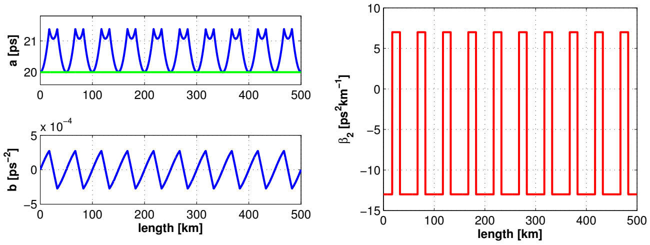
Remark 1: Theorem 1 implies that, both in transient and steady-state, one can achieve arbitrary close tracking of the reference system by increasing the adaptation gain. The reader is referred to [13, Section 3.2] for details on the stability proof, the derivation of the performance bounds, and a discussion of how these bounds can be used for ensuring transient response with *desired* specifications.

V. Simulation Results

In this section, we apply the active dispersion management scheme developed in this paper to the transmission of 20 ps Gaussian pulses of peak power 5 mW ($E_0 = 100$ ps mW) along a 500 km fiber. The nominal value for the parameter $\kappa(z)$ is $\kappa_0 = 5{\cdot}10^{-3}\ \mathrm{km}^{-1}\mathrm{mW}^{-1}$, while $\kappa_{\min}$ and $\kappa_{\max}$ are taken, respectively, as $2\cdot 10^{-3}\ \mathrm{km}^{-1}\mathrm{mW}^{-1}$ and $8\cdot 10^{-3}\ \mathrm{km}^{-1}\mathrm{mW}^{-1}$. To better illustrate the benefits of active dispersion management, we first consider the application of a passive dispersion managed transmission system.

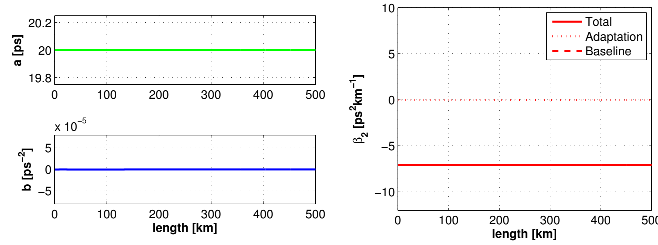
A. Pulse Propagation with Passive Dispersion Management

Based on the nominal value of $\kappa(z)$, we now design a periodic map for passive dispersion compensation. We consider compensation elements of 50 km comprising fiber segments with negative, positive, and negative dispersions.



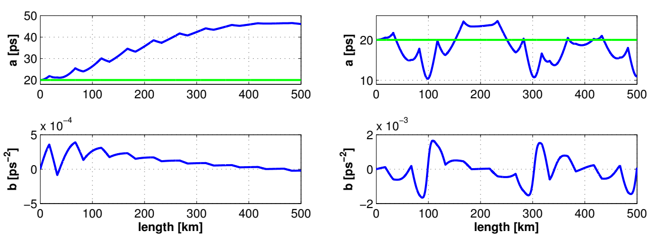
(a) Soliton propagation (b) Group-velocity dispersion

Fig. 1: Passive dispersion management. Propagation of a 20 ps pulse along nominal fiber ($\kappa(z) \equiv \kappa_0$).



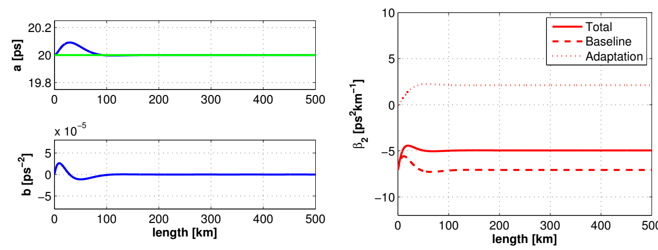
(a) Soliton propagation (b) Group-velocity dispersion

Fig. 3: Active dispersion management. Propagation of a 20 ps pulse along nominal fiber ($\kappa(z) \equiv \kappa_0$).



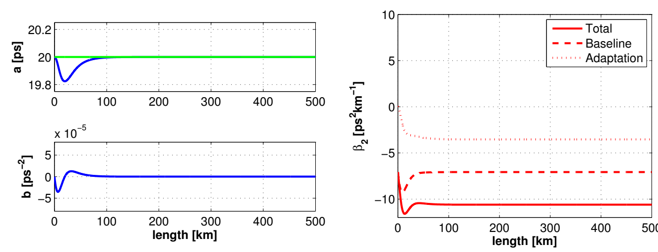
(a) Low nonlinearity: $\kappa(z) \equiv 0.7\kappa_0$ (b) High nonlinearity: $\kappa(z) \equiv 1.5\kappa_0$

Fig. 2: Passive dispersion management. Propagation of a 20 ps pulse along off-nominal fibers.



(a) Soliton propagation (b) Group-velocity dispersion

Fig. 4: Active dispersion management. Propagation of a 20 ps pulse along off-nominal fiber ($\kappa(z) \equiv 0.7\kappa_0$).



(a) Soliton propagation (b) Group-velocity dispersion

Fig. 5: Active dispersion management. Propagation of a 20 ps pulse along off-nominal fiber ($\kappa(z) \equiv 1.5\kappa_0$).

The dispersion values alternate between -13 ps^{2}km^{-1} and $+7$ ps^{2}km^{-1}, and the segment with positive dispersion is centered in the compensation element and has a length of 15.32 km. Figure 1 shows the evolution of $a(z)$ and $b(z)$ along the fiber. Although the pulse experiences periodic breathing along each element, it has desired width and zero chirp at the output of each compensation element.

This passive approach may fail to work if the nominal value κ_0 is not close enough to the actual value of $\kappa(z)$. For instance, Figure 2a shows the evolution of $a(z)$ and $b(z)$ along the fiber for the case $\kappa(z) \equiv 3.5 \cdot 10^{-3}$ km^{-1}mW^{-1} (that is, $\kappa(z)$ is 30% smaller than its nominal value κ_0). As it can be observed, the pulse broadens up to 45 ps, which may result in transmission errors if it "spills over" the designated window. Similarly, Figure 2b illustrates the case in which $\kappa(z)$ is 50% larger than its nominal value ($\kappa(z) \equiv 7.5 \cdot 10^{-3}$ km^{-1}mW^{-1}). The dispersion map is not able to reproduce the desired pulse at the end of the fiber, and the transmitted pulse exhibits a final width of 11 ps.

B. Pulse Propagation with Active Dispersion Management

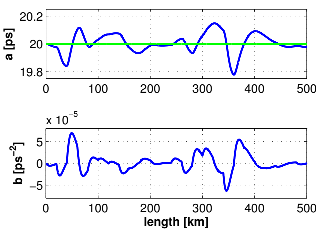
Next, we consider the active dispersion management approach developed in Section III and IV. For this particular application, we select the following control parameters:

$$k^\top = \left[\, 1.81 \;\; 0.20 \,\right] \cdot 10^{-2}, \qquad k_g = 1.81 \cdot 10^{-2},$$
$$K_{\rm sp} = \left[\begin{smallmatrix} 0 & 0 \\ 38.40 & 0.1440 \end{smallmatrix}\right], \quad C(s) = \frac{1}{(\frac{s}{0.1}+1)(\frac{s}{0.3}+1)(\frac{s}{3}+1)},$$
$$\Gamma = 5 \cdot 10^4, \qquad Q = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 100 \end{smallmatrix}\right], \qquad \Theta_1 = \left[\, -100, \; 100 \,\right],$$
$$\Sigma_1 = \left[\, -10^{-3}, \; 10^{-3} \,\right], \qquad \Theta_2 = \left[\, -3 \cdot 10^{-3}, \; 3 \cdot 10^{-3} \,\right].$$

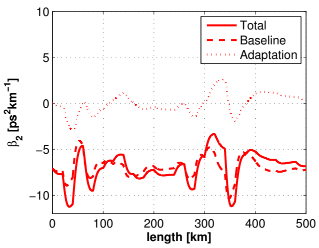
First, we show that the active dispersion management control scheme is able to transmit the soliton with desired width and zero frequency chirp along the fiber for the nominal case, that is when $\kappa(z) \equiv \kappa_0$. Figure 3 presents the evolution of the parameters $a(z)$ and $b(z)$ as well as the resulting dispersion control signal. The nonadaptive control law is able to set the group-velocity dispersion to ensure the correct transmission of the pulse, while the contribution of the adaptive augmentation loop remains zero as expected.

Next, we present simulation results to illustrate the performance of the active dispersion management scheme for off-nominal conditions. First, we consider the same scenarios discussed in the previous section for passive dispersion management. Figures 4 and 5 show, respectively, the pulse propagation for the cases $\kappa(z) \equiv 3.5 \cdot 10^{-3}$ km^{-1}mW^{-1} and $\kappa(z) \equiv 7.5 \cdot 10^{-3}$ km^{-1}mW^{-1}. In these two scenarios, the adaptive controller adjusts the dispersion so as to ensure an error-free transmission at the end of the optical fiber.

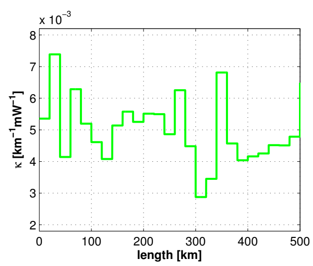
The active dispersion management scheme can ensure a correct soliton transmission even for spatially-varying $\kappa(z)$. Figures 6 and 7 show, respectively, the pulse propagation along fibers in which $\kappa(z)$ has a piecewise constant distribution, and a biased sinusoidal pattern. These simulation results demonstrate that the $\mathcal{L}_1$ adaptive controller is able to compensate for the uncertainties along the fiber and regulate the width and chirp of the pulse around the desired values, thus ensuring a correct transmission.



(a) Soliton propagation

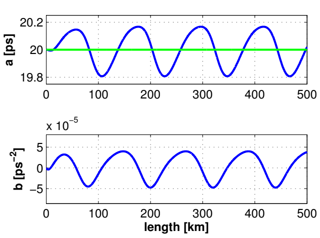


(b) Group-velocity dispersion

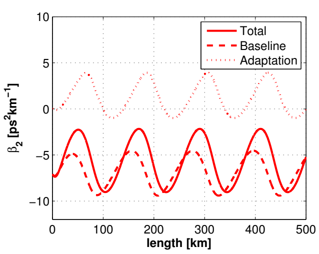


(c) Parameter $\kappa(z)$

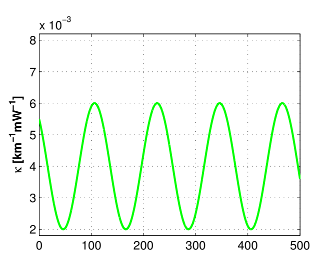
Fig. 6: Active dispersion management. Propagation of a 20 ps pulse along off-nominal fiber with spatially-varying piecewise constant nonlinearity.



(a) Soliton propagation



(b) Group-velocity dispersion



(c) Parameter $\kappa(z)$

Fig. 7: Active dispersion management. Propagation of a 20 ps pulse along off-nominal fiber with spatially-varying sinusoidal nonlinearity.

VI. Conclusions and Future Work

Using a reduced variational model, this paper proposed the application of an $\mathcal{L}_1$ adaptive control architecture to the problem of active dispersion management for propagation of Gaussian-shaped pulses along uncertain optical fibers. The proposed feedback scheme does not require *a priori* precise knowledge of the characteristics of the fiber, and can be used to shape soliton pulse width along the fiber by controlling transient spatial dynamics. The approach relies on the assumption that dispersion can be accurately tuned and pulse deformation can be precisely measured in real time along the fiber.

Simulation results demonstrated that using tunable dispersion fibers under feedback control can successfully propagate pulse-width solitons in the presence of spatially-varying uncertainties. This is in contrast to current passive dispersion management techniques, which may fail to work appropriately under these conditions.

Future work will address delay and spatial quantization effects caused by the sensors and actuators that could currently be practically used along optical fibers in active dispersion management transmission systems.

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