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Relaxed Convergence Conditions for Multi-Agent Systems under a Class of Consensus Algorithms

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Abstract— **This paper presents sufficient conditions for the convergence for a group of single integrator agents with a directed and static information flow graph, under a special class of control inputs. Using a novel approach based on the smallest order of the nonzero derivative, it is shown that under some mild conditions the convex hull of the agents will be contracting. The finite intersection property is then used to prove the convergence of the agents to a common fixed point. The results obtained in this work are more general than the ones reported in the literature. An illustrative example is provided to verify the proposed convergence conditions.**

I. Introduction

Cooperative control of multi-agent systems has received considerable attention in the past decade as an effective means of motion coordination [1], [2], [3]. Some recent developments in this area of research are surveyed in [4], and its important applications in mobile robots, sensor networks, air traffic control, etc. are discussed in [5], [3], [6]. In this type of control strategy, a group of local controllers are employed to achieve prescribed objectives cooperatively with minimum information exchange. Typical objectives in cooperative formation control problem include consensus, rendezvous, containment, and formation [7], [8], [9], [10].

In the classical consensus problem, it is desired to find a state update rule for the agents such that some quantity of interest in every agent converges to a common fixed value in the steady state. Various design objectives and constraints are investigated in the literatures in the past few years [11], [7], [12]. Linear time-invariant (LTI) consensus algorithms for multi-agent systems subject to switching communication topologies and time-delay are proposed in [7]. The work [12] presents both discrete and continuous-time consensus protocols for a group of agents which exchange information over limited and unreliable communication links and switching topology. Preserving the connectivity of the underlying network of agents is another important related problem which has been studied in some detail recently [13], [14], [5], [15]. As for design constraints, collision avoidance is one of the most important problems in real-world applications of autonomous vehicles [16], [17], [5].

The convergence of the system state under the control strategies discussed above is typically determined by finding an appropriate Lyapunov function. Constructing a proper Lyapunov function, however, is a complicated task in general. To overcome this problem, in some recent papers the stability of general distributed consensus algorithms (instead of specific ones) is investigated [18], [19], [20], [21]. Graphical conditions are introduced in [18] for the exponential stability of a class of continuous linear time-varying (LTV) systems whose A-matrix in the state-space representation is Metzler with zero row sums. Nonlinear consensus algorithms prove effective when certain criteria such as connectivity preservation and collision avoidance are to be regarded [14], [5], [15]. In [19], the convergence of a class of discrete-time nonlinear consensus algorithms with time-dependent communication links is shown under a convexity assumption and some conditions on the communication graph. As the continuous-time counterpart of [19], the work [20] studies the state agreement problem for coupled nonlinear differential equations with switching vector fields. It is shown that under a strict sub-tangentiality condition and uniformly quasi-strongly connectedness assumption for the interaction digraph, the system has the asymptotic state-agreement property. The case of static interaction digraphs with somewhat relaxed conditions are investigated in [21]. Sufficient conditions for the convergence of a class of nonlinear distributed consensus algorithms for the case of undirected and static information flow graphs are provided in [22]. These conditions require the nonlinear weights appearing in the corresponding control law to be positive unless the agent coincides with all its neighbors.

The present paper deals with the same class of continuous-time nonlinear consensus algorithms considered in [22] for single integrator agents, and provides more relaxed convergence conditions. Similar to the formulation given in [22], the control input of each agent is considered as a state-dependent combination of the relative positions of its neighbors in the information flow graph, which is assumed to be static and directed. Using a novel approach, the convex hull of the agents under the proposed relaxed conditions is shown to have the same nestedness property as [22]. A result from general topology on finite intersection property is then used to prove the convergence of the agents to a common fixed point. The proposed convergence conditions are also more general than the ones in [21], [20], under the additional (mild) assumption that the weights are analytic for the static interaction graph. This generality is discussed in details, and is illustrated by a consensus example for which the convergence cannot be deduced from existing results.

The remainder of this paper is organized as follows. The problem is formulated in Section II, where some useful notations and definitions are also introduced. The convergence conditions for the consensus algorithms introduced

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in Section II are presented in Section III. Simulations are provided in Section IV to demonstrate the convergence under the proposed conditions. Finally, concluding remarks are drawn in Section V.

II. Problem Formulation

In this section, some standard definitions and a formulation similar to [22] are presented.

Definition 1: The function $f : \mathbb{R} \to \mathbb{R}^m$ is said to be of class C^k if the derivatives $f^{(1)}, \ldots, f^{(k)}$ exist and are continuous ($f^{(k)}$ is the k^{th} derivative of f). In particular, a function f of class C^∞ is called a smooth function.

Definition 2: For a smooth function $f : \mathbb{R} \to \mathbb{R}$, the smallest natural number n for which $f^{(n)}(t) \neq 0$ is called the *index* of f at time t, and is denoted by $\rho(f(t))$.

Definition 3: For a smooth function $f : \mathbb{R} \to \mathbb{R}$, the *extended index* of f at time t, denoted by $\tilde{\rho}(f(t))$, is defined as the smallest nonnegative integer n for which $f^{(n)}(t) \neq 0$. Note that by definition $f^{(0)}(t)$ is the same as $f(t)$.

Definition 4: A function $f : \mathbb{R}^m \to \mathbb{R}$, is called *analytic* on $\mathbb{R}^m$, written $f \in C^\omega(\mathbb{R}^m)$, if for any $\alpha \in \mathbb{R}^m$ the function f may be expressed as a convergent power series in some neighborhood of α (see [23]). Note that the set of analytic functions and smooth functions are not equivalent.

Definition 5: A set-valued function $S(\cdot)$ is said to be *nested* if for every $t_1, t_2 \in \mathbb{R}$, $0 \le t_1 \le t_2$, the relation $S(t_2) \subseteq S(t_1)$ holds.

Definition 6: In a digraph G, a vertex v is said to be *reachable* from a vertex u, if there is a directed path from u to v. The set of all reachable vertices from the vertex u in G is denoted by $R_u(G)$.

Definition 7: A digraph G is said to be *quasi-strongly connected* if for every pair of distinct vertices u and v of G, there is a vertex from which both u and v are reachable (see [24]).

Definition 8: A group of agents $1, \ldots, n$ is said to converge to a consensus if $q_i(t) \to \bar{q}$ as $t \to \infty$, for any $i \in \mathbb{N}_n := \{1, \ldots, n\}$, where $q_i(t) \in \mathbb{R}^m$ denotes the state of agent i at time t, and $\bar{q}$ is a constant.

Definition 9: For a function $q : \mathbb{R} \to \mathbb{R}^m$, the point $\bar{p} \in \mathbb{R}^m$ is said to be a *positive limit point* of $q(\cdot)$ if there exists a sequence $\{t_n\}$ with $t_n \to \infty$ as $n \to \infty$, such that $q(t_n) \to \bar{p}$ as $n \to \infty$. The set of all positive limit points of $q(\cdot)$ is called the *positive limit set* of $q(\cdot)$.

Definition 10: A family $\mathscr{A} = \{A_\alpha\}_{\alpha \in I}$ of subsets of a set X is said to have the finite intersection property if every finite sub-family $\{A_1, A_2, \ldots, A_n\}$ of $\mathscr{A}$ satisfies $\bigcap_{i=1}^{n} A_i \neq \emptyset$ (see [25]).

Consider a set of n single-integrator agents in the 2D plane, each represented by

$$\dot{q}_i(t) = u_i(t), \quad i \in \mathbb{N}_n \tag{1}$$

where $q_i(t) \in \mathbb{R}^2$ is the position of agent i at time t, and u_i is the corresponding control signal. Note that for brevity, the time argument is omitted hereafter in some of the time-dependent functions. Let the information flow graph for the network be denoted by $G = (V, E)$, with $V = \{1, \ldots, n\}$ representing the set of n vertices (associated with the n agents), and $E \subseteq V \times V$ representing the corresponding edges. The information flow graph G is assumed to be static and directed. There is a directed edge from vertex j to vertex i in G if and only if $(j,i) \in E$. The set of the neighbors of vertex i in G is defined as $N_i = \{j | (j,i) \in E\}$, and its indegree is $d_i = |N_i|$. Each agent can only incorporate its own position and the position of its neighbors in its control law. In this paper, distributed control laws of the following form are considered

$$u_i = -\sum_{j \in N_i} \beta_{ij}(q_i - q_j) \quad , \quad i \in \mathbb{N}_n \tag{2}$$

where the coefficients $\beta_{ij} : \mathbb{R}^{2(d_i+1)} \to \mathbb{R}$, $i \in \mathbb{N}_n$, $j \in N_i$, are state-dependent. More specifically, each coefficient β_{ij} is a function of the position of agent i and the positions of the neighbors of agent i in G.

Problem Statement: It is desired to obtain sufficient conditions (less conservative than the existing results) on the coefficients β_{ij} in (2), which guarantee the convergence of the agents to a consensus.

III. Main Results

Consider again a set of n agents in the 2D plane with the differential equations of the form (1), and let them evolve according to the control laws given by (2). The aim of this section is to show that under the following assumptions on the coefficients β_{ij} in (2), the agents converge to a consensus.

Assumption 1: The state-dependent coefficients β_{ij} in (2) are analytic, real and nonnegative for all $i \in \mathbb{N}_n$ and $j \in N_i$.

Assumption 2: The system (1) with the control law of the form (2) has no solution in which the convex hull of the agents:

1) is not a singleton,
2) is fixed with at least one fixed agent at each vertex.

Denote with $S(t)$ the convex hull of the agents at time t, i.e. $S(t) = \mathrm{Conv}(\{q_i(t) | i \in \mathbb{N}_n\})$. Four lemmas are presented in the sequel, which will later be used to prove the nestedness property for $S(t)$.

Lemma 1: Consider a function $f : \mathbb{R} \to \mathbb{R}$, with the property that $f^{\rho(f(t))}(t) > 0$, for some t. Then, there exists $\delta > 0$ such that $f(t) < f(t+\tau)$, $\forall \tau \in (0, \delta]$.

Proof. The proof is straightforward, and is omitted here. ■

Remark 1: if $f^{\rho(f(t))}(t) < 0$, then one can show that there exists $\delta > 0$ for which $f(t) > f(t+\tau)$, $\forall \tau \in (0, \delta]$.

In order to show the nestedness property for the convex hull $S(t)$, it is required to investigate the behavior of the agents on the boundary of the convex hull. Consider a line l passing through the boundary of $S(t)$ at some time $t \ge 0$. Denote with e_l the unit vector perpendicular to l, in the direction of the half-plane containing $S(t)$. Note that the intersection of $S(t)$ with l is either an edge or a vertex (Fig. 1 shows the case when the intersection is an edge). Define $f_l : \mathbb{R}^2 \to \mathbb{R}$ as the projection of x on e_l, i.e. $f_l(x) = < x, e_l >$. Let agent i be on l at time t. Denote with $N_i^l(t)$ the set of those neighbors of i which lie on l, and with $\bar{N}_i^l(t)$ the set

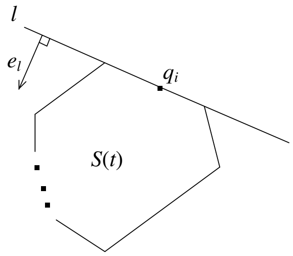

Fig. 1: $S(t)$ is the convex hull of the agents at time t, q_i is the position of an agent on l, and e_l is the unit vector perpendicular to l in the direction of the half-plane containing $S(t)$.

of those neighbors do not lie on l. Now, define $\eta_{i1}^l(t)$ and $\eta_{i2}^l(t)$ as follows

$$\eta_{i1}^l(t) = \begin{cases} \min_{j \in N_i^l(t)} \{\tilde{\rho}(\beta_{ij}) + \rho(f_l(q_j))\}, & N_i^l(t) \neq \emptyset \\ \infty, & N_i^l(t) = \emptyset \end{cases} \tag{3}$$

and

$$\eta_{i2}^l(t) = \begin{cases} \min_{j \in \bar{N}_i^l(t)} \{\tilde{\rho}(\beta_{ij})\}, & \bar{N}_i^l(t) \neq \emptyset \\ \infty, & \bar{N}_i^l(t) = \emptyset \end{cases} \tag{4}$$

where in calculating $\tilde{\rho}(\beta_{ij})$, the coefficient β_{ij} is regarded as an implicit function of time; clearly $\eta_{i1}^l(t) \geq 1$ and $\eta_{i2}^l(t) \geq 0$. Define also

$$\eta_i^l(t) = \min\{\eta_{i1}^l(t), \eta_{i2}^l(t)\} \tag{5}$$

Using Lemmas 2-4 and the definitions given above, the behavior of the agents on the boundary of $S(t)$ is described in the sequel.

Lemma 2: Consider a line l passing through the boundary of $S(t)$ at some time $t \geq 0$, and assume that $q_i(t)$ belongs to l, for some $i \in \mathbb{N}_n$. Then, the following statements are true:
i) If $\eta_i^l = 0$, then $f_l(\dot{q}_i) > 0$.
ii) If $\eta_i^l \geq 1$, then $f_l(q_i^{(k)}) = 0$, for $k = 1, \ldots, \eta_i^l$.
Proof.
Part (i): First, note that for any $j \in N_i^l$, the function $f_l(q_j - q_i)$ is equal to zero, and for any $j \in \bar{N}_i^l$, it is strictly positive. Also, $\beta_{ij} \geq 0$ for any $j \in N_i$, according to Assumption 1. The relation $\eta_i^l = 0$ implies that $\eta_{i2}^l = 0$, which yields that $\bar{N}_i^l \neq \emptyset$, and that there exists an agent $v \in \bar{N}_i^l$ for which $\beta_{iv} > 0$. Therefore, using (1) and (2) one can write

$$\begin{aligned} f_l(\dot{q}_i) &= \sum_{j \in \bar{N}_i^l} \beta_{ij} f_l(q_j - q_i) \\ &\geq \beta_{iv} f_l(q_v - q_i) \\ &> 0 \end{aligned} \tag{6}$$

Part (ii): It is straightforward to show that

$$f_l(q_i^{(k+1)}) = \sum_{j \in N_i} \sum_{r=0}^{k} \beta_{ij}^{(k-r)} (f_l(q_j^{(r)}) - f_l(q_i^{(r)})) \binom{k}{r} \tag{7}$$

where $\beta_{ij}^{(k-r)}$ is the $(k-r)^{th}$ derivative of β_{ij} with respect to time (note that β_{ij} is an implicit function of time). Assume now that $k < \eta_i^l$; this means that $k - r < \eta_i^l \leq \eta_{i2}^l$, and hence $\beta_{ij}^{(k-r)} = 0$ for $j \in \bar{N}_i^l$. On the other hand, since $k < \eta_i^l \leq \eta_{i1}^l$, one can easily show that $\beta_{ij}^{(k-r)} f_l(q_j^{(r)}) = 0$, for $j \in N_i^l$ and $1 \leq r \leq k$. Using these results along with the fact that $f_l(q_j - q_i) = 0$ for $j \in N_i^l$, equation (7) reduces to

$$f_l(q_i^{(k+1)}) = -\sum_{j \in N_i^l} \sum_{r=1}^{k} \beta_{ij}^{(k-r)} f_l(q_i^{(r)}) \binom{k}{r} \tag{8}$$

The rest of the proof follows by a simple induction. ∎

Lemma 3: Consider a line l passing through the boundary of $S(t)$ at some time $t \geq 0$, and assume that $q_i(t) \in l$, for some $i \in \mathbb{N}_n$. If $\rho(f_l(q_i)) < \infty$, then $f_l(q_i^{(\rho(f_l(q_i)))}) > 0$.

Proof. Since $\rho(f_l(q_i)) < \infty$, thus $\eta_i^l < \infty$ according to Lemma 2. Before getting to the proof, let some useful results on $f_l(q_i^{(\eta_i^l+1)})$ be obtained, assuming $1 \leq \eta_i^l < \infty$. Using Lemma 2 and taking an approach similar to the one used to derive (8) from (7), one can show that

$$\begin{aligned} f_l(q_i^{(\eta_i^l+1)}) &= \sum_{j \in N_i^l} \sum_{r=1}^{\eta_i^l} \beta_{ij}^{(\eta_i^l - r)} f_l(q_j^{(r)}) \binom{\eta_i^l}{r} \\ &+ \sum_{j \in \bar{N}_i^l} \beta_{ij}^{(\eta_i^l)} f_l(q_j - q_i) \end{aligned} \tag{9}$$

There are three possible cases for η_i^l, η_{i1}^l, and η_{i2}^l:
Case (i): $\eta_i^l = \eta_{i2}^l < \eta_{i1}^l$. In this case, (9) reduces to

$$f_l(q_i^{(\eta_i^l+1)}) = \sum_{\substack{j \in \bar{N}_i^l \\ \tilde{\rho}(\beta_{ij}) = \eta_i^l}} \beta_{ij}^{(\eta_i^l)} f_l(q_j - q_i) \tag{10}$$

On the other hand, the relation $\tilde{\rho}(\beta_{ij}) = \eta_i^l \geq 1$ implies that $\beta_{ij} = 0$. If $\beta_{ij}^{(\tilde{\rho}(\beta_{ij}))} < 0$, then it is straightforward to show using Remark 1 that β_{ij} is negative in a right-sided vicinity of t (again, β_{ij} is regarded here as an implicit function of time). However, this is in contradiction with Assumption 1; therefore $\beta_{ij}^{(\tilde{\rho}(\beta_{ij}))} > 0$, and it results from (10) that $f_l(q_i^{(\eta_i^l+1)}) > 0$.
Case (ii): $\eta_i^l = \eta_{i1}^l < \eta_{i2}^l$. In this case, (9) reduces to

$$f_l(q_i^{(\eta_i^l+1)}) = \sum_{\substack{j \in N_i(l) \\ \tilde{\rho}(\beta_{ij}) + \rho(f_l(q_j)) = \eta_i^l}} \beta_{ij}^{(\tilde{\rho}(\beta_{ij}))} f_l(q_j^{(\rho(f_l(q_j)))}) \binom{\eta_i^l}{\tilde{\rho}(\beta_{ij})} \tag{11}$$

If $\beta_{ij} \neq 0$, then $\tilde{\rho}(\beta_{ij}) = 0$ and $\beta_{ij}^{(\tilde{\rho}(\beta_{ij}))} = \beta_{ij} > 0$. If on the other hand $\beta_{ij} = 0$, then the inequality $\beta_{ij}^{(\tilde{\rho}(\beta_{ij}))} > 0$ still holds as shown in case (i).
Case (iii): $\eta_i^l = \eta_{i1}^l = \eta_{i2}^l$. In this case, (9) yields

$$\begin{aligned} f_l(q_i^{(\eta_i^l+1)}) &= \sum_{\substack{j \in N_i(l) \\ \tilde{\rho}(\beta_{ij}) + \rho(f_l(q_j)) = \eta_i^l}} \beta_{ij}^{(\tilde{\rho}(\beta_{ij}))} f_l(q_j^{(\rho(f_l(q_j)))}) \binom{\eta_i^l}{\tilde{\rho}(\beta_{ij})} \\ &+ \sum_{\substack{j \in \bar{N}_i^l \\ \tilde{\rho}(\beta_{ij}) = \eta_i^l}} \beta_{ij}^{(\eta_i^l)} f_l(q_j - q_i) \\ &> \sum_{\substack{j \in N_i(l) \\ \tilde{\rho}(\beta_{ij}) + \rho(f_l(q_j)) = \eta_i^l}} \beta_{ij}^{(\tilde{\rho}(\beta_{ij}))} f_l(q_j^{(\rho(f_l(q_j)))}) \binom{\eta_i^l}{\tilde{\rho}(\beta_{ij})} \end{aligned} \tag{12}$$

(note that the inequalities $\beta_{ij}^{(\tilde{\rho}(\beta_{ij}))} > 0$ and $f_l(q_j - q_i) > 0$, $\forall j \in \bar{N}_i^l$, are used in deriving (12)).

From the results presented in cases (ii) and (iii), one can easily conclude that if $\eta_i^l = \eta_{i1}^l$, then

$$f_l(q_i^{(\eta_i^l+1)}) \geq \sum_{\substack{j\in N_i(l)\\ \tilde{\rho}(\beta_{ij})+\rho(f_l(q_j))=\eta_i^l}} \alpha_{ij} f_l(q_j^{(\rho(f_l(q_j)))}) \tag{13}$$

for some positive coefficients α_{ij}'s.

It is desired now to use induction on $\rho(f_l(q_i))$ together with the results developed so far in the paper to prove the lemma. For $\rho(f_l(q_i)) = 1$, if $\eta_i^l \geq 1$ then it results from Lemma 2 that $f_l(\dot{q}_i) = 0$, which is a contradiction; therefore, $\eta_i^l = 0$, and hence $f_l(\dot{q}_i) > 0$, according to Lemma 2. Assume now that the statement of the lemma holds for $\rho(f_l(q_i)) \leq k$, for some $k \geq 1$; it is desired to prove that it holds for $\rho(f_l(q_i)) = k+1$ as well. Lemma 2 implies that $1 \leq \eta_i^l < k+1$. If $\eta_{i2}^l < \eta_{i1}^l$, then it results from case (i) as well as Lemma 2 that $\rho(f_l(q_i)) = \eta_i^l + 1$ and $f_l(q_i^{(\eta_i^l+1)}) > 0$. If on the other hand $\eta_{i2}^l \geq \eta_{i1}^l$ (i.e. $\eta_i^l = \eta_{i1}^l$), then (13) holds. Moreover, for any j in the summation domain of (13) the relation $\rho(f_l(q_j)) \leq \eta_i^l < k+1$ holds, and hence due to the assumption of induction $f_l(q_j^{(\rho(f_l(q_j)))}) > 0$. This along with (13) yields $f_l(q_i^{(\eta_i^l+1)}) > 0$, and then it is implies from Lemma 2 that $\rho(f_l(q_i)) = \eta_i^l + 1$. This completes the proof. ■

Corollary 1: Consider a line l passing through the boundary of $S(t)$ at some time $t \geq 0$, and assume that $q_i(t) \in l$, for some $i \in \mathbb{N}_n$. Then, $\rho(f_l(q_i)) = \eta_i^l + 1 = \min\{\eta_{i1}^l, \eta_{i2}^l\} + 1$, where η_{i1}^l and η_{i2}^l are defined in (3) and (4).

Proof. The proof follows directly from Lemma 3 (as its by-product). ■

Lemma 4: Consider a line l passing through the boundary of $S(t)$ at some time $t \geq 0$. Given $q_i(t) \in l$, if $f_l(q_i(t))$ has a finite index, then there exists $\delta_i > 0$ such that for any $\tau \in (0, \delta_i]$ the inequality $f_l(q_i(t)) < f_l(q_i(t+\tau))$ holds; otherwise, $f_l(\dot{q}_i) \equiv 0$.

Proof. If $\rho(f_l(q_i)) < \infty$, then according to Lemma 3, $f_l(q_i^{(\rho(f_l(q_i)))}) > 0$. Therefore, it results from Lemma 1 that there exists $\delta_i > 0$ such that for any $\tau \in (0, \delta_i]$ the inequality $f_l(q_i(t)) < f_l(q_i(t+\tau))$ holds. This means that agent i will move towards the interior of the half plane (defined by l) containing $S(t)$.

Now, consider the case where $\rho(f_l(q_i)) = \infty$. Since β_{ij}'s are analytic, according to Theorem 39.12 in [26], q_i is also analytic, implying that $f_l(q_i)$ is analytic as well. Therefore, it is resulted from $\rho(f_l(q_i)) = \infty$ that $f_l(q_i) \equiv f_l(q_i(t))$, which means that q_i has been on l from the beginning and will stay on it at all times. ■

Theorem 1: Under Assumption 1, the convex hull of the agents is nested.

Proof. Consider the agents at any arbitrary time $t \geq 0$. By applying Lemma 4 to all the edges on the boundary of $S(t)$, one can easily show that there exists $\delta(t) > 0$ for which

$$q_i(t+\tau) \in S(t), \quad \forall i \in \mathbb{N}_n,\ \forall \tau \in [0, \delta(t)] \tag{14}$$

implying that $S(t+\tau) \subseteq S(t)$, for any $\tau \in [0, \delta(t)]$. Using this and an approach similar to the one used in the proof of nestedness in Theorem 1 in [22], the nestedness property of $S(t)$ can be deduced. ■

The following result from [25] will be used in the proof of the main theorem.

Theorem 2: A topological space is compact if and only if each family of closed sets which has the finite intersection property has a non-void intersection.

In the sequel, sufficient conditions are provided for convergence to consensus, as the most important contribution of the paper.

Theorem 3: Consider a set of n agents in the 2D plane with dynamics of the form (1), evolved under the control laws given by (2). Under Assumptions 1-2, the agents converge to a consensus.

Proof. Define $\mu_1(q(t))$ and $\mu_2(q(t))$ as the area and the diameter of $S(t)$, respectively, where $q(t) = (q_1(t), \ldots, q_n(t))$. Using the nestedness property of $S(t)$, it is straightforward to show that there exists nonnegative real numbers a_1 and a_2 for which $\lim_{t\to\infty} \mu_1(q(t)) = a_1$ and $\lim_{t\to\infty} \mu_2(q(t)) = a_2$. Moreover, for any $p \in L^+$, $\mu_1(p) = a_1$ and $\mu_2(p) = a_2$, where L^+ denotes the positive limit set of $q(t)$ (see the proof of Theorem 1 in [22]). It is desired now to show that $a_1 = 0$. If $a_1 > 0$, then the invariant property of L^+ (see Lemma 4 in [22]) along with the nestedness property of the convex hull of the agents, and the fact that $\mu_1(p) = a_1$ for any $p \in L^+$, yields that starting from any $p(0) = (p_1(0), \ldots, p_n(0)) \in L^+$, the convex hull $S(t)$ will remain fixed, i.e. $S(t) \equiv S(0)$. Consider an agent, say agent i, at a vertex of $S(0)$ and let l_1 and l_2 be the two lines passing through the two edges connected to this vertex on the boundary of $S(0)$. Using Lemma 4 (once with $l = l_1$ and then with $l = l_2$) it can be concluded that either agent i moves away from this vertex or $f_{l_1}(\dot{p}_i) \equiv f_{l_2}(\dot{p}_i) \equiv 0$, the latter case implying that agent i stays fixed at that vertex. Therefore, in order for $S(t)$ to remain fixed, there should be at least one fixed agent at each vertex of $S(0)$, which contradicts Assumption 2. This contradiction yields $a_1 = 0$, i.e. if $p = (p_1, \ldots, p_n)$ is a positive limit point, then p_i's are collinear. Using this property and following a similar argument, it is concluded that a_2 ia also 0, i.e. $p_1 = \ldots = p_n$ for any $p = (p_1, \ldots, p_n) \in L^+$.

To complete the proof, note that since $S(t)$ is nested, it satisfies the finite intersection property (see Definition 10), and hence according to Theorem 2, $\bigcap_{t\geq 0} S(t) = Q \neq \emptyset$. On the other hand, $a_2 = 0$ implies that the diameter of $S(t)$ approaches 0 as $t \to \infty$, meaning that Q is a single point. On the other hand, since $Q \in S(t)$, hence $\|q_i(t) - Q\| \leq \mu_2(q(t))$, which in turn implies that $q_i(t) \to Q$ as $t \to \infty$ because $\mu_2(q(t)) \to 0$ as $t \to \infty$. In other words, the agents converge to a fixed single point, and this completes the proof. ■

Assumption 2 is essential in the above theorem, but it is not straightforward to verify it, in general. The following proposition will prove useful in verifying the condition of this assumption.

Proposition 1: Let the condition of Assumption 1 hold, and assume the convex hull of the agents is fixed. Then for a fixed agent, say agent i, at a vertex of this convex hull, and for every $j \in N_i$, either $q_j \equiv q_i$ or $\beta_{ij} \equiv 0$.

Proof. First note that under Assumption 1, Lemmas 2-4 and Theorem 1 still hold. Consider the agents at some $t \geq 0$, and let l_1 and l_2 be the two lines passing through the two edges on the boundary of the convex hull connected to the vertex at which q_i is fixed. Using Corollary 1 for both l_1 and l_2, one can conclude that $\tilde{\rho}(\beta_{ij}) = \infty$ for $j \in \bar{N}_i(l_1) \cup \bar{N}_i(l_2)$, implying that β_{ij} is identically zero because it is analytic. The only remaining neighbors that are not in $\bar{N}_i(l_1) \cup \bar{N}_i(l_2)$ are those for which $q_j(t) = q_i(t)$. For such a neighbor, if $\tilde{\rho}(\beta_{ij}) = \infty$ then $\beta_{ij} \equiv 0$ similarly; if on the other hand $\tilde{\rho}(\beta_{ij})$ is finite, then $\rho(f_{l_1}(q_j(t))) = \rho(f_{l_2}(q_j(t))) = \infty$, and consequently $f_{l_1}(\dot{q}_j) \equiv f_{l_2}(\dot{q}_j) \equiv 0$. This implies that $\dot{q}_j \equiv 0$, and hence $q_j \equiv q_i$. ■

The next proposition characterizes the main advantage of this work over [21], [20], [22].

Proposition 2: Consider a set of n agents in the 2D plane with dynamics of the form (1), and a quasi-strongly connected information flow graph. Let the control law be of the form (2), where the corresponding coefficients satisfy the conditions of Assumption 1. Define $Q_i = \{q_j | j \in N_i \cup \{i\}\}$, and assume that if agent i is at a vertex of $\mathrm{Conv}(Q_i)$ and Q_i is not a singleton, then $\dot{q}_i \not\equiv 0$. Then the agents converge to a consensus.

Proof. It suffices to show that the conditions of the proposition imply that Assumption 2 holds. Suppose that there is a solution for which Assumption 2 does not hold, and let agent i be a fixed agent at a vertex of the convex hull associated with such a solution. Clearly, q_i is also a vertex of $\mathrm{Conv}(Q_i)$ at all times. This, along with the fact that $\dot{q}_i \equiv 0$, results that Q_i should be a singleton at all times, and hence $q_j \equiv q_i$ for all $j \in N_i$. Repeating the same argument, one can conclude that $q_j \equiv q_i$ for all agents j from which i is reachable in G. Now, consider two fixed agents i_1 and i_2 at two distinct vertices of the convex hull. Since G is quasi-strongly connected, there is an agent j from which both i_1 and i_2 are reachable in G, implying that $q_{i_1} \equiv q_{i_2}$. This contradicts the initial assumption that agents i_1 and i_2 are located at two distinct vertices of the convex hull, and completes the proof. ■

Remark 2: It is important to note that [21], [20] do not guarantee convergence to a consensus under the setting of Proposition 2. More precisely, [21], [20] require that when an agent takes a certain position with respect to some of its neighbors, its velocity must be nonzero. However, the present work only requires that in such configurations the velocity of the above-mentioned agent is not identically zero in order to deduce convergence to a consensus. On the other hand, it is to be noted that [21], [20] do not require the control coefficients β_{ij}'s to be analytic. They only require that the control signals u_i's are continuous functions of the state.

Remark 3: The results obtained in this work are more general than the ones in [22] in the sense that it is concerned with directed information flow graphs (as opposed to undirected information flow graphs considered in [22]). Moreover, [22] requires all the control coefficients β_{ij}'s to be positive if Q_i is not a singleton (which also requires the velocity of agent i to be nonzero). According to Proposition 2, however, some or all of these coefficients can be zero at some time instants, as long as the velocity of corresponding agent is not identically zero, i.e. $\dot{q}_i \not\equiv 0$.

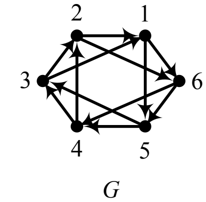

Fig. 2: The information flow graph G in Example 1 for the case of $n = 6$

IV. Simulation Results

Example 1: Consider a swarm of n agents in a plane with the dynamics of the form (1) and the control inputs given by

$$\begin{aligned} u_i &= -\|q_i - q_{i+1}\|^2 (q_i - q_{i+1}) \\ &\quad -(1 - \|q_i - q_{i+2}\|^2)^2 (q_i - q_{i+2}) \end{aligned} \tag{15}$$

where $i \in \mathbb{N}_n$, $q_{n+1} = q_1$, and $q_{n+2} = q_2$. It can be easily verified that Assumption 1 holds for the above control law. Therefore, to show the convergence of the agents to a consensus, it suffices to show that Assumption 2 also holds. Suppose that there exists a solution to (1) under the control inputs given by (15), for which Assumption 2 does not hold. Assume also that agent i is fixed at a vertex of the fixed convex hull corresponding to this solution, for some $i \in \mathbb{N}_n$. Proposition 1 implies that either $\|q_i - q_{i+1}\|^2 \equiv 0$ or $q_{i+1} \equiv q_i$, either case resulting in $q_{i+1} \equiv q_i$. Similarly, one can show that $q_{i+2} \equiv q_{i+1}$. Repeating the same argument, it can be concluded that all agents should coincide with agent i, which is a contradiction because a solution which does not satisfy Assumption 2 should not be a singleton. Therefore, Assumptions 1 and 2 both hold, and convergence to a consensus is consequently deduced from Theorem 3.

The information flow graph G and the trajectories of the agents under the given control law for this example for the case of $n = 6$ are depicted in Figs. 2 and 3, respectively. The convex hull of the agents at three time instants $t_0 = 0$ sec, $t_1 = 0.3$ sec, and $t_2 = 1.25$ sec are also drawn in Fig. 3. It can be observed from this figure that $S(t_2) \subseteq S(t_1) \subseteq S(t_0)$. This is in accordance with the nestedness property of $S(t)$ which results from Theorem 1. The norms of the control inputs u_i, $i \in \mathbb{N}_6$ are also plotted in Fig. 4.

Remark 4: It is important to note that convergence to a consensus for the above example cannot be deduced from [21], [20], [22].

V. Conclusions

This paper presents conditions for the convergence of a class of continuous-time nonlinear consensus algorithms for single integrator agents. The main contribution of this work is to develop less restictive convergence conditions, as an extension to the authors' recent work in [22]. The control

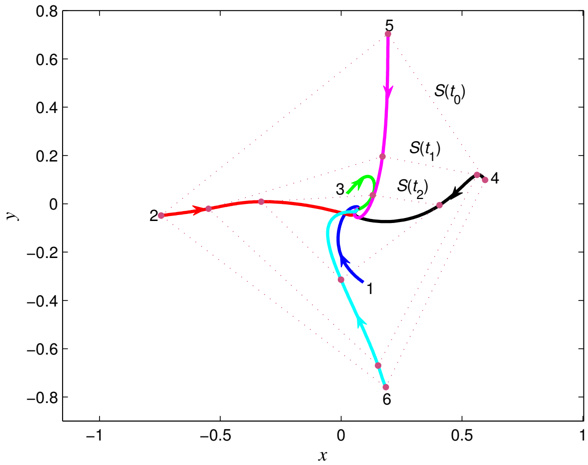

Fig. 3: The agents' planar motion for the case of $n = 6$

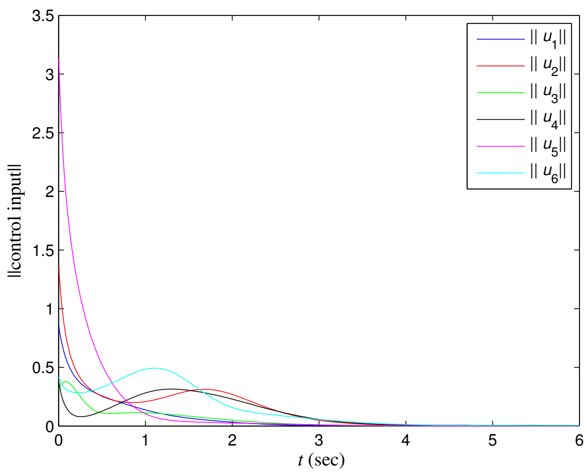

Fig. 4: The norms of the control inputs u_i, $i \in \mathbb{N}_6$

input of each agent is assumed to have the same form as in [22], while the information flow graph is assumed to be directed. It is shown that the convex hull of the agents preserves the nestedness property under the proposed mild conditions. The convergence to a fixed point is subsequently proved using a Lasalle-like approach as well as the finite intersection property of convex hulls. The results are shown to be more general than the ones reported in the literature, and simulations for a consensus problem confirm the validity of the results.

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