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State-Feedback Stabilizability, Optimality, and Convexity in Switched Positive Linear Systems

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***Abstract*— The present paper is concerned with state-feedback stabilizability in discrete-time switched positive linear systems. Necessary and sufficient conditions for state-feedback exponential stabilizability, in this class of switched systems, are presented. It is shown that, a switched positive linear system is state-feedback exponentially stabilizable if and only if an associated sequence, whose elements are computable via linear programming, has an element smaller than one. Also, a switched positive linear system is state-feedback exponentially stabilizable if and only if there exits a product of their modes matrices whose spectral radius is smaller than one. Equivalently, the state-feedback exponential stabilizability of a switched positive linear system is shown to be equivalent to the solvability of an associated dynamic programming equation on a given convex cone. That associated dynamic programming equation it is shown to have at most one solution. This unique solution, of the associated dynamic programming equation, is shown to be concave, monotonic, positively homogeneous, and the optimal cost functional of a related optimal control problem (involving the switched positive linear system) whose complete solution is also presented in this communication.**

I. Introduction

In this paper, we use the term switched system to refer to a dynamical system described by a differential or difference equation whose right hand side is dynamically selected from a given finite set of functions, and this selection is governed by a function (of the time) termed switching signal.

Stability and stabilizability problems, concerning switched systems, have lately been extensively investigated, and some of the vast research in this area is documented in various surveys [4], [3], [11] and monographs [13], [10], [5], [12].

The topic of the present communication is concerned with the problem of finding necessary and sufficient conditions for the existence of a state-feedback that exponentially stabilizes a discrete-time switched linear systems. In [6] we presented complete and general solutions for that problem. That paper includes three different (but equivalent) necessary and sufficient conditions for the existence of a state-feedback that exponentially stabilizes a general switched linear system. Convex analysis was also already used in [6] in order to obtain a sufficient (conservative) state-feedback stabilizability condition based on solving convex programming problems. Further extensions and refinements were also included in [7] and [8]. In the present work we specialize some of those results and techniques (developed in [6] and [7]) in order to address, in a convex (but not conservative) manner, the state-feedback stabilizability problem in discrete-time switched positive linear systems.

The organization of the paper is as follows. Mathematical preliminaries are in section II. Different but equivalent necessary and sufficient conditions for state-feedback exponential stabilizability of the switched positive linear systems are presented and proved in section III. A sequence, whose elements are computable by solving linear programming problems, is associated to the switched positive linear system, and we prove that the state-feedback stabilizability of the switched system is equivalent to the existence of an element smaller than one in that sequence. The state-feedback stabilizability of the switched system is also proved to be equivalent to the solvability of an associated dynamic programming equation on some specific convex cone. Results regarding the solvability of the associated dynamic programming equation are in section IV. Some observations regarding stabilizing state-feedback mappings and their corresponding Lyapunov functions are included in section V. The complete solution of a related optimal control problem is presented in section VI. Summary and concluding remarks are in section VII.

Most of the notation used through the paper is standard. $\mathbb{Z}^+$ denote the non-negative integers. For $k \in \mathbb{Z}^+$, we use $\mathbb{Z}^{[0,k]}$ to also denote the set $\mathbb{Z}^{[0,k]} = \{0, \ldots, k\}$. We use l_+^n to denote the set of all the sequences $\{x_k\} \subset \mathbb{R}^n$, $k \in \mathbb{Z}^+$. For $x \in \mathbb{R}^n$ we use $\|x\| = \sqrt{\sum_{i=1}^n x_i^2}$, $\|x\|_1 = \sum_{i=1}^n |x_i|$, $\|x\|_\infty = \max_{i\in\{1,\ldots,n\}} |x_i|$, and we denote by $|x| \in (\mathbb{R}^n)^+$ the vector whose i-element is $|x_i|$. $\mathbf{1} \in (\mathbb{R}^n)^+$ is the vector with all its elements equal to 1. For $x \in \mathbb{R}^n$, $y \in \mathbb{R}^n$, and $X \in \mathbb{R}^{n\times n}$, we use $x \geq 0$, $x \geq y$, and $X \geq 0$ to mean $x \in (\mathbb{R}^n)^+$, $x - y \in (\mathbb{R}^n)^+$, and $X \in (\mathbb{R}^{n\times n})^+$ respectively. $\rho(X)$ is the spectral radius of $X \in \mathbb{R}^{n\times n}$, and $\lambda_{PF}(X)$ is the Perron-Frobenius eigenvalue of $X \in (\mathbb{R}^{n\times n})^+$.

II. Preliminaries

Let $N \in \mathbb{Z}^+$, $N > 0$, be given. We denote by $\mathcal{Q}$ the set $\mathcal{Q} = \{1, \ldots, N\}$. Let us introduce the following sets of control functions (or switching signals)

$$\begin{aligned} \mathcal{Q}_k &= \{q \mid q : \mathbb{Z}^{[0,k-1]} \longrightarrow \mathcal{Q}\}, \ k \in \mathbb{Z}^+, \ k > 0, \\ \mathcal{Q}_\infty &= \{q \mid q : \mathbb{Z}^+ \longrightarrow \mathcal{Q}\}. \end{aligned}$$

Let $A_i \in \mathbb{R}^{n\times n}$, $i \in \{1, \ldots, N\}$, be given matrices. The present article is concerned with the dynamical system

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described by

$$x(k+1) = A_{q(k)}x(k)\ ,\ k \in \mathbb{Z}^+\ ,\ x(0) = x_0 \in \mathbb{R}^n\ ,\ q \in \mathcal{Q}_\infty\ , \tag{1}$$

which will be referred as the switched linear system (1). And in case that $A_i \geq 0$, $\forall i \in \{1, \ldots, N\}$, that is, when all the matrices are element-wise non-negative, we refer to such system as the switched positive linear system (1). The motion of such a controlled dynamical system will be denoted by $x(\cdot\,; x_0, q)$. It is clear that the non-negative orthant, $(\mathbb{R}^n)^+$, is an invariant set for the switched positive linear system (1).

To each mapping $\kappa : \mathbb{R}^n \longrightarrow \mathcal{Q}$ we associate the diagonal (or static) operator $\mathcal{F}_\kappa : l_+^n \longrightarrow \mathcal{Q}_\infty$ defined by

$$\mathcal{F}_\kappa(x)(k) = \kappa(x(k))\ ,\ k \in \mathbb{Z}^+ .$$

It is clear that if we also associate to each mapping $\kappa : \mathbb{R}^n \longrightarrow \mathcal{Q}$ the (closed-loop) dynamical system described by

$$x_{cl}(k+1) = A_{\kappa(x_{cl}(k))}x_{cl}(k)\ ,\ k \in \mathbb{Z}^+\ ,\ x_{cl}(0) = x_0 \in \mathbb{R}^n\ , \tag{2}$$

then, it follows that $x(\cdot\,; x_0, \mathcal{F}_\kappa(x)) = x_{cl}(\cdot\,; x_0)$.

The following definitions, [6], are used in this work.

Definition 1: The switched system (1) is state-feedback exponentially stabilizable whenever there exist a mapping $\kappa : \mathbb{R}^n \longrightarrow \mathcal{Q}$ and scalars $\alpha \geq 1$ and $0 < \beta < 1$ such that the motions of the associated (closed-loop) dynamical system (2) satisfy

$$\|x_{cl}(k; x_0)\| \leq \alpha\beta^k\|x_0\|\ ,\ k \in \mathbb{Z}^+\ ,\ x_0 \in \mathbb{R}^n .$$

Definition 2: The switched system (1) is uniformly exponentially convergent whenever there exist scalars $\alpha \geq 1$ and $0 < \beta < 1$ that obey the following property:
For each $x_0 \in \mathbb{R}^n$ there exists $q_{x_0} \in \mathcal{Q}_\infty$ such that the corresponding motion of (1) satisfies

$$\|x(k; x_0, q_{x_0})\| \leq \alpha\beta^k\|x_0\|\ ,\ k \in \mathbb{Z}^+ .$$

In [6] it was proved that the above two concepts are in fact equivalent. That is, we proved that:

Theorem 1 ([6]): The switched linear system (1) is state-feedback exponentially stabilizable if and only if, it is uniformly exponentially convergent.

III. Convex Conditions for State-Feedback Stabilizability in Switched Positive Linear Systems

We associate to the sets $\mathcal{Q}_k$, $k \in \mathbb{Z}^+$, $k > 0$, of control functions, the following sets $\mathcal{S}_k$, $k \in \mathbb{Z}^+$, $k > 0$, of matrices:

$$\mathcal{S}_k = \{S \in \mathbb{R}^{n\times n}\ :\ S = A_{q(k-1)} \ldots A_{q(0)}\,, q \in \mathcal{Q}_k\} .$$

We also associate to the switched positive linear system (1), the sequence of functions $\{\Pi_k\}$, where $\Pi_k : (\mathbb{R}^n)^+ \longrightarrow \mathbb{R}^+$, $k \in \mathbb{Z}^+$, $k > 0$, is defined by

$$\Pi_k(x_0) = \min_{q\in\mathcal{Q}_k} \|x(k; x_0, q)\|_1 = \min_{S\in\mathcal{S}_k} \mathbf{1}^*Sx_0\ , \tag{3}$$

and the sequence $\{\pi_k\}$, where $\pi_k \in \mathbb{R}^+$, $k \in \mathbb{Z}^+$, $k > 0$, is defined by

$$\begin{aligned}\pi_k &= \max_{x_0\in\{x_0\in(\mathbb{R}^n)^+:\|x_0\|_1\leq 1\}} \Pi_k(x_0) \\ &= \max_{x_0\in\{x_0\in(\mathbb{R}^n)^+:\|x_0\|_1\leq 1\}} \min_{S\in\mathcal{S}_k} \mathbf{1}^*Sx_0\ .\end{aligned} \tag{4}$$

Some useful simple properties of these sequences $\{\Pi_k\}$ and $\{\pi_k\}$ are included in the next Fact.

Fact 1: For each given $k \in \mathbb{Z}^+$, $k > 0$, it follows that

(1) Π_k is concave.

(2) $\Pi_k(\lambda x_0) = \lambda\Pi_k(x_0)\ ,\ \lambda \in (\mathbb{R})^+\ ,\ x_0 \in (\mathbb{R}^n)^+ .$

(3) $\pi_k = \max_{x_0\in\{x_0\in(\mathbb{R}^n)^+:\|x_0\|_1=1\}} \Pi_k(x_0)\,.$

(4) $\pi_k = \max_{\|x_0\|_1\leq 1} \min_{q\in\mathcal{Q}_k} \|x(k; x_0, q)\|_1\,.$

(5) $\pi_k \leq \min_{S\in\mathcal{S}_k} \max_{j\in\{1,\ldots,n\}} \|(S)_j\|_1$
$= \min_{S\in\mathcal{S}_k} \|S^*\mathbf{1}\|_\infty\,.$

(6) For each given $h \in \mathbb{Z}^+$, $h > 0$, it follows that

$$\pi_{hk} \leq \left(\pi_k\right)^h .$$

We further associate to the switched positive linear system (1) the sequence $\{\delta_k\}$, where $\delta_k \in \mathbb{R}^+$, $k \in \mathbb{Z}^+$, $k > 0$, is defined via the following linear programming problem

$$\delta_k = \min_{\nu\in\mathbb{R}^+,\ \lambda\in(\mathbb{R}^{N^k})^+ :\ \sum_{q\in\mathcal{Q}_k}\lambda_q=1\,,\ \sum_{q\in\mathcal{Q}_k}\lambda_q S_q^*\mathbf{1}\leq\nu\mathbf{1}} \nu\,. \tag{5}$$

The following result is a consequence of the convexity of the functions $-\Pi_k$, $k \in \mathbb{Z}^+$, $k > 0$.

Lemma 1: Consider the switched positive linear system (1) and the associated sequences $\{\pi_k\}$ and $\{\delta_k\}$. It is always verified that

$$\pi_k = \delta_k\ ,\ k \in \mathbb{Z}^+\ ,\ k > 0\ .$$

Proof: Notice that, for each given $k \in \mathbb{Z}^+$, $k > 0$, we have that

$$-\pi_k = \min_{x_0\in\{x_0\in(\mathbb{R}^n)^+\,:\,\mathbf{1}^*x_0\leq 1\}} -\Pi_k(x_0)\ = \min_{(t_0,x_0)\in\{t_0\in\mathbb{R},x_0\in(\mathbb{R}^n)^+\,:\,\mathbf{1}^*x_0\leq 1\,,\,-\mathbf{1}^*S_qx_0\leq t_0\,,\forall q\in\mathcal{Q}_k\}} t_0\ .$$

The proof follows by observing that the convex programming problem in (5) (in fact a linear programming problem) is related, via Lagrange duality [1], with the last convex programming problem (which is also a linear programming problem). Clearly, the Lagrangian [1] associated with the above convex optimization problem is the function

$$\begin{aligned}&\mathcal{L}(t_0, x_0, \eta, \nu, \lambda) = \\ &-\nu + \big(1 - \sum_{q\in\mathcal{Q}_k} \lambda_q\big)t_0 + \big(\nu\mathbf{1}^* - \eta^* - \sum_{q\in\mathcal{Q}_k} \lambda_q\mathbf{1}^*S_q\big)x_0\ , \\ &\eta \in (\mathbb{R}^n)^+\ ,\ \nu \in \mathbb{R}^+\ ,\ \lambda \in (\mathbb{R}^{N^k})^+ ,\end{aligned}$$

and the Lagrange dual function is

$$\begin{aligned}&g(\eta, \nu, \lambda) = \inf_{(t_0,x_0)\in\mathbb{R}\times\mathbb{R}^n} \mathcal{L}(t_0, x_0, \eta, \nu, \lambda)\ = \\ &\begin{cases} -\nu\,, & \sum_{q\in\mathcal{Q}_k}\lambda_q = 1\,,\ \nu\mathbf{1} - \sum_{q\in\mathcal{Q}_k}\lambda_q S_q^*\mathbf{1} = \eta \\ -\infty\,, & \text{otherwise}\end{cases}\ , \\ &\eta \in (\mathbb{R}^n)^+\ ,\ \nu \in \mathbb{R}^+\ ,\ \lambda \in (\mathbb{R}^{N^k})^+ .\end{aligned}$$

Thus, the Lagrange dual optimization problem is

$$\sup_{\eta\in(\mathbb{R}^n)^+\,,\,\nu\in\mathbb{R}^+\,,\,\lambda\in(\mathbb{R}^{N^k})^+} g(\eta, \nu, \lambda)\ = \max_{\nu\in\mathbb{R}^+,\ \lambda\in(\mathbb{R}^{N^k})^+ :\ \sum_{q\in\mathcal{Q}_k}\lambda_q=1\,,\ \sum_{q\in\mathcal{Q}_k}\lambda_q S_q^*\mathbf{1}\leq\nu\mathbf{1}} -\nu\ ,$$

whose optimal value (as clearly follows from (5)) is $-\delta_k$. Since the Slater condition for the convex (primal) optimization problem is satisfied, it then follows [1] that strong duality is achieved. That is, $-\pi_k = -\delta_k$. ■

The main result in the section is presented next.

Theorem 2: For the switched positive linear system (1) the following assertions are equivalent:

(i) The switched positive linear system (1) is state-feedback exponentially stabilizable.

(ii) There exists $k_0 \in \mathbb{Z}^+$, $k_0 > 0$, such that $\delta_{k_0} < 1$.

(iii) $\lim_{k\to+\infty} \delta_k = 0$.

(iv) There exists $k_0 \in \mathbb{Z}^+$, $k_0 > 0$, such that $\min_{S\in\mathcal{S}_{k_0}} \rho(S) < 1$.

(v) $\lim_{k\to+\infty} \min_{S\in\mathcal{S}_k} \rho(S) = 0$.

(vi) There exists a concave function $W : (\mathbb{R}^n)^+ \longrightarrow \mathbb{R}^+$ satisfying
- $W(\lambda x_0) = \lambda W(x_0)\,,\ \lambda \in \mathbb{R}^+\,,\ x_0 \in (\mathbb{R}^n)^+$,
- $W(x_0) \leq W(y_0)\ ,\ x_0 \in (\mathbb{R}^n)^+\ ,$ $(y_0 - x_0) \in (\mathbb{R}^n)^+$,
- $\|x_0\|_1 \leq W(x_0) \leq \gamma\|x_0\|_1\,,\ x_0 \in (\mathbb{R}^n)^+$, for some $\gamma > 1$,

which solves the following associated dynamic programming equation:

$$W(x_0) = \mathbf{1}^* x_0 + \min_{q\in\mathcal{Q}} W(A_q x_0)\,,\ x_0 \in (\mathbb{R}^n)^+\,. \tag{6}$$

Moreover, any state-feedback mapping $\kappa : \mathbb{R}^n \longrightarrow \mathcal{Q}$ defined by

$$\kappa(x_0) \in \arg\min_{q\in\mathcal{Q}} W(A_q|x_0|)\,,\ x_0 \in \mathbb{R}^n\,, \tag{7}$$

with W as in (vi), exponentially stabilizes the switched positive linear system (1). And furthermore, the function $W(|\cdot|)$ is a Lyapunov function for the exponential stability of the trivial solution of the associated closed-loop dynamical system (2).

Proof: This proof is organized as follows: It is proved that (i) $\Longrightarrow$ (ii) $\Longrightarrow$ (vi) $\Longrightarrow$ (i). We also prove that (ii) $\Longrightarrow$ (iii) $\Longrightarrow$ (v) (notice that it trivially follows that (v) $\Longrightarrow$ (iv)), and finally (iv) $\Longrightarrow$ (ii).

((i) $\Longrightarrow$ (ii).) By assumption there exist a mapping $\kappa : \mathbb{R}^n \longrightarrow \mathcal{Q}$ and scalars $\alpha \geq 1$ and $0 < \beta < 1$ such that the motions of the associated (closed-loop) system (2) satisfy

$$\frac{1}{\sqrt{n}}\|x_{cl}(k;x_0)\|_1 \leq \|x_{cl}(k;x_0)\| \leq \alpha\beta^k\|x_0\| \leq \alpha\beta^k\|x_0\|_1\,,\ k \in \mathbb{Z}^+\,,\ x_0 \in \mathbb{R}^n\,.$$

Choose $k_0 \in \mathbb{Z}^+$, $k_0 > 0$, such that $\sqrt{n}\alpha\beta^{k_0} < 1$. And define the following family of control functions:

$$q_{x_0} = \mathcal{F}_\kappa(x_{cl}(\cdot\,;x_0))\,,\ x_0 \in \mathbb{R}^n\,,\ \|x_0\|_1 \leq 1\,.$$

Then, we have that

$$\min_{q\in\mathcal{Q}_{k_0}} \|x(k_0;x_0,q)\|_1 \leq \|x(k_0;x_0,q_{x_0})\|_1 = \|x_{cl}(k_0;x_0)\|_1 \leq \sqrt{n}\alpha\beta^{k_0}\,,\ x_0 \in \mathbb{R}^n\,,\ \|x_0\|_1 \leq 1\,.$$

It therefore follows from Lemma 1 and from Fact 1 that

$$\delta_{k_0} = \pi_{k_0} = \max_{\|x_0\|_1\leq 1}\ \min_{q\in\mathcal{Q}_{k_0}} \|x(k_0;x_0,q)\|_1 \leq \sqrt{n}\alpha\beta^{k_0} < 1\,.$$

((ii) $\Longrightarrow$ (iii).) By assumption there exists $k_0 \in \mathbb{Z}^+$, $k_0 > 0$, such that $\delta_{k_0} = \pi_{k_0} < 1$. It can be assumed, without loss of generality, that $k_0 > 1$. Notice that in case that $k_0 = 1$ we can appeal to Fact 1 (property (6)) to define a new $k_0^{\text{new}} = hk_0$ with $h \in \mathbb{Z}^+$, $h > 1$. Thus, $k_0^{\text{new}} > 1$, and moreover $\pi_{k_0^{\text{new}}} \leq (\pi_{k_0})^h < 1$. Now, for a given $x_0 \in (\mathbb{R}^n)^+$, we will consider the optimal control problem

$$\min_{q\in\mathcal{Q}_{k_0}} \|x(k_0;x_0,q)\|_1\,, \tag{8}$$

and we will use $\hat{q}_{k_0,x_0}$ to denote a solution for that problem. Therefore, for any given $x_0 \in (\mathbb{R}^n)^+$

$$\|x(k_0;x_0,\hat{q}_{k_0,x_0})\|_1 = \min_{q\in\mathcal{Q}_{k_0}} \|x(k_0;x_0,q)\|_1 = \Pi_{k_0}(x_0) \leq \pi_{k_0}\|x_0\|_1\,.$$

Let us define

$$M_{k_0} = \max\{1\,,\ \max_{S\in\bigcup_{j=1}^{k_0-1}\mathcal{S}_j} \|S^*\mathbf{1}\|_\infty\}\,.$$

Let $x_0 \in (\mathbb{R}^n)^+$ be given, and let $h \in \mathbb{Z}^+$, $h > 0$, be given. Let $\tilde{q}_{hk_0,x_0} \in \mathcal{Q}_{hk_0}$ be a control function made up by concatenating solutions of the optimal control problem (8) with the following initial conditions:

$$\hat{x}_0 = x_0\,,\ \hat{x}_1 = x(k_0;\hat{x}_0,\hat{q}_{k_0,\hat{x}_0})\,,\dots\,,\ \hat{x}_{h-1} = x(k_0;\hat{x}_{h-2},\hat{q}_{k_0,\hat{x}_{h-2}})\,.$$

That is, using the above notation, the control function $\tilde{q}_{hk_0,x_0}$ is defined by

$$\tilde{q}_{hk_0,x_0}(jk_0 + i) = \hat{q}_{k_0,\hat{x}_j}(i)\,,\ i \in \{0,\dots,(k_0-1)\}\,,\ j \in \{0,\dots,(h-1)\}\,.$$

Now, it is easy to see that, with the above defined control function $\tilde{q}_{hk_0,x_0}$ the following inequalities are satisfied:

$$\|x(k;x_0,\tilde{q}_{hk_0,x_0})\|_1 \leq M_{k_0}\pi_{k_0}^j\|x_0\|_1\,,\ k \in \mathbb{Z}^+\,,\ k \in \{jk_0,\dots,jk_0+(k_0-1)\}\,,\ j \in \{0,\dots,(h-1)\}\,,\ \text{and}$$

$$\|x(k;x_0,\tilde{q}_{hk_0,x_0})\|_1 \leq M_{k_0}\pi_{k_0}^h\|x_0\|_1\,,\ k = hk_0\,.$$

It is then clear that the above expression implies that $\lim_{k\to+\infty}\delta_k = \lim_{k\to+\infty}\pi_k = 0$. Given $\epsilon > 0$ arbitrary, we choose $j_0 \in \mathbb{Z}^+$, $j_0 > 0$, such that $M_{k_0}\pi_{k_0}^{j_0} < \epsilon$. Then, for any $k \in \mathbb{Z}^+$, $k \geq j_0 k_0$, it is verified that (where we have chosen $h \in \mathbb{Z}^+$, $h > 0$, such that $k \leq hk_0$; thus $j \geq j_0$)

$$\delta_k = \pi_k = \max_{z\in(\mathbb{R}^n)^+:\|z\|_1\leq 1} \Pi_k(z) = \Pi_k(x_0) = \min_{q\in\mathcal{Q}_k} \|x(k;x_0,q)\|_1 \leq \|x(k;x_0,\tilde{q}_{hk_0,x_0})\|_1 \leq M_{k_0}\pi_{k_0}^j \leq M_{k_0}\pi_{k_0}^{j_0} < \epsilon\,,$$

where x_0 denotes an optimal solution for the problem $\max_{z\in(\mathbb{R}^n)^+:\|z\|_1\leq 1}\Pi_k(z)$.

((ii) $\Longrightarrow$ (vi).) (In this part of the proof we continue with the line of reasoning developed at the previous part, ((ii) $\Longrightarrow$ (iii)).) For each $k \in \mathbb{Z}^+$, $k > 0$, we now define the cost functional $J_k : (\mathbb{R}^n)^+ \times \mathcal{Q}_k \longrightarrow \mathbb{R}^+$ by

$$J_k(x_0,q) = \sum_{i=0}^{k} \|x(i;x_0,q)\|_1 = \sum_{i=0}^{k} \mathbf{1}^* x(i;x_0,q)\,. \tag{9}$$

For any given $x_0 \in (\mathbb{R}^n)^+$ we will consider the following family of optimal control problems (where $k \in \mathbb{Z}^+$, $k > 0$):

$$\min_{q \in \mathcal{Q}_k} J_k(x_0, q)\,, \tag{10}$$

and we will denote by $U_k(x_0)$ the optimal values of those problems. It immediately follows that the functions $U_k : (\mathbb{R}^n)^+ \longrightarrow \mathbb{R}^+$, $k \in \mathbb{Z}^+$, $k > 0$, are concave and continuous in the whole non-negative orthant $(\mathbb{R}^n)^+$, and also verify the following two properties:

$$U_k(\lambda x_0) = \lambda U_k(x_0)\ ,\ \lambda \in \mathbb{R}^+\ ,\ x_0 \in (\mathbb{R}^n)^+\ , \tag{11}$$

$$U_k(x_0) \le U_k(y_0)\,,\, x_0 \in (\mathbb{R}^n)^+\,,\, (y_0 - x_0) \in (\mathbb{R}^n)^+\,. \tag{12}$$

Moreover, for any given $x_0 \in (\mathbb{R}^n)^+$, we have that (choosing $h \in \mathbb{Z}^+$, $h > 0$, such that $k \le hk_0$)

$$U_k(x_0) = \min_{q \in \mathcal{Q}_k} J_k(x_0, q) \le \min_{q \in \mathcal{Q}_{hk_0}} J_{hk_0}(x_0, q) \le$$
$$J_{hk_0}(x_0, \tilde{q}_{hk_0,x_0}) = \sum_{i=0}^{hk_0} \|x(i; x_0, \tilde{q}_{hk_0,x_0})\|_1 \le$$
$$\sum_{j=0}^{h} k_0 M_{k_0} \pi_{k_0}^j \|x_0\|_1 \le \frac{k_0 M_{k_0}}{(1 - \pi_{k_0})} \|x_0\|_1\,,\ k \in \mathbb{Z}^+\,,\, k > 0\,.$$

It was therefore proved that

$$\|x_0\|_1 \le U_k(x_0) \le \frac{k_0 M_{k_0}}{(1 - \pi_{k_0})} \|x_0\|_1\,,$$
$$x_0 \in (\mathbb{R}^n)^+\,,\ k \in \mathbb{Z}^+\,,\ k > 0\,. \tag{13}$$

It is also easy to see that the following property is verified:

$$U_{k+1}(x_0) \ge U_k(x_0)\,,\ x_0 \in (\mathbb{R}^n)^+\,,\ k \in \mathbb{Z}^+\,,\ k > 0\,. \tag{14}$$

It then follows that, for each given $x_0 \in (\mathbb{R}^n)^+$, the limit $\lim_{k \to +\infty} U_k(x_0)$ exists. That fact lead us to the introduction of the function $W : (\mathbb{R}^n)^+ \longrightarrow \mathbb{R}^+$ defined by

$$W(x_0) = \lim_{k \to +\infty} U_k(x_0)$$

which, as can be easily verified, inherits all of the aforementioned properties the functions U_k have. That is, the function W is concave and also verifies

$$W(\lambda x_0) = \lambda W(x_0)\ ,\ \lambda \in \mathbb{R}^+\ ,\ x_0 \in (\mathbb{R}^n)^+\ ,$$
$$W(x_0) \le W(y_0)\ ,\ x_0 \in (\mathbb{R}^n)^+\ ,\ (y_0 - x_0) \in (\mathbb{R}^n)^+\ ,$$
$$\|x_0\|_1 \le W(x_0) \le \frac{k_0 M_{k_0}}{(1 - \pi_{k_0})} \|x_0\|_1\,,\ x_0 \in (\mathbb{R}^n)^+\,.$$

Furthermore, since

$$\begin{aligned} U_{k+1}(x_0) &= \big(\|x_0\|_1 + \min_{q \in \mathcal{Q}} U_k(x(1; x_0, q))\big) \\ &= \big(\|x_0\|_1 + \min_{q \in \mathcal{Q}} U_k(A_q x_0)\big)\,, \\ &\quad x_0 \in (\mathbb{R}^n)^+\,,\ k \in \mathbb{Z}^+\,,\ k > 0\,, \end{aligned}$$

it then follows that W is a solution of the following dynamic programming equation:

$$W(x_0) = \mathbf{1}^* x_0 + \min_{q \in \mathcal{Q}} W(A_q x_0)\,,\ x_0 \in (\mathbb{R}^n)^+\,.$$

((vi) $\Longrightarrow$ (i).) (Without lose of generality we use $\gamma = \frac{k_0 M_{k_0}}{(1-\pi_{k_0})} > 1$ in this part of the proof.) We now claim that any mapping $\kappa : \mathbb{R}^n \longrightarrow \mathcal{Q}$ defined via

$$\kappa(x_0) \in \arg\min_{q \in \mathcal{Q}} W(A_q |x_0|)\,,\ x_0 \in \mathbb{R}^n$$

is an exponentially stabilizing state-feedback mapping. Clearly, using the function $W(|\cdot|) : \mathbb{R}^n \longrightarrow \mathbb{R}^+$ as a Lyapunov candidate we immediately obtain that

$$W(|A_{\kappa(x_0)} x_0|) - W(|x_0|) \le$$
$$W(A_{\kappa(x_0)} |x_0|) - W(|x_0|) = -\|x_0\|_1\,,\ x_0 \in \mathbb{R}^n\,,$$

which means that $W(|\cdot|)$ is indeed a Lyapunov function for the exponential stability of the trivial solution of the associated closed-loop dynamical system (2). It is now easy to verify that, when using the above state-feedback mapping, the motions of the associated closed-loop dynamical system (2) satisfy

$$\|x_{cl}(k; x_0)\|_1 \le W(|x_{cl}(k; x_0)|) \le$$
$$\Big(1 - \frac{(1 - \pi_{k_0})}{k_0 M_{k_0}}\Big)^k W(|x_0|) \le$$
$$\frac{k_0 M_{k_0}}{(1 - \pi_{k_0})} \Big(1 - \frac{(1 - \pi_{k_0})}{k_0 M_{k_0}}\Big)^k \|x_0\|_1\,,\ x_0 \in \mathbb{R}^n\,,$$

implying that

$$\|x_{cl}(k; x_0)\| \le \alpha \beta^k \|x_0\|\,,\ k \in \mathbb{Z}^+\,,\ x_0 \in \mathbb{R}^n\,,$$
$$\text{with}\,,\quad \alpha = \frac{\sqrt{n}\, k_0 M_{k_0}}{(1 - \pi_{k_0})}\ ,\ \beta = \Big(1 - \frac{(1 - \pi_{k_0})}{k_0 M_{k_0}}\Big)\ .$$

((iii) $\Longrightarrow$ (v).) That $\lim_{k \to +\infty}\ \min_{S \in \mathcal{S}_k} \rho(S) = 0$, is a direct consequence of the fact that

$$\delta_k = \min_{\lambda \in (\mathbb{R}^{N^k})^+,\, \nu \in \mathbb{R}^+ :\, \sum_{q \in \mathcal{Q}_k} \lambda_q = 1\,,\, \sum_{q \in \mathcal{Q}_k} \lambda_q S_q^* \mathbf{1} \le \nu \mathbf{1}} \nu =$$
$$\min_{\lambda \in (\mathbb{R}^{N^k})^+ :\, \sum_{q \in \mathcal{Q}_k} \lambda_q = 1}\ \max_{j \in \{1,\ldots,n\}} \|\big(\sum_{q \in \mathcal{Q}_k} \lambda_q S_q\big)_j\|_1 \ge$$
$$\min_{\lambda \in (\mathbb{R}^{N^k})^+ :\, \sum_{q \in \mathcal{Q}_k} \lambda_q = 1}\ \frac{1}{n} \sum_{i,j=1}^{n} \big(\sum_{q \in \mathcal{Q}_k} \lambda_q S_q\big)_{ij} =$$
$$\min_{S \in \mathcal{S}_k} \frac{1}{n} \sum_{i,j=1}^{n} \big(S\big)_{ij} \ge \frac{1}{n} \min_{S \in \mathcal{S}_k}\ \max_{j \in \{1,\ldots,n\}} \|\big(S\big)_j\|_1 \ge$$
$$\frac{1}{n} \min_{S \in \mathcal{S}_k} \lambda_{PF}(S) = \frac{1}{n} \min_{S \in \mathcal{S}_k} \rho(S)\ .$$

((iv) $\Longrightarrow$ (ii).) If there is $k_0 \in \mathbb{Z}^+$, $k_0 > 0$, such that $\min_{S \in \mathcal{S}_{k_0}} \rho(S) < 1$, then, for $h \in \mathbb{Z}^+$, $h > 0$, big enough, and $k_1 = hk_0$ it will be that $\min_{S \in \mathcal{S}_{k_1}} \|S^* \mathbf{1}\|_\infty < 1$. The proof is now completed by invoking Fact 1 (property (5)) and Lemma 1. ■

Remark 1: It immediately follows from Theorem 2 that, when the switched positive linear system (1) is state-feedback exponentially stabilizable, an exponentially stabilizing state-feedback mapping $\kappa : \mathbb{R}^n \longrightarrow \mathcal{Q}$, given by (7), can always be chosen with the following property:

$$\kappa(\lambda x_0) = \kappa(x_0)\,,\ \lambda \in (\mathbb{R} \setminus \{0\})\,,\ x_0 \in \mathbb{R}^n\,.$$

That particular observation is also valid for general switched linear system as it was already pointed out in [6].
It is important to point out here, that in [2], it was already proved that: The switched positive linear system (1) is uniformly exponentially convergent if and only if $\min_{S\in\mathcal{S}_k}\rho(S)<1$, for some $k\in\mathbb{Z}^+$, $k>0$. Notice that, this result jointly with Theorem 1 provide an alternative proof for the equivalence (i) $\Longleftrightarrow$ (iv) in Theorem 2.

IV. On the Solvability of the Associated Dynamic Programming Equation

This section is devoted to present results, which are related with Theorem 2, and which are concerned with the solvability of the associated dynamic programming equation (6), with the number of solutions this equation has, and also with the properties of their solutions. Let us begin by introducing the following sets, $\mathcal{L}^+$ and $\mathcal{L}^{++}$, of functions as

$$\mathcal{L}^+=\{\Phi:(\mathbb{R}^n)^+\longrightarrow\mathbb{R}^+ \,:\exists\,\gamma_2>0\,: \\ \Phi(x_0)\le\gamma_2\mathbf{1}^*x_0\,,\,\forall x_0\in(\mathbb{R}^n)^+\}\,,$$

$$\mathcal{L}^{++}=\{\Phi:(\mathbb{R}^n)^+\longrightarrow\mathbb{R}^+ \,:\exists\,\gamma_1>0\,,\,\gamma_2>0\,: \\ \gamma_1\mathbf{1}^*x_0\le\Phi(x_0)\le\gamma_2\mathbf{1}^*x_0\,,\,\forall x_0\in(\mathbb{R}^n)^+\}\,.$$

Theorem 3: The dynamic programming equation (6) associated to the switched positive linear system (1) has a solution W inside the convex cone $\mathcal{L}^+$, if and only if, the switched positive linear system (1) is state-feedback exponentially stabilizable. Moreover:

(1) The convex cone $\mathcal{L}^+$ admits at most one solution of the dynamic programming equation (6).

(2) If $W\in\mathcal{L}^+$ is the solution of the associated dynamic programming equation (6), then, $W\in\mathcal{L}^{++}$ and it has the following properties:

- (i) W is concave and continuous on $(\mathbb{R}^n)^+$.
- (ii) It is verified that

$$W(\lambda x_0)=\lambda W(x_0)\,,\,\lambda\in\mathbb{R}^+\,,\,x_0\in(\mathbb{R}^n)^+.$$

- (iii) It is also verified that

$$W(x_0)\le W(y_0)\,,\,x_0\in(\mathbb{R}^n)^+\,,\,(y_0-x_0)\in(\mathbb{R}^n)^+.$$

- (iv) For each given $k\in\mathbb{Z}^+$, $k>0$, it is

$$0\le W(x_0)-U_k(x_0)\le \\ \frac{\left(\frac{k_0M_{k_0}}{(1-\delta_{k_0})}-1\right)}{\left(1+\frac{(1-\delta_{k_0})}{k_0M_{k_0}M_2}\right)^k}\,\mathbf{1}^*x_0\,,\,x_0\in(\mathbb{R}^n)^+,$$

where, in the last expression, $k_0\in\mathbb{Z}^+$, $k_0>1$, is such that $\delta_{k_0}<1$,

$$M_l=\max\{1\,,\max_{S\in\bigcup_{j=1}^{l-1}\mathcal{S}_j}\|S^*\mathbf{1}\|_\infty\}\,,\,l\in\mathbb{Z}^+\,,\,l>1\,,$$

and where the approximating functions U_k, $k\in\mathbb{Z}^+$, $k>0$, can be expressed as follows:

$$U_k(x_0)=\mathbf{1}^*x_0+\min_{q\in\mathcal{Q}_k}c_q^*x_0\;,\;x_0\in(\mathbb{R}^n)^+,$$

$$c_q=\sum_{l=1}^{k}(A_{q(l-1)}\dots A_{q(0)})^*\mathbf{1}\,,\,q\in\mathcal{Q}_k\,.$$

Proof: In Part 1 we prove the necessary and sufficient condition for existence of solution in $\mathcal{L}^+$. In Part 2, we prove the rest of the statement.

Part 1.- (Sufficiency.) It was already proved, in Theorem 2, that if the switched positive linear system (1) is state-feedback exponentially stabilizable, then, there exists $W\in\mathcal{L}^{++}\subset\mathcal{L}^+$ that solves the dynamic programming equation.
(Necessity.) If $W\in\mathcal{L}^+$ and solves the dynamic programming equation (6), then, it immediately follows that $W\in\mathcal{L}^{++}$, with $\gamma_1=1$. Thus, any state-feedback mapping $\kappa:(\mathbb{R}^n)^+\longrightarrow\mathcal{Q}$ obeying $\kappa(x_0)\in\arg\min_{q\in\mathcal{Q}}W(A_qx_0)$, $x_0\in(\mathbb{R}^n)^+$, exponentially stabilizes, within the non-negative orthant $(\mathbb{R}^n)^+$, the switched positive linear system (1). Clearly, it follows from

$$W(A_{\kappa(x_0)}x_0)-W(x_0)=-\|x_0\|_1\,,\;\forall x_0\in(\mathbb{R}^n)^+$$

that for the motions of the associated closed-loop dynamical system (2) it is

$$\|x_{cl}(k;x_0)\|_1\le\gamma_2\big(1-\frac{1}{\gamma_2}\big)^k\|x_0\|_1\,,\;x_0\in(\mathbb{R}^n)^+\,.$$

Choosing $k_0\in\mathbb{Z}^+$, $k_0>0$, such that $\gamma_2\big(1-\frac{1}{\gamma_2}\big)^{k_0}<1$, and defining the following family of control functions:

$$q_{x_0}=\mathcal{F}_\kappa(x_{cl}(\cdot\,;x_0))\,,\;x_0\in(\mathbb{R}^n)^+\,,\;\|x_0\|_1\le1\,,$$

we obtain, for $x_0\in(\mathbb{R}^n)^+$, $\|x_0\|_1\le1$, that

$$\min_{q\in\mathcal{Q}_{k_0}}\|x(k_0;x_0,q)\|_1\le\|x(k_0;x_0,q_{x_0})\|_1= \\ \|x_{cl}(k_0;x_0)\|_1\le\gamma_2\big(1-\frac{1}{\gamma_2}\big)^{k_0}\,.$$

It therefore follows from Lemma 1 that

$$\delta_{k_0}=\pi_{k_0}=\max_{x_0\in(\mathbb{R}^n)^+:\|x_0\|_1\le1}\;\min_{q\in\mathcal{Q}_{k_0}}\|x(k_0;x_0,q)\|_1\le \\ \gamma_2\big(1-\frac{1}{\gamma_2}\big)^{k_0}<1\,.$$

Now, by invoking Theorem 2, it follows that the switched positive linear system (1) is state-feedback exponentially stabilizable.

Part 2.- Assume $W\in\mathcal{L}^+$ is solution of the dynamic programming equation (6). Then, since the switched positive linear system (1) is state-feedback stabilizable, it follows from Theorem 2 that there exists $k_0\in\mathbb{Z}^+$, $k_0>1$, such that $\delta_{k_0}<1$. As in the part ((ii) $\Longrightarrow$ (vi)) of the proof of Theorem 2, we consider the costs functionals $J_k:(\mathbb{R}^n)^+\times\mathcal{Q}_k\longrightarrow\mathbb{R}^+$, $k\in\mathbb{Z}^+$, $k>0$, defined by (9), and for given $x_0\in(\mathbb{R}^n)^+$ we consider the family of optimal control problems defined in (10) and we will denote by $U_k(x_0)$ the optimal values of these problems. It follows (as it was shown in the ((ii) $\Longrightarrow$ (vi)) part of the proof of Theorem 2) that the functions $U_k:(\mathbb{R}^n)^+\longrightarrow\mathbb{R}^+$ are concave and continuous on $(\mathbb{R}^n)^+$, and they also obey properties (11), (12), (14), and (13). As a result of all that, we can define a function $U_\infty : (\mathbb{R}^n)^+ \longrightarrow \mathbb{R}^+$ by

$$U_\infty(x_0) = \lim_{k\to+\infty} U_k(x_0)$$

which therefore will also be concave and will verify

$$U_\infty(\lambda x_0) = \lambda U_\infty(x_0)\ ,\ \lambda \in \mathbb{R}^+\ ,\ x_0 \in (\mathbb{R}^n)^+\ ,$$
$$U_\infty(x_0) \le U_\infty(y_0)\ ,\ x_0 \in (\mathbb{R}^n)^+\ ,\ (y_0 - x_0) \in (\mathbb{R}^n)^+\ ,$$
$$U_\infty(x_0) \ge U_k(x_0)\ ,\ x_0 \in (\mathbb{R}^n)^+\ ,\ k \in \mathbb{Z}^+\ ,\ k > 0\ ,$$
$$\text{and}\ ,\quad \mathbf{1}^* x_0 \le U_\infty(x_0) \le \frac{k_0 M_{k_0}}{(1-\delta_{k_0})}\mathbf{1}^* x_0\ ,\ x_0 \in (\mathbb{R}^n)^+\ .$$

Further, since

$$U_{k+1}(x_0) \;=\; \big(\mathbf{1}^* x_0 + \min_{q\in\mathcal{Q}} U_k(A_q x_0)\big)\ , \quad x_0 \in (\mathbb{R}^n)^+\ ,\ k \in \mathbb{Z}^+\ , \tag{15}$$

it then follows that

$$U_\infty(x_0) = \mathbf{1}^* x_0 + \min_{q\in\mathcal{Q}} U_\infty(A_q x_0)\ ,\ x_0 \in (\mathbb{R}^n)^+\ .$$

We now claim that $U_\infty = W$. By assumption $W \in \mathcal{L}^{++}$, that is, there are $\gamma_1 > 0$, $\gamma_2 > 0$, such that

$$\gamma_1 \mathbf{1}^* x_0 \le W(x_0) \le \gamma_2 \mathbf{1}^* x_0\ ,\ x_0 \in (\mathbb{R}^n)^+\ . \tag{16}$$

Without loss of generality, we will assume that $\gamma_1 \le 1$, and $\gamma_2 > 1$. We now use (15) and (16) to invoke Lemma 2 (stated after this proof) from which it is concluded that

$$\frac{-(\gamma_1^{-1}-1)\gamma_2}{(1+(\gamma_2 M_2)^{-1})^k}\mathbf{1}^* x_0 \le W(x_0) - U_k(x_0) \le \frac{(1-\gamma_2^{-1})\gamma_2}{(1+(\gamma_2 M_2)^{-1})^k}\mathbf{1}^* x_0\ ,\ x_0 \in (\mathbb{R}^n)^+\ ,\ k \in \mathbb{Z}^+\ ,\ k > 0\ .$$

The above bounds imply that for each given $x_0 \in (\mathbb{R}^n)^+$

$$\lim_{k\to+\infty} U_k(x_0) = W(x_0)\ .$$

Hence, $U_\infty = W$, and the claim was proved. The above bounds also imply that

$$\lim_{k\to+\infty}\ \sup_{x_0\in(\mathbb{R}^n)^+ :\, \|x_0\|_1 \le 1} |W(x_0) - U_k(x_0)| = 0\ ,$$

and therefore, the continuity of W, on the whole non-negative orthant $(\mathbb{R}^n)^+$, follows from the continuity of the functions U_k, $k \in \mathbb{Z}^+$, $k > 0$. The final properties of the solution $W \in \mathcal{W}^+$ follow now from the properties we have already established on U_∞, and (then) by setting $\gamma_2 = \frac{k_0 M_{k_0}}{(1-\delta_{k_0})}$ on the above upper bound. ∎

In the proof of Theorem 3 we have used the next result that we have adapted from [9] to fit in the setting of the present discussion.

Lemma 2: Consider the switched positive linear system (1). Let $W : (\mathbb{R}^n)^+ \longrightarrow \mathbb{R}^+$ satisfying

$$\gamma_1 \mathbf{1}^* x_0 \le W(x_0) \le \gamma_2 \mathbf{1}^* x_0\ ,\ \forall x_0 \in (\mathbb{R}^n)^+\ ,$$

for some given $1 \ge \gamma_1 > 0$, $\gamma_2 > 1$, be a solution of the associated dynamic programming equation (6). Let $\{U_k\}$, $k \in \mathbb{Z}^+$, be the sequence of functions generated by

$$U_{k+1}(x_0) \;=\; \big(\mathbf{1}^* x_0 + \min_{q\in\mathcal{Q}} U_k(A_q x_0)\big)\ , \quad U_0(x_0) = \mathbf{1}^* x_0\ ,\ x_0 \in (\mathbb{R}^n)^+\ ,\ k \in \mathbb{Z}^+\ .$$

Then, under these conditions,

$$\frac{-(\gamma_1^{-1}-1)}{(1+(\gamma_2 M_2)^{-1})^k} W(x_0) \le W(x_0) - U_k(x_0) \le \frac{(1-\gamma_2^{-1})}{(1+(\gamma_2 M_2)^{-1})^k} W(x_0)\ ,\ x_0 \in (\mathbb{R}^n)^+\ ,\ k \in \mathbb{Z}^+\ ,$$

where $M_2 = \max\{1\,, \max_{q\in\mathcal{Q}} \|A_q^* \mathbf{1}\|_\infty\}$.

V. On Stabilizing State-Feedback Mappings and Their Corresponding Lyapunov Functions

Straightforward but important conclusions regarding stabilizing state-feedback mappings and their corresponding Lyapunov functions are obtained from Theorem 3 and summarized in the next result.

Corollary 1: Assume the switched positive linear system (1) is state-feedback exponentially stabilizable; that is, there exists $k_0 \in \mathbb{Z}^+$, $k_0 > 1$, such that $\delta_{k_0} < 1$. Let $k_1 \in \mathbb{Z}^+$, $k_1 > 0$, be such that $\epsilon_1 = \frac{\left(\frac{k_0 M_{k_0}}{(1-\delta_{k_0})} - 1\right)}{\left(1 + \frac{(1-\delta_{k_0})}{k_0 M_{k_0} M_2}\right)^{k_1}} < 1$, where M_l, $l \in \mathbb{Z}^+$, $l > 1$, is as in Theorem 3. Let $k \in \mathbb{Z}^+$, $k \ge k_1$, be given, but arbitrary, and consider the corresponding finite subsets $\{p_q\}_{q\in\mathcal{Q}_{k+1}} \subset (\mathbb{R}^n)^+$ and $\{c_q\}_{q\in\mathcal{Q}_k} \subset (\mathbb{R}^n)^+$ defined as

$$p_q = \sum_{l=1}^{k+1} (A_{q(l-1)} \ldots A_{q(0)})^* \mathbf{1}\ ,\ q \in \mathcal{Q}_{k+1}\ , \tag{17}$$

$$c_q = \sum_{l=1}^{k} (A_{q(l-1)} \ldots A_{q(0)})^* \mathbf{1}\ ,\ q \in \mathcal{Q}_{k}\ . \tag{18}$$

Then, under these conditions, the following holds:

- Every state-feedback mapping $\kappa_k : \mathbb{R}^n \longrightarrow \mathcal{Q}$ given by
$$\kappa_k(x_0) \in \big(\arg\min_{q\in\mathcal{Q}_{k+1}} p_q^* |x_0|\big)(0)\ ,\ x_0 \in \mathbb{R}^n\ , \tag{19}$$
exponentially stabilizes the switched system (1).
- Also, the function $U_k(|\cdot|)$ which can be expressed as
$$U_k(|x_0|) = \mathbf{1}^* |x_0| + \min_{q\in\mathcal{Q}_k} c_q^* |x_0|\ ,\ x_0 \in \mathbb{R}^n\ , \tag{20}$$
is a Lyapunov function for the exponential stability of the trivial solution of the associated closed-loop system (2) (that uses the corresponding feedback mapping κ_k).

Proof: Let $W \in \mathcal{L}^{++}$ be the solution of the associated dynamic programming equation (6) (that by virtue of Theorems 2 and 3 it is known to exist). It is a consequence of Theorem 3 that for each $k \in \mathbb{Z}^+$, such that $k \ge k_1$, it is

$$0 \le W(|x_0|) - U_k(|x_0|) \le \epsilon_1 \|x_0\|_1\ ,\ \forall x_0 \in \mathbb{R}^n\ ,$$
$$\Longrightarrow\ U_{k+1}(|x_0|) - U_k(|x_0|) \le \epsilon_1 \|x_0\|_1\ ,\ \forall x_0 \in \mathbb{R}^n\ .$$

Then, using properties (12) and (15) (enjoyed by the functions U_k) jointly with the underlying assumptions, we obtain, for each $k \in \mathbb{Z}^+$, $k \ge k_1$, that

$$U_k(|A_{\kappa_k(x_0)} x_0|) - U_k(|x_0|) \le U_k(A_{\kappa_k(x_0)} |x_0|) - U_k(|x_0|) = \min_{q\in\mathcal{Q}} U_k(A_q |x_0|) - U_k(|x_0|) \le -(1-\epsilon_1)\|x_0\|_1\ , \forall x_0 \in \mathbb{R}^n.$$

Finally, noticing that the functions U_k obey property (13) ($\Longrightarrow U_k \in \mathcal{L}^{++}$) completes the proof. ■

Remark 2: Expressions (19) and (20) constitute explicit formulas for a stabilizing state-feedback mapping and its corresponding Lyapunov function respectively. And the sets of vectors, $\{p_q\}_{q\in\mathcal{Q}_{k+1}}$ in (17) and $\{c_q\}_{q\in\mathcal{Q}_k}$ in (18), constitute representations for that state-feedback mapping κ_k and for the corresponding Lyapunov function $U_k(|\cdot|)$ respectively. Concerning these representations, it is important to remark that, typically, these sets of vectors include vectors that are redundant, and therefore they can be eliminated by means of some selection procedure. For instance, it is clear that a vector, in $\{p_q\}_{q\in\mathcal{Q}_{k+1}}$, that belongs to the convex hull of a different subset of vectors in $\{p_q\}_{q\in\mathcal{Q}_{k+1}}$ is redundant.

Remark 3: We further remark that in order to find a stabilizing state-feedback mapping it is enough to find $k \in \mathbb{Z}^+$, $k > 0$, satisfying the condition

$$U_{k+1}(x_0) - U_k(x_0) < 1\,,\ \forall x_0 \in (\mathbb{R}^n)^+ :\ \|x_0\|_1 = 1\,.$$

It immediately follows from the proof of Corollary 1 that, for such a k, the state-feedback mapping κ_k in (19) is stabilizing and $U_k(|\cdot|)$ is its corresponding Lyapunov function.

VI. On a Related Optimal Control Problem

In this section we present a complete solution for an optimal control problem involving the switched positive linear system (1) which is closely related with the state-feedback exponential stabilization problem in a sense that will be precisely stated next, in Theorem 4. We begin with the following definition.

Definition 3: The switched linear system (1) is said to be uniformly l_1 bounded whenever there exists a scalar $\gamma \geq 1$ that obeys the following property:
For each $x_0 \in \mathbb{R}^n$ there exists $q_{x_0} \in \mathcal{Q}_\infty$ such that the corresponding motion of (1) satisfies

$$\sum_{i=0}^{k} \|x(i;x_0,q_{x_0})\|_1 \leq \gamma\|x_0\|_1\,,\ k \in \mathbb{Z}^+\,. \tag{21}$$

In relation with the switched positive linear system (1) we introduce the following cost functional $J_\infty : (\mathbb{R}^n)^+ \times \mathcal{Q}_\infty \longrightarrow (\mathbb{R}^{(\text{ext})})^+$ defined as

$$J_\infty(x_0,q) = \lim_{k\to+\infty} \sum_{i=0}^{k} \|x(i;x_0,q)\|_1\,. \tag{22}$$

For each given $x_0 \in (\mathbb{R}^n)^+$, we will consider, and also solve in this section, the following optimal control problem:

$$\inf_{q\in\mathcal{Q}_\infty} J_\infty(x_0,q)\ , \tag{23}$$

for which it will be denoted by $U : (\mathbb{R}^n)^+ \longrightarrow (\mathbb{R}^{(\text{ext})})^+$ the optimal cost functional

$$U(x_0) = \inf_{q\in\mathcal{Q}_\infty} J_\infty(x_0,q)\,. \tag{24}$$

We further associate, to the switched positive linear system (1), the sequence $\{\upsilon_k\}$, $k \in \mathbb{Z}^+$, $k > 0$, defined as

$$\upsilon_k = \max_{x_0\in(\mathbb{R}^n)^+ :\, \|x_0\|_1\leq 1} U_k(x_0)\,. \tag{25}$$

Regarding the above posed optimal control problem (23) we have the following important result.

Theorem 4: For the switched positive linear system (1) and the optimal control problem (23) the following assertions are equivalent:

- (i) The optimal cost functional U, defined in (24), is continuous at $x_0 = 0$.
- (ii) The switched positive linear system (1) is uniformly l_1 bounded.
- (iii) The switched positive linear system (1) is state-feedback exponentially stabilizable.
- (iv) The associated sequence $\{\upsilon_k\}$, $k \in \mathbb{Z}^+$, $k > 0$, is bounded.

Further, in case the above conditions are satisfied, we have

$$U(x_0) = \min_{q\in\mathcal{Q}_\infty} J_\infty(x_0,q) = W(x_0)\,,\ x_0 \in (\mathbb{R}^n)^+\,,$$

where $W \in \mathcal{L}^{++}$ is the solution of the associated dynamic programming equation (6). Moreover,

$$\hat{q}_{x_0} = \mathcal{F}_\kappa(x(\cdot;x_0,\mathcal{F}_\kappa(x)))\,,\ x_0 \in (\mathbb{R}^n)^+$$
$$\text{with}\,,\quad \kappa(x_0) \in \arg\min_{q\in\mathcal{Q}} W(A_q x_0)\,,\ x_0 \in (\mathbb{R}^n)^+$$

is an optimal solution for the optimal control problem under consideration.

Proof: **((ii) $\Longrightarrow$ (i).)** The continuity of U at $x_0 = 0$ follows from

$$0 \leq U(x_0) \leq J_\infty(x_0,q_{x_0}) \leq \gamma\|x_0\|_1\,,\ x_0 \in (\mathbb{R}^n)^+\,.$$

((i) $\Longrightarrow$ (ii).) By hypothesis, there is $\delta > 0$ such that $U(x_0) \leq 1$, $\forall x_0 \in (\mathbb{R}^n)^+ : \|x_0\|_1 \leq \delta$. Since U is positively homogeneous, we have that

$$U(x_0) \leq \frac{1}{\delta}\|x_0\|_1\,,\ \forall x_0 \in (\mathbb{R}^n)^+\,,$$

from which, choosing $\gamma = \frac{2}{\delta}$, the condition (21) is satisfied for each $x_0 \in (\mathbb{R}^n)^+$, and therefore for each $x_0 \in \mathbb{R}^n$.
((ii) $\Longrightarrow$ (iv).) By hypothesis, for each $x_0 \in (\mathbb{R}^n)^+$, we have

$$U_k(x_0) \leq J_k(x_0,q_{x_0}) \leq \gamma\|x_0\|_1\,,\ k \in \mathbb{Z}^+\,,\ k > 0\,,$$

implying that $\upsilon_k \leq \gamma$, $k \in \mathbb{Z}^+$, $k > 0$.
((iv) $\Longrightarrow$ (iii)) Invoking Lemma 3

$$\delta_k \leq \upsilon_k \prod_{j=1}^{k}(\frac{\upsilon_j - 1}{\upsilon_j}) \leq \gamma(1-\frac{1}{\gamma})^k\,,\ k \in \mathbb{Z}^+\,,\ k > 0\,,$$

hence there exists $k_0 \in \mathbb{Z}^+$, $k_0 > 0$, obeying $\delta_{k_0} < 1$. Theorem 2 completes this part of the proof.
((iii) $\Longrightarrow$ (ii)) An exponentially stabilizing mapping $\kappa : \mathbb{R}^n \longrightarrow \mathcal{Q}$ (will be associated with scalars $\alpha \geq 1$ and $0 < \beta < 1$, and) generates control functions $q_{x_0} \in \mathcal{Q}_\infty$ via $q_{x_0} = \mathcal{F}_\kappa(x(\cdot;x_0,\mathcal{F}_\kappa(x)))$, $x_0 \in \mathbb{R}^n$, for which the motions of (1) satisfy

$$\|x(k;x_0,q_{x_0})\| \leq \alpha\beta^k\|x_0\|\,,\ k \in \mathbb{Z}^+\,,\ x_0 \in \mathbb{R}^n\,.$$

Then, condition (21) holds with $\gamma = \frac{\sqrt{n}\,\alpha}{(1-\beta)}$.
The last part of the Theorem is now proved. By assumption there is $W \in \mathcal{L}^{++}$, the unique solution of the associated dynamic programming equation (6). Consider a state-feedback

mapping $\kappa : \mathbb{R}^n \longrightarrow \mathcal{Q}$ defined as in (7), and consider the family of control functions $\hat{q}_{x_0} \in \mathcal{Q}_\infty$, $x_0 \in \mathbb{R}^n$, generated when closing the loop with the above feedback, that is $\hat{q}_{x_0} = \mathcal{F}_\kappa(x(\cdot; x_0, \mathcal{F}_\kappa(x)))$, $x_0 \in \mathbb{R}^n$. Then, for each given $x_0 \in (\mathbb{R}^n)^+$, we have that

$$J_\infty(x_0, \hat{q}_{x_0}) = \lim_{k \to +\infty} \sum_{i=0}^{k} \|x(i; x_0, \hat{q}_{x_0})\|_1 = \lim_{k \to +\infty} \big(W(x_0) - W(x(k+1; x_0, \hat{q}_{x_0}))\big) = W(x_0) .$$

Now, for each given $x_0 \in (\mathbb{R}^n)^+$, let $q_{x_0} \in \mathcal{Q}_\infty$ be any control function for which $J_\infty(x_0, q_{x_0})$ is finite. It then follows that $\lim_{i \to +\infty} \|x(i; x_0, q_{x_0})\|_1 = 0$, implying that $\lim_{i \to +\infty} W(x(i; x_0, q_{x_0})) = 0$. It also follows that

$$J_\infty(x_0, q_{x_0}) = \lim_{k \to +\infty} \sum_{i=0}^{k} \|x(i; x_0, q_{x_0})\|_1 \geq \lim_{k \to +\infty} \sum_{i=0}^{k} \big(W(x(i; x_0, q_{x_0})) - W(x(i+1; x_0, q_{x_0}))\big) = \lim_{k \to +\infty} \big(W(x_0) - W(x(k+1; x_0, q_{x_0}))\big) = W(x_0) ,$$

which completes the proof of the Theorem. ■

The next result provides with a needed relation between the associated sequences $\{\upsilon_k\}$ and $\{\delta_k\}$.

Lemma 3: Let $x_0 \in (\mathbb{R}^n)^+$ and $k \in \mathbb{Z}^+$, $k > 0$, be given. Let $\hat{q}_{k,x_0} \in \mathcal{Q}_k$ be such that

$$U_k(x_0) = J_k(x_0, \hat{q}_{k,x_0}) = \min_{q \in \mathcal{Q}_k} J_k(x_0, q) .$$

Then, $\|x(k; x_0, \hat{q}_{k,x_0})\|_1 \leq \upsilon_k \prod_{j=1}^{k} (\frac{\upsilon_j - 1}{\upsilon_j}) \|x_0\|_1$,
and therefore, $\delta_k = \pi_k \leq \upsilon_k \prod_{j=1}^{k} (\frac{\upsilon_j - 1}{\upsilon_j})$.

Proof: Let us use $x_i = x(i; x_0, \hat{q}_{k,x_0})$, $i = 0, \ldots, k$, and $z_i = \|x(i; x_0, \hat{q}_{k,x_0})\|_1$, $i = 0, \ldots, k$. By the optimality of $\hat{q}_{k,x_0} \in \mathcal{Q}_k$ it follows that

$$\sum_{i=j}^{k} z_i = U_{k-j}(x_j) \leq \upsilon_{k-j} \|x_j\|_1 = \upsilon_{k-j} z_j , \quad j \in \{0, \ldots, k-1\} .$$

To compute an upper bound for z_k, we use the above k linear inequalities to pose and solve the following linear programming problem (in the variables $z_1, \ldots, z_k$):

$$\begin{aligned} &\max \ z_k \\ \text{s.t.} \quad &X_k \begin{pmatrix} z_1 & \ldots & z_k \end{pmatrix}^* \leq b_k , \\ &z_1 \geq 0 , \ \ldots , \ z_k \geq 0 , \end{aligned}$$

where $X_k \in \mathbb{R}^{k \times k}$, $b_k \in \mathbb{R}^k$ are

$$X_k = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 & 1 \\ (1-\upsilon_{k-1}) & 1 & 1 & \cdots & 1 & 1 \\ 0 & (1-\upsilon_{k-2}) & 1 & \cdots & 1 & 1 \\ \vdots & & & & & \vdots \\ 0 & 0 & 0 & \cdots & (1-\upsilon_1) & 1 \end{pmatrix} ,$$

$$b_k = \begin{pmatrix} (\upsilon_k - 1) z_0 & 0 & 0 & \cdots & 0 \end{pmatrix}^* .$$

The proof of the Lemma is now consequence of the fact that the optimal value for that linear programming problem is exactly given by $\upsilon_k \prod_{j=1}^{k} (\frac{\upsilon_j - 1}{\upsilon_j}) z_0$. ■

VII. Summary and Concluding Remarks

It has been proved that a discrete-time switched positive linear system is state-feedback exponentially stabilizable if and only if for the associated sequence $\{\delta_k\}$, whose elements are computable by solving linear programming problems, there exists $k_0 \in \mathbb{Z}^+$, $k_0 > 0$, with the property that $\delta_{k_0} < 1$. The last was also proved to be equivalent to the existence of $k_0 \in \mathbb{Z}^+$, $k_0 > 0$, for which $\min_{S \in \mathcal{S}_{k_0}} \rho(S) < 1$. It was also shown that a switched positive linear system is state-feedback exponentially stabilizable if and only if an associated dynamic programming equation has a solution W on a convex cone. Such a solution, W, which was shown to be unique, defines a stabilizing state-feedback mapping $\kappa : \mathbb{R}^n \longrightarrow \mathcal{Q}$ via $\kappa(x) \in \arg\min_{q \in \mathcal{Q}} W(A_q|x|)$. Such a mapping κ can always be chosen to be conic-wise constant. The function $W : (\mathbb{R}^n)^+ \longrightarrow \mathbb{R}^+$ is continuous, concave, monotonic, positively homogeneous, can be uniformly approximated on compacts using a finite set of linear functionals, and moreover, $W(|\cdot|)$ is a Lyapunov function for the exponential stability of the trivial solution of the associated closed-loop dynamical system. It was also shown that a switched positive linear system is state-feedback exponentially stabilizable if and only if it is uniformly l_1 bounded. Further, the state-feedback exponential stabilizability of the switched positive linear system was related to an optimal control problem whose complete solution was also presented. The optimal cost functional for that optimal control problem was shown to be the above function W, and a state-feedback mapping κ, as above, was shown to generate an optimal control.

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