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$\mathcal{L}_1$ Adaptive Control for Positive LTI Systems

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***Abstract*— Positive systems represent a class of systems whose state variables are nonnegative. This paper presents an $\mathcal{L}_1$ adaptive controller for such systems. The control objective is to force the system output track a given nonnegative reference signal. In the analysis of the $\mathcal{L}_1$ adaptive controller, we first show the positivity of the constructed reference system, and then demonstrate that the controlled system states track those of the reference system. We provide the uniform transient and steady-state performance bounds on the system state and the control signals, which can be systematically improved by increasing the adaptation rate, under the assumption that the unknown system parameters are positive. Simulations are presented to corroborate our results.**

I. Introduction

Positive systems can be seen as a special class of systems where the state variables are nonnegative for all time [12]. Positive systems are pervasive in engineering applications and in nature. For example, many cellular systems that describe transportation, accumulation, and drainage processes of elements and compounds like hormones, glucose, insulin and metals are positive systems. In biomedical applications, the dynamical evolution of virus populations under drug treatment is a positive system since the cell populations and the drug doses can never be negative [1]. In industrial engineering, many systems that involve chemical reactions and heat exchangers are also examples of positive systems [5].

The control of positive systems has been of great interest for many decades. In [4], necessary and sufficient conditions are obtained for stabilizability of positive LTI systems. In [2], the authors develop optimal output feedback controllers for set-point regulation of linear non-negative dynamical systems. In [3], the servomechanism problem of nonnegative constant reference signals for stable MIMO positive LTI systems with unmeasurable unknown constant nonnegative disturbances under strictly nonnegative control inputs is solved using a clamping LQ regulator.

For control of uncertain systems with guaranteed performance, $\mathcal{L}_1$ adaptive control has proven to be a promising direction. Previous work [7]–[10] has shown that $\mathcal{L}_1$ adaptive controllers guarantee uniform performance bounds for system's both input and output signals. These bounds are inversely proportional to the adaptation rate, and it has been shown that increasing the adaptation rate does not compromise the robustness, i.e., the time-delay margin is guaranteed to be bounded away from *zero* [7], [11]. These results motivate the developments of this paper, which aims at quantification of the performance bounds of $\mathcal{L}_1$ controller for positive systems.

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In this paper, we keep the same structure of the $\mathcal{L}_1$ adaptive controller as in [8] and construct one that retains the nonnegative components of those studied earlier. The reshaping of the control signal introduces new tracking error. Using the properties of $\mathcal{L}_1$ norm, we analyze the tracking and the prediction errors for positive systems and show the desired transient performance of the output signal.

The paper is organized as follows. Section II states some preliminary definitions and the problem formulation. We present the $\mathcal{L}_1$ adaptive controller for nonnegative systems in Section III, and analyze its performance in Section IV. In Section V, we use numerical examples to illustrate the controller. Section VI concludes the paper and discusses future work.

II. Problem Formulation

In this section, we formulate the $\mathcal{L}_1$ adaptive control problem for positive systems. We first review some basic definitions and facts on nonnegative matrices and positive systems.

Definition 1: A matrix $A = [a_{i,j}] \in \mathbb{R}^{n\times n}$ is a *nonnegative* matrix, if all its entries are nonnegative.

Definition 2: A matrix $A \in \mathbb{R}^{n\times n}$ is *Metzler*, if all its off-diagonal elements are nonnegative.

Definition 3 ([5]): A linear system

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t), \quad x(0) = x_0, \\ y(t) &= Cx(t) + Du(t), \end{aligned} \tag{1}$$

where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{r\times n}$, $D \in \mathbb{R}^{r\times m}$ is a *positive* linear system, if for every nonnegative initial state and for every nonnegative input, the state of the system and the output remain nonnegative.

A positive LTI system can be characterized by its system matrices. The following theorem gives a necessary and sufficient condition for a system to be positive.

Theorem 1 ([5]): A linear system in (1) is *positive* if and only if the matrix A is *Metzler*, and B, C and D are nonnegative matrices.

In this paper, we consider the following positive LTI system

$$\begin{aligned} &\dot{x}(t) = Ax(t) + bu_+(t)\,, \quad x(0) = x_0\,, \\ &y(t) = c^\top x(t)\,, \end{aligned} \tag{2}$$

where $x(t) \in \mathbb{R}^n_+$ is the system state vector, $u_+(t) \in \mathbb{R}_+$ is the control signal, A is an unknown $n \times n$ matrix, (A, b) is controllable, b, $c \in \mathbb{R}^n_+$ are known constant vectors, and $y(t) \in \mathbb{R}$ is the regulated output.

Since we assume that system (2) is positive, by Theorem 1, we have A Metzler, and b and c nonnegative. To facilitate our analysis, we make the following assumptions.

We first assume that A is Hurwitz. To see the necessity of the assumption, we refer to the following result from Lemma 6 in [4].

Lemma 1 ([4]): Consider a positive system (A, b), where A is an unstable Metzler matrix, $b \geq 0$. There does not exist $k \geq 0$ such that $A + bk^\top$ is stable.

Assumption 1: The system matrix A is Hurwitz.

Then we introduce the widely used matching assumption and further assume some rough knowledge of the unknown parameter θ.

Assumption 2 (Matching Assumption): Given a Hurwitz $A_m \in \mathbb{R}^{n\times n}$, there exists a parameter vector θ, such that $A = A_m + b\theta^\top$. Further assume that the unknown parameter θ belongs to a given compact convex set, $\theta \in \Theta_B \subset \mathbb{R}^n$.

With Assumptions 1 and 2 at hand, we can rewrite the system as

$$\begin{aligned} \dot{x}(t) &= A_m x(t) + b\left(\theta^\top x(t) + u_+(t)\right), \quad x(0) = x_0, \\ y(t) &= c^\top x(t). \end{aligned} \tag{3}$$

The objective is to design an adaptive nonnegative control $u_+(t)$ which ensures that the system output $y(t)$ tracks a given nonnegative reference signal $r(t)$ with quantifiable bounds both in transient and steady-state.

III. $\mathcal{L}_1$ Adaptive Controller

In this section, we introduce the $\mathcal{L}_1$ adaptive controller for the system in (3). The $\mathcal{L}_1$ adaptive controller consists of the state predictor, the adaptive law and the control law.

For the linearly parameterized system in (3), we consider the following state predictor

$$\begin{aligned} \dot{\hat{x}}(t) &= A_m\hat{x}(t) + b\left(\hat{\theta}^\top(t)x(t) + u_+(t)\right) \\ \hat{y}(t) &= c^\top\hat{x}(t), \quad \hat{x}(0) = x_0, \end{aligned} \tag{4}$$

where $\hat{x}(t) \in \mathbb{R}^n$, $\hat{y}(t) \in \mathbb{R}$ are the state and the output of the state predictor and $\hat{\theta}(t) \in \mathbb{R}^n$ is an estimate of the parameter θ. The projection-type adaptive law for $\hat{\theta}(t)$ is given by

$$\dot{\hat{\theta}}(t) = \Gamma \mathrm{Proj}(\hat{\theta}(t), -x(t)\tilde{x}^\top(t)Pb), \ \hat{\theta}(0) = \hat{\theta}_0, \tag{5}$$

where $\tilde{x}(t) \triangleq \hat{x}(t) - x(t)$ is the prediction error, $\Gamma > 0$ is the adaptation rate, the projection ensures that $\hat{\theta}(t) \in \Theta_B$ for all $t \geq 0$, and $P = P^\top > 0$ is the solution to the algebraic Lyapunov equation $A_m^\top P + PA_m = -Q$ for some $Q > 0$.

The control signal is defined by

$$\begin{aligned} u_+(t) &= \hat{\eta}_{rC}(t)\mathbf{I}_{\{\hat{\eta}_r(t)\geq 0\}}, \ \hat{\eta}_{rC}(s) = C(s)\hat{\eta}_{r+}(s), \\ \hat{\eta}_{r+}(t) &= \hat{\eta}_r(t)\mathbf{I}_{\{\hat{\eta}_r(t)\geq 0\}}, \ \hat{\eta}_r(s) = -\hat{\eta}(s) + k_g r(s), \end{aligned} \tag{6}$$

where $k_g \triangleq 1/(c^\top H(0))$,

$$H(s) \triangleq (s\mathbb{I} - A_m)^{-1}b, \ \hat{\eta}(t) \triangleq \hat{\theta}^\top(t)x(t), \tag{7}$$

and $\mathbf{I}_{\{\hat{\eta}_r(t)\geq 0\}}$ is an indicator function. Here $u_+(t)$ is the filtered positive part of the estimated signal $\hat{\eta}_r(t)$. Let the difference between $u_+(t)$ and the unconstrained adaptive control signal $u_c(t)$ be

$$\Delta_u(t) \triangleq u_+(t) - u_c(t), \tag{8}$$

where $u_c(s) \triangleq C(s)\hat{\eta}_r(s)$. This difference represents the control deficiency, caused by the positive control constraint.

In the design of the control law, $C(s)$ is a low-pass filter, which is stable and strictly proper with DC gain $C(0) = 1$. Its state-space realization assumes zero initialization. The selection of ω_c in $C(s)$ needs to satisfy the following $\mathcal{L}_1$-norm condition

$$\|G(s)\|_{\mathcal{L}_1}\theta_{1\max} < 1, \tag{9}$$

where

$$\begin{aligned} \theta_{1\max} &\triangleq \max\nolimits_{\theta\in\Theta_B}\|\theta\|_1, \\ G(s) &\triangleq H(s)(1 - C(s)), \ H(s) = (s\mathbb{I} - A_m)^{-1}b. \end{aligned}$$

If we pick a first order low-pass filter

$$C(s) = \frac{\omega_c}{s + \omega_c}, \tag{10}$$

where $\omega_c > 0$ is the bandwidth of the filter, the $\mathcal{L}_1$-norm condition in (9) reduces to

$$A_g \triangleq \begin{bmatrix} A_m + b\theta^\top & b\omega_c \\ -\theta^\top & -\omega_c \end{bmatrix} \text{ is Hurwitz.} \tag{11}$$

The $\mathcal{L}_1$ adaptive controller consists of (4), (5) and (6), subject to the condition in (9).

IV. Analysis of $\mathcal{L}_1$ Adaptive Controller

In this section, we analyze the performance of the $\mathcal{L}_1$ adaptive controller. We first analyze the error between the state predictor and the real system, and obtain the intermediate result on the prediction error. We then introduce a reference system, and show that the input and the output signals of the closed-loop system track those of the reference system with uniform transient and steady-state performance bounds.

A. Prediction Error

Let

$$\tilde{x}(t) \triangleq \hat{x}(t) - x(t), \quad \tilde{\theta}(t) \triangleq \hat{\theta}(t) - \theta. \tag{12}$$

From (3) and (4) we obtain the prediction error dynamics

$$\dot{\tilde{x}}(t) = A_m\tilde{x}(t) + b\tilde{\theta}^\top(t)x(t), \quad \tilde{x}(0) = 0. \tag{13}$$

The prediction error is bounded as follows.

Lemma 2 ([8]): For the system in (3) and the controller defined by (6), we have the following uniform bound

$$\|\tilde{x}\|_{\mathcal{L}_\infty} \leq \sqrt{\frac{\theta_{2\max}}{\lambda_{\min}(P)\Gamma}}, \ \theta_{2\max} \triangleq 4\max_{\theta\in\Theta_B}\|\theta\|_2^2. \tag{14}$$

To further prove the asymptotic convergence of $\tilde{x}(t)$ to zero, $x(t)$ needs to be uniformly bounded.

Consider a scalar system, where x, θ, u_+ are scalars in (3). The system dynamics (3) shows that the closed-loop system depends on the control $u_+(t)$. As described in (6), $u_+(t)$

is a truncated signal of the designed adaptive control $u_c(t)$. Thus, the closed-loop system switches between two cases: truncated case, where the adaptive control cannot be an input to the system; and untruncated case, where $u_+(t) = u_c(t)$ controls the system.

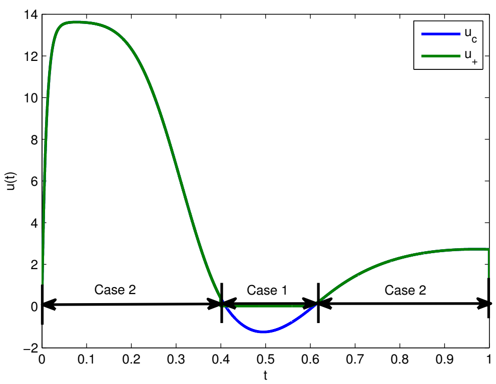

Fig. 1. Two cases of the control signal

1) Truncated case, $\hat{\eta}_{rC}(t) < 0$, $u_+(t) = 0$: In this case, no control signal enters the system, and by (2) and (6), the closed-loop dynamics reduce to

$$\dot{x}(t) = ax(t).$$

Then, Assumption 1 implies that the system is stable. The Lyapunov function $V_1(t) \triangleq \frac{1}{2}x^2(t)$ has a negative definite derivative, $\dot{V}_1(t) < 0$, and $|x(t)|$ is decreasing.

2) Untruncated case, $\hat{\eta}_{rC}(t) \geq 0$, $u_+(t) = u_c(t)$: This is the case, where the system receives the regular adaptive control signal. By (2) and (6), the closed-loop dynamics are given by

$$\dot{x}(t) = a_m x(t) + b\left(\theta x(t) + u_c(t)\right), \quad x(0) = x_0,$$

and the control law is

$$u_c(s) = C(s)\left(\hat{\eta}(s) + k_g r(s)\right),$$

where $\hat{\eta}(s)$ is defined in (6). Further, present $u_c(s)$ in two components

$$\begin{aligned} u_c(s) &= C(s)\left(\theta x(s) + \tilde{\eta}(s) + k_g r(s)\right) \\ &= C(s)\left(\theta x(s) + k_g r(s)\right) + C(s)\tilde{\eta}(s) \\ &= u_n(s) + \tilde{\eta}_C(s), \end{aligned} \tag{15}$$

where $\tilde{\eta}(t) \triangleq \tilde{\theta}(t)x(t)$, $u_n(s) \triangleq C(s)\left(\theta x(s) + k_g r(s)\right)$, and $\tilde{\eta}_C(s) \triangleq C(s)\tilde{\eta}(s)$. Consider a first-order low-pass filter given by (10). We can denote the state of the low-pass filter by $x_c(t)$, and rewrite the closed-loop dynamics as

$$\begin{aligned} \dot{x}(t) &= a_m x(t) + b\theta x(t) + b\omega_c x_c(t) + b\tilde{\eta}_C(t), \ x(0) = x_0, \\ \dot{x}_c(t) &= -\omega_c x_c(t) - \theta x(t) + k_g r(t), \quad x_c(0) = 0. \end{aligned}$$

The dynamics can be rewritten in compact form

$$\dot{x}_a(t) = A_g x_a(t) + b_g \begin{bmatrix} b\tilde{\eta}_C(t) \\ k_g r(t) \end{bmatrix}, \ x_a(0) = x_{a0},$$

$$x_a(t) \triangleq \begin{bmatrix} x(t) \\ x_c(t) \end{bmatrix}, \ b_g = \mathbb{I}_2, \ x_{a0} = \begin{bmatrix} x_0 \\ 0 \end{bmatrix},$$

where A_g is defined in (11), $x_a(t)$ is the augmented state of the closed-loop system, and $\tilde{\eta}_C(t)$ is the inverse Laplace transform of $\tilde{\eta}_C(s)$, defined in (15).

If the selection of $C(s)$ satisfies the simplified $\mathcal{L}_1$ condition in (11), the closed-loop system can be considered as a stable system with input $\begin{bmatrix} b\tilde{\eta}_C(t) & k_g r(t) \end{bmatrix}^\top$. Note that the Laplace transform of (13) gives

$$\tilde{x}(s) = H(s)\tilde{\eta}(s), \tag{16}$$

which leads to

$$\tilde{\eta}_C(s) = C(s)\tilde{\eta}(s) = C(s)\frac{1}{H(s)}\tilde{x}(s) = C(s)\frac{s - a_m}{b}\tilde{x}(s).$$

Since $C(s)$ is strictly proper and stable, $C(s)\frac{s-a_m}{b}$ is proper and stable, and thus has a finite $\mathcal{L}_1$-norm. Then, $\tilde{\eta}_C(t)$ is bounded by

$$\begin{aligned} |\tilde{\eta}_C(t)| \leq& \|\tilde{\eta}_C\|_{\mathcal{L}_\infty} \leq \left\| C(s)\frac{s - a_m}{b} \right\|_{\mathcal{L}_1} \|\tilde{x}\|_{\mathcal{L}_\infty} \\ =& \left\| C(s)\frac{s - a_m}{b} \right\|_{\mathcal{L}_1} \sqrt{\frac{\theta_{2\max}}{\lambda_{\min}(P)\Gamma}} = \frac{\gamma_{\tilde{\eta}_C}}{\sqrt{\Gamma}}, \end{aligned} \tag{17}$$

where $\gamma_{\tilde{\eta}_C} \triangleq \|C(s)\frac{s-a_m}{b}\|_{\mathcal{L}_1}\sqrt{\frac{\theta_{2\max}}{\lambda_{\min}(P)}}$. This bound can be arbitrarily reduced by increasing the adaptation rate Γ. Consider the Lyapunov function candidate

$$V_2(t) = x_a^\top(t) P_g x_a(t)$$

with the derivative

$$\begin{aligned} \dot{V}_2(t) =& x_a^\top(t) P_g \dot{x}_a(t) + \dot{x}_a^\top(t) P_g x_a(t) \\ =& x_a^\top(t) P_g A_g x_a(t) + x_a^\top(t) A_g^\top P_g x_a(t) \\ &+ 2x_a^\top(t) P_g b_g \begin{bmatrix} b\tilde{\eta}_C(t) & k_g r(t) \end{bmatrix}^\top \\ \leq& -x_a^\top(t) Q_g x_a(t) \\ &+ 2\|x_a(t)\|_2 \|P_g\|_2 \left\| \begin{bmatrix} b\tilde{\eta}_C(t) & k_g r(t) \end{bmatrix} \right\|_2 \\ \leq& -\lambda_{\min}(Q_g)\|x_a^\top(t)\|_2^2 \\ &+ 2\|x_a(t)\|_2 \|P_g\|_2 \sqrt{|b\tilde{\eta}_C(t)|^2 + |k_g r(t)|^2} \end{aligned}$$

where P_g is the solution to the Lyapunov equation $A_g^\top P_g + P_g A_g = -Q_g$ for some positive definite Q_g. The bound in (17) ensures that

$$\begin{aligned} \dot{V}_2(t) \leq& -\lambda_{\min}(Q_g)\|x_a^\top(t)\|_2^2 \\ &+ 2\|x_a(t)\|_2 \|P_g\|_2 \sqrt{\frac{b^2\gamma_{\tilde{\eta}_C}^2}{\Gamma} + k_g^2 r_{\max}^2}, \end{aligned}$$

where $r_{\max} \triangleq \max_{t\geq 0}|r(t)|$, $\gamma_{\tilde{\eta}_C}$ is defined in (17). Thus, $x_a(t)$ is ultimately bounded. Let

$$\rho = \frac{2\|P_g\|_2 \sqrt{\frac{b^2\gamma_{\tilde{\eta}_C}^2}{\Gamma} + k_g^2 r_{\max}^2}}{\lambda_{\min}(Q_g)}.$$

When $\|x_a(t)\|_2 > \rho$, we have $\dot{V}_2(t) < 0$. The analysis of the two cases above lead to a positive invariant set of x_a. Let

$$\mathcal{A} = \{x_a : \|x_a\|_2 \leq \rho\}.$$

Since the state-space realization of the low-pass filter C(s) assumes zero initialization, we have $x_c(0) = 0$. If $|x_0| \leq \rho$, then $\|x_{a0}\|_2 \leq \rho$. So $x_a(t) \in \mathcal{A}$.

If $\|x_a(t_1)\|_2 \leq \rho$, consider the two cases.

$i)$ In the truncated case shown above, $|x(t)|$ decreases. Since $\hat{\eta}_{rC}(t) < 0$, the input to the low-pass filter is zero, and the state space realization of the low-pass filter reduces to $\dot{x}_c(t) = -\omega_c x_c(t)$, and $|x_c(t)|$ decreases. Then, $\|x_a(t)\|_2$ also decreases. If $\|x_a(t_1)\|_2 \leq \rho$, $t_1 \leq t_2$, then $\|x_a(t_2)\|_2 \leq \rho$.

$ii)$ In the untruncated case, it is shown that $\|x_a(t)\|_2 > \rho$ implies $\dot{V}_2(t) < 0$. Then if $\|x_a(t_1)\|_2 \leq \rho$, $t_1 \leq t_2$, the state stays within the bound $\|x_a(t_2)\|_2 \leq \rho$.

Thus, in both cases, if $x_a(0) \in \mathcal{A}$, it stays in $\mathcal{A}$. This implies that $x(t)$ is bounded, which leads to the following asymptotic convergence result.

Lemma 3: Consider the system in (3) for $n = 1$, i.e. $x \in \mathbb{R}$. If $|x_0| \leq \rho$, the controller in (6) ensures that the prediction error (12) converges to zero asymptotically

$$\lim_{t\to\infty} \tilde{x}(t) = 0.$$

Proof: Consider the following Lyapunov function

$$V(t) = \tilde{x}^\top(t)P\tilde{x}(t) + \tilde{\theta}^\top(t)\Gamma^{-1}\tilde{\theta}(t).$$

By the proof of Lemma 2 in [8], $\dot{V}(t) \leq 0$, so $V(t)$ monotonically decreases. Moreover, the quadratic Lyapunov function has a lower bound $V(t) \geq 0$, so $V(t)$ converges to a limit when $t \to \infty$.

Further, it can be checked that the second derivative $\ddot{V}(t)$ is bounded. So $\dot{V}(t)$ is uniformly continuous. By Barbalat's Lemma, $\lim_{t\to\infty} \dot{V}(t) = 0$. Recall that

$$\dot{V}(t) \leq -\tilde{x}^\top(t)Q\tilde{x}(t) \leq 0\,,\ Q > 0\,.$$

The sandwich theorem from [13] implies $\lim_{t\to\infty} \tilde{x}(t) = 0$. ∎

B. Reference System

Consider the reference system

$$\begin{aligned}\dot{x}_{\text{ref}}(t) =& A_m x_{\text{ref}}(t) + b\left(\theta^\top x_{\text{ref}}(t) + u_{\text{ref}}(t)\right)\,,\\ y_{\text{ref}}(t) =& c^\top x_{\text{ref}}(t)\,,\quad x_{\text{ref}}(0) = x_0\,,\end{aligned} \tag{18}$$

with the following controller structure, including the control deficiency,

$$u_{\text{ref}}(s) = C(s)\left(-\theta^\top x_{\text{ref}}(s) + k_g r(s)\right) + \Delta_u(s)\,, \tag{19}$$

where $\Delta_u(s)$ is defined in (8).

Lemma 4: If A_g in (11) is Hurwitz, then the closed loop reference system in (18) and (19) is BIBO stable.

Proof: Let the state of the low-pass filter be $x_{\text{ref}c}$. Then the state space realization of $C(s)$ can be written as

$$\dot{x}_{\text{ref}c}(t) = -\omega_c x_{\text{ref}c}(t) + \left(-\theta^\top x_{\text{ref}}(t) + k_g r(t)\right)\,. \tag{20}$$

By (18) and (20), we have the state space model of the closed loop reference system

$$\begin{bmatrix}\dot{x}_{\text{ref}}(t)\\ \dot{x}_{\text{ref}c}(t)\end{bmatrix} = \begin{bmatrix}A_m + b\theta^\top & b\omega_c\\ -\theta^\top & -\omega_c\end{bmatrix}\begin{bmatrix}x_{\text{ref}}(t)\\ x_{\text{ref}c}(t)\end{bmatrix} + \begin{bmatrix}b\Delta_u(t)\\ k_g r(t)\end{bmatrix}.$$

Thus, if A_g is Hurwitz, the reference system is BIBO stable. ∎

C. Tracking Error

Theorem 2: Consider the system in (3) and the adaptive controller in (4)-(6). The tracking error is upper bounded by

$$\|x - x_{ref}\|_{\mathcal{L}_\infty} \leq \gamma_x\,, \tag{21}$$

$$\|u_+ - u_{ref}\|_{\mathcal{L}_\infty} \leq \gamma_u\,, \tag{22}$$

where

$$\gamma_x \triangleq \|H_{\tilde{x}e}(s)\|_{\mathcal{L}_1}\sqrt{\frac{\theta_{2\max}}{\lambda_{\min}(P)\Gamma}}\,, \tag{23}$$

$$\gamma_u \triangleq \|C(s)\frac{1}{c_0^\top H(s)}c_0^\top\|_{\mathcal{L}_1}\sqrt{\frac{\theta_{2\max}}{\lambda_{\min}(P)\Gamma}} + \|C(s)\|\theta_{1\max}\gamma_x,$$

$$H_{\tilde{x}e}(s) \triangleq -\left(\mathbb{I} + (\mathbb{I} - G(s)\theta^\top)^{-1}[G(s)\theta^\top + (C(s)-1)\mathbb{I}]\right).$$

Proof: Denote the tracking error of the state

$$e(t) = x(t) - x_{ref}(t). \tag{24}$$

We have

$$x(t) = \hat{x}(t) - \tilde{x}(t).$$

From (4), we further write $\hat{x}(s)$ as

$$\begin{aligned}\hat{x}(s) =& H(s)\hat{\eta}(s) + H(s)u_+(s) + (s\mathbb{I} - A_m)^{-1}x_0\\ =& H(s)\hat{\eta}(s) + H(s)u_c(s) + H(s)\Delta_u(s)\\ &+ (s\mathbb{I} - A_m)^{-1}x_0,\end{aligned}$$

where $\hat{\eta}(t)$ is defined in (7), $\Delta_u(t)$ is defined in (8) and $H(s) = (s\mathbb{I} - A_m)^{-1}b$. Substituting

$$u_c(s) = C(s)\hat{\eta}_r(s) = C(s)(-\hat{\eta}(s) + k_g r(s))$$

into the equation above yields

$$\begin{aligned}\hat{x}(s) =& G(s)\hat{\eta}(s) + H(s)C(s)k_g r(s) - H(s)\Delta_u(s)\\ &+ (s\mathbb{I} - A_m)^{-1}x_0,\end{aligned}$$

where $G(s) = (1 - C(s))H(s)$. Recall that $\hat{\eta}(t) = \hat{\theta}^\top(t)x(t)$. We can further rewrite it as

$$\begin{aligned}\hat{\eta}(t) =& \hat{\theta}^\top(t)x(t)\\ =& \theta^\top x(t) + \tilde{\theta}^\top(t)x(t)\\ =& \theta^\top\hat{x}(t) - \theta^\top\tilde{x}(t) + \tilde{\theta}^\top(t)x(t).\end{aligned}$$

Now $\hat{x}(s)$ takes the form:

$$\begin{aligned}\hat{x}(s) =& (\mathbb{I} - G(s)\theta^\top)^{-1}H(s)C(s)k_g r(s)\\ &+ (\mathbb{I} - G(s)\theta^\top)^{-1}[-G(s)\theta^\top - (C(s)-1)\mathbb{I}]\tilde{x}(s)\\ &+ (\mathbb{I} - G(s)\theta^\top)^{-1}H(s)\Delta_u(s)\\ &+ (\mathbb{I} - G(s)\theta^\top)^{-1}(s\mathbb{I} - A_m)^{-1}x_0.\end{aligned}$$

Hence,

$$\begin{aligned} x(s) =& \hat{x}(s) - \tilde{x}(s) \\ =& (\mathbb{I} - G(s)\theta^\top)^{-1} H(s)C(s)k_g r(s) \\ & - \left(\mathbb{I} + (\mathbb{I} - G(s)\theta^\top)^{-1}[G(s)\theta^\top + (C(s)-1)\mathbb{I}]\right)\tilde{x}(s) \\ & + (\mathbb{I} - G(s)\theta^\top)^{-1} H(s)\Delta_u(s) \\ & + (\mathbb{I} - G(s)\theta^\top)^{-1}(s\mathbb{I} - A_m)^{-1} x_0. \end{aligned}$$

On the other hand, by (18) and (19), we have

$$\begin{aligned} x_{\text{ref}}(s) =& (\mathbb{I} - G(s)\theta^\top)^{-1} H(s)C(s)k_g r(s) \\ & + (\mathbb{I} - G(s)\theta^\top)^{-1} H(s)\Delta_u(s) \\ & + (\mathbb{I} - G(s)\theta^\top)^{-1}(s\mathbb{I} - A_m)^{-1} x_0\,. \end{aligned}$$

So the error dynamics are described by

$$e(s) = x(s) - x_{\text{ref}}(s) = H_{\tilde{x}e}(s)\tilde{x}(s)\,, \tag{25}$$

where $H_{\tilde{x}e}(s)$ is defined in (23). Thus, $e(t)$ can be bounded by

$$\|e\|_{\mathcal{L}_\infty} \leq \|H_{\tilde{x}e}(s)\|_{\mathcal{L}_1}\|\tilde{x}\|_{\mathcal{L}_\infty}\,.$$

This bound, together with Lemma 2, proves the first bound in (21).

Further, equations (6) and (19) yield

$$u_+(s) - u_{ref}(s) = C(s)(\tilde{\eta}(s) + \theta^\top e(s)).$$

Since $H(s)$ is stable and proper, there exists c_0, such that $c_0^\top H(s)$ is minimum phase with relative degree 1 [8]. Thus, we can rewrite the equation above as

$$u_+(s) - u_{ref}(s) = C(s)\left(\frac{1}{c_0^\top H(s)} c_0^\top H(s)\tilde{\eta}(s) + \theta^\top e(s)\right). \tag{26}$$

Substitute (16) into (26) to get

$$u_+(s) - u_{ref}(s) = -C(s)\frac{1}{c_0^\top H(s)} c_0^\top \tilde{x}(s) + C(s)\theta^\top e(s).$$

Thus,

$$\begin{aligned} \|u_+ - u_{ref}\|_{\mathcal{L}_\infty} \leq& \left\|C(s)\frac{1}{c_0^\top H(s)} c_0^\top\right\|_{\mathcal{L}_1}\|\tilde{x}\|_{\mathcal{L}_\infty} \\ & + \|C(s)\|_{\mathcal{L}_1}\theta_{1\max}\|e\|_{\mathcal{L}_\infty}, \end{aligned} \tag{27}$$

where $\theta_{1\max}$ is defined in (9).

To obtain the tracking error of the input signal, we substitute (21) into (27) and arrive at (22). ■

V. Simulation

In this section, we use a numerical example to demonstrate the performance of the $\mathcal{L}_1$ adaptive controller for a positive system. Consider the system in (2) with

$$A = \begin{bmatrix} -1 & 0.6 \\ 0.2 & -1.4 \end{bmatrix}, b = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, c = \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

$$x_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \theta = \begin{bmatrix} 4 \\ 4.5 \end{bmatrix},$$

and let $\Theta_B = \{\theta_1 \in [-10, 10], \theta_2 \in [-10, 10]\}$, which gives $\theta_{1\max} = 10$. Let the low-pass filter be $C(s) = \frac{\omega_c}{s+\omega_c}$.

To select the bandwidth of $C(s)$, we recall that the selection of $C(s)$ should verify the $\mathcal{L}_1$ condition (9). Let

$$\lambda(\omega_c) = \|G(s)\|_{\mathcal{L}_1}\theta_{1\max}.$$

We plot $\lambda(\omega_c)$ and ω_c in Figure 2.

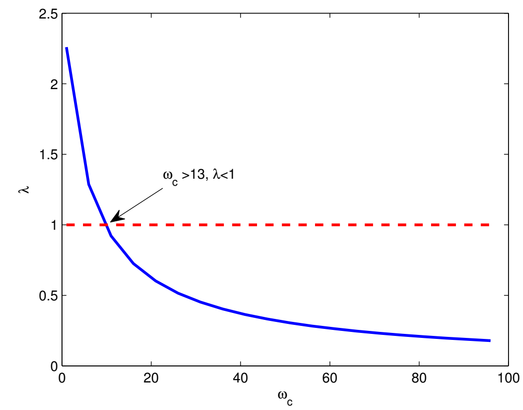

Fig. 2. $\lambda(\omega_c)$ and constant 1.

Notice that for $\omega_c > 13$, we have $\lambda < 1$. Let $\omega_c = 15$, and $C(s) = \frac{15}{s+15}$, which leads to

$$A_g = \begin{bmatrix} -1 & 0.6 & 0 \\ 0.2 & -1.4 & 15 \\ -4 & -4.5 & -15 \end{bmatrix}.$$

We can verify that A_g is Hurwitz, with eigenvalues -1.5221 and $-7.9389 \pm 4.1935i$. Let the adaptation rate be $\Gamma = 10^3$. We now show the tracking performance of the closed-loop system to nonnegative reference signals.

Figures 5 and 6 show the output $y(t)$ and the control $u_+(t)$ of the system response to step references. For scaled reference signals, the $\mathcal{L}_1$ adaptive controller leads to scaled system outputs. Note that for the above references we did not redesign or retune the adaptive controller.

VI. Conclusion

In this paper, we have used the $\mathcal{L}_1$ adaptive controller to solve the tracking problem for positive LTI systems with uniform performance bounds. Positive systems play an important role in population models and biomedical systems. In our future work, we intend to extend the adaptive control results to nonlinear positive systems and apply results to study drug treatment design for different immunodeficiency diseases.

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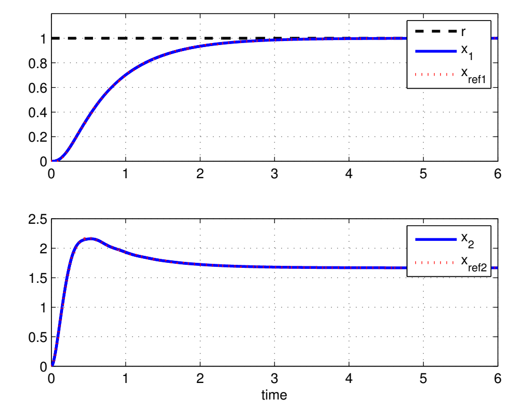

Fig. 3. $x(t)$ for step reference $r(t) = 1$.

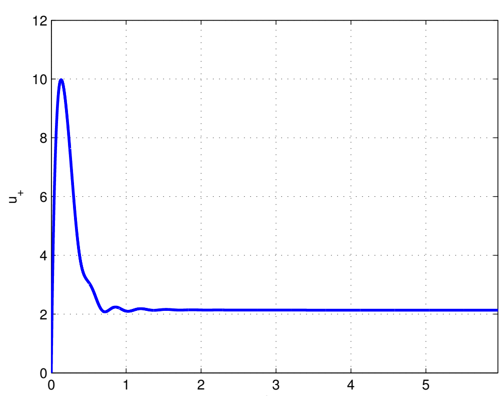

Fig. 4. $u_+(t)$ for step reference $r(t) = 1$.

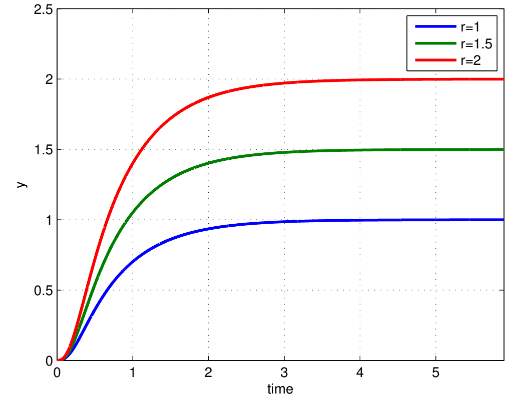

Fig. 5. $y(t)$ for step references $r(t) = 1, 2, 3$.

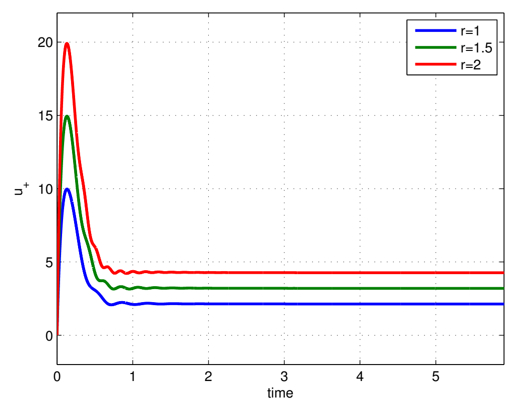

Fig. 6. $u_+(t)$ for step references $r(t) = 1, 2, 3$.