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A new parameter estimation algorithm for non-uniformly multirate sampled-data systems

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***Abstract*— For the input-output representation of non-uniformly multirate discrete-time systems, a coupled least squares algorithm is derived to estimate the model parameters with the advantage of avoiding the computation of matrix inversion. Moreover, The proposed algorithm has good convergence properties. The simulation test verifies the effectiveness of the algorithm.**

Index terms: Recursive identification; parameter estimation; stochastic gradient; least squares; discretization; multirate systems; non-uniform sampling

I. Introduction

In process industries, several sampling rates often co-exist in a control system so that a better tradeoff between performance and implementation cost can be achieved [1]. For example, in a polymer reactor or a distillation tower, composition, density or molecular weight distribution are measured at much lower frequencies than flow rates, temperatures and pressures. Such systems with more than one sampling rate are called multirate systems.

Research on multirate systems is active in recent years and many achievements have been reported in the control and identification fields. In the area of process control, Li *et al.* proposed an inferential control scheme for dual-rate systems [1] and studied the application of dual-rate modeling in the petroleum refinery [2]. Rossiter *et al.* discussed the dual-rate predictive control scheme for dual-rate systems [3]. Sheng *et al.* presented a scheme for designing filters to achieve fast state estimation in the H_2 and H_∞ settings using the linear matrix inequality solution [4]. Gao *et al.* investigated the problem of robust H_∞ control for sampled-data systems with uncertain parameters and probabilistic sampling [5]. Yu *et al.* studied the l_2–l_∞ filtering problems for multirate systems [6]. In the literature of system identification, Li *et al.* used a sub-space method to estimate the parameters of the lifted state-space models for general dual-rate systems [7]. Ding and Chen presented a hierarchical identification approach to estimate the parameters and states of the lifted state-space models for such general dual-rate systems [8]. Recently, the polynomial transformation technique was used to obtain a dual-rate model for dual-rate system identification [9], [10].

In multirate systems, it is generally assumed that all variables are uniformly sampled at constant intervals. However, this is not always true in many practical cases. For the non-uniformly multirate sampled-data systems, the sampling intervals for the input and/or output channels are non-equidistant in time. Systems with missing measurements can be seen as a kind of non-uniformly sampled systems. A lot of work has been done in this field [11]–[14]. Another non-uniform sampling pattern exists typically for the cases when the manual sampling or laboratory analysis is required [15]. In this area, Sheng *et al.* proposed a generalized predictive control (GPC) design [16]; Li *et al.* gave a Kalman filter-based method for state estimation, fault detection and isolation for a class of periodically non-uniformly sampled systems [15]. Ding *et al.* studied the reconstruction of continuous-time systems, the controllability and observability, the computation of single-rate models and the state-space model identification for non-uniformly sampled systems [17], [18]; Xie *et al.* studied a stochastic gradient method for the non-uniformly sampled systems and used the multi-innovation technique to improve the convergence rate [19].

In [19], the authors only considered the non-uniform sampling scheme for the system output. In that case, an equivalent multiple-input single-output system model can be obtained by using the discretization technique. Different from the work in [19], in this paper, we consider the non-uniform sampling scheme for both the system input and output. In this case, the converted model becomes an equivalent multi-input multi-output one. To avoid computing the matrix inverse at each recursion of the recursive least squares algorithm [20], we propose a new algorithm to estimate the parameters of the multi-input multi-output representation. It is important to acknowledge that a number of previous work has been done on this problem. For example, the hierarchical least squares and hierarchical gradient methods for multivariable systems [21], the multi-innovation stochastic gradient algorithms [22], [23]. Recently, Ding *et al.* presented a partially coupled stochastic gradient algorithm for the non-uniformly sampled systems [24], which is similar to the proposed algorithm in this paper; however, only one part of the parameters are coupled in that algorithm.

The rest of the paper is organized as follows. Section II derives the identification model of the non-uniformly sampled systems. Section III gives the recursive least squares algorithm for the non-uniformly sampled systems. Section IV presents the coupled least squares algorithm. Section V provides an illustration example. Finally, Section VI offers some concluding remarks.

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II. THE IDENTIFICATION MODEL

Consider a non-uniformly sampled multirate system depicted in Fig. 1, where P_c is a continuous-time process with

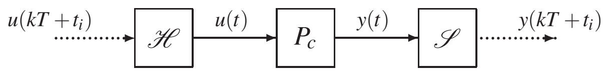

Fig. 1. The non-uniformly sampling systems

the state-space representation:

$$P_c: \quad \begin{cases} \dot{x}(t) = Ax(t) + Bu(t), \\ y(t) = Cx(t) + Du(t), \end{cases} \tag{1}$$

$x(t) \in \mathbb{R}^n$ is the state vector, $u(t) \in \mathbb{R}^1$ and $y(t) \in \mathbb{R}^1$ are the input and output of P_c, respectively; $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^n$, $C \in \mathbb{R}^{1\times n}$ and $D \in \mathbb{R}^1$ are constant or constant matrices; $\mathscr{H}$ denotes a non-uniform zero-order hold (ZOH) with the following description:

$$u(t) = \begin{cases} u(kT), & kT \leqslant t < kT + t_1, \\ u(kT+t_1), & kT+t_1 \leqslant t < kT + t_2, \\ \quad\vdots & \\ u(kT+t_r), & kT+t_{r-1} \leqslant t < (k+1)T. \end{cases} \tag{2}$$

$\mathscr{S}$ denotes a non-uniform sampler. The sampling intervals are $\tau_1, \tau_2, \cdots, \tau_r$, and then are repeated. Thus the sampling scheme is periodical with a large period $T = \tau_1 + \tau_2 + \cdots + \tau_r = t_r$, which can be termed as the frame period. $t = kT + t_i = kT + t_{i-1} + \tau_i$ are the updating and sampling intervals. So the control input u is updated r times at the instants $t = kT + t_i$ $(i = 0, 1, \cdots, r-1)$ over each framework period T, i.e., over the $(k+1)$th period $[kT, (k+1)T)$, and the output y is sampled r times at the instants $t = kT + t_i$ $(i = 0, 1, \cdots, r-1)$ within each framework period T.

Referring to [17], [18], discretizing P_c and using (2) yield the following discrete-time state-space model for non-uniformly sampled systems:

$$\begin{aligned} x(kT+T) &= G_r x(kT) + \sum_{j=1}^{r} \exp[A(T-t_j)] F_{\tau_j} u(kT + t_{j-1}) \\ &= G_r x(kT) + \sum_{j=1}^{r} F_j u(kT+t_{j-1}) \\ &= G_r x(kT) + F\underline{u}(kT), \end{aligned} \tag{3}$$

$$\begin{aligned} y(kT+t_{i-1}) &= CG_{i-1}x(kT) + \sum_{j=1}^{i-1} D_{i-1,j} u_j(kT+t_{j-1}) \\ &\quad + Du(kT+t_{i-1}) \\ &= C_{i-1}x(kT) + D_{i-1}\underline{u}(kT), \ i = 1, 2, \cdots, r, \end{aligned} \tag{4}$$

where $\underline{u}(kT) := [u(kT), u(kT+t_1), \cdots, u(kT+t_{r-1})]^{\mathrm{T}}$ is the non-uniformly stacked input vector, and

$$\begin{aligned} G_i &:= \exp(At_i) \in \mathbb{R}^{n\times n}, \ i = 1, 2, \cdots, r, \\ F &:= [F_1, F_2, \cdots, F_r] \in \mathbb{R}^{n\times r}, \\ F_i &:= \exp(A(T-t_i))F_{\tau_i} = G_r G_i^{-1} F_{\tau_i} \in \mathbb{R}^n, \\ F_{\tau_i} &:= \int_0^{\tau_i} \exp(At)\mathrm{d}tB \in \mathbb{R}^n, \\ C_i &:= CG_i \in \mathbb{R}^{1\times n}, \ i = 0, 1, \cdots, (r-1), \\ D_i &:= [D_{i1}, D_{i2}, \cdots, D_{ii}, D, 0, \cdots, 0] \in \mathbb{R}^{1\times r}, \\ D_{ij} &:= CG_i G_j^{-1} F_{\tau_1} \in \mathbb{R}^1, \ j = 1, 2, \cdots, i. \end{aligned}$$

Let z be a forward shift operator (z^{-1} be a backward shift operator): $zx(kT+t_i) = x(kT+T+t_i)$ and $z^{-1}x(kT+t_i) = x(kT-T+t_i)$. From (3) and (4), we have

$$\begin{aligned} y(kT+t_{i-1}) &= [C_{i-1}(zI_n - G_r)^{-1}F + D_{i-1}]\underline{u}(kT) \\ &= \left[\frac{z^{-n}C_{i-1}\,\mathrm{adj}[zI_n - G_r]F}{z^{-n}\det[zI_n - G_r]} + D_{i-1}\right]\underline{u}(kT) \\ &= \frac{\beta_i(z)}{\alpha(z)}\underline{u}(kT), \ i = 1, 2, \cdots, r, \end{aligned} \tag{5}$$

where I_n is an $n \times n$ identity matrix, $\alpha(z)$ is the characteristic polynomial of order n and $\beta_i(z)$ is a row vector polynomial with

$$\begin{aligned} \alpha(z) &:= z^{-n}\det[zI_n - G_r] \\ &= 1 + \alpha_1 z^{-1} + \alpha_2 z^{-2} + \cdots + \alpha_n z^{-n}, \ \alpha_i \in \mathbb{R}^1, \\ \beta_i(z) &:= z^{-n}C_{i-1}\,\mathrm{adj}[zI_n - G_r]F + D_{i-1}\alpha(z) \\ &= \beta_{i0} + \beta_{i1}z^{-1} + \beta_{i2}z^{-2} + \cdots + \beta_{in}z^{-n}, \ \beta_{ij} \in \mathbb{R}^{1\times r}. \end{aligned}$$

Taking into account the disturbance noise $e_i(kT)$ in (5), the output at $t = kT + t_{i-1}$ can be written as

$$y(kT+t_{i-1}) = \frac{\beta_i(z)}{\alpha(z)}\underline{u}(kT) + e_i(kT), \ i = 1, 2, \cdots, r. \tag{6}$$

Here, the noise term $e_i(kT)$ is considered to be a colored noise in the following form

$$e_i(kT) = \frac{1}{\alpha(z)} v_i(kT) \ i = 1, 2, \cdots, r,$$

where $v(kT) := [v_1(kT), v_2(kT), \cdots, v_r(kT)]^{\mathrm{T}} \in \mathbb{R}^r$ is a white noise vector.

Define the stacked output vector:

$$\underline{y}(kT) := \begin{bmatrix} y(kT) \\ y(kT+t_1) \\ y(kT+t_2) \\ \vdots \\ y(kT+t_{r-1}) \end{bmatrix} \in \mathbb{R}^r,$$

and the output information matrix $\psi(kT)$ and input information vector $\varphi(kT)$ as

$$\begin{aligned} \psi(kT) &:= [\underline{y}(kT-T), \underline{y}(kT-2T), \cdots, \underline{y}(kT-nT)] \in \mathbb{R}^{r\times n}, \\ \varphi(kT) &:= [\underline{u}^{\mathrm{T}}(kT), \underline{u}^{\mathrm{T}}(kT-T), \cdots, \underline{u}^{\mathrm{T}}(kT-nT)]^{\mathrm{T}} \in \mathbb{R}^{n_0}, \\ n_0 &:= (n+1)r. \end{aligned}$$

Define the parameter vector $\alpha \in \mathbb{R}^n$ and parameter matrix $\theta \in \mathbb{R}^{n_0 \times r}$ as

$$\alpha := [\alpha_1, \alpha_2, \cdots, \alpha_n]^{\mathrm{T}} \in \mathbb{R}^n,$$
$$\theta := [\theta_1, \theta_2, \cdots, \theta_r] \in \mathbb{R}^{n_0 \times r}.$$

Then we can get a stacked identification model:

$$\underline{y}(kT) + \psi(kT)\alpha = \theta^{\mathrm{T}}\varphi(kT) + v(kT). \tag{7}$$

The objective of this paper is to present a new coupled least squares identification method to estimate the parameters in α and θ of the input-output representation in (7) for the non-uniformly sampled systems, based on the given non-uniform input-output data $\{u(kT+t_i), y(kT+t_i)$: $i = 0,1,\cdots,r-1,\ k = 0,1,2,\cdots\}$.

III. The recursive least squares algorithm

For the convenience of deriving the coupled least squares algorithm, we first give the recursive least squares algorithm for the non-uniformly sampled systems. Let us introduce some notations here. The symbols $\lambda_{\max}[X]$ and $\lambda_{\min}[X]$ represent the maximum and minimum eigenvalues of the square matrix X, respectively; The norm of the matrix X is defined by $\|X\|^2 := \mathrm{tr}[XX^{\mathrm{T}}]$; $|X| = \det[X]$ denotes the determinant of a square matrix X; $\mathbf{1}_n$ represents an n-dimensional column vector whose elements are all 1; p_0 is a large positive number, e.g., $p_0 = 10^6$; $\otimes$ denotes the Kronecker product, if $A = [a_{ij}] \in \mathbb{R}^{m\times n}$, $B = [b_{ij}] \in \mathbb{R}^{p\times q}$, then $A \otimes B = [a_{ij}B] \in \mathbb{R}^{(mp)\times(nq)}$; $\mathrm{col}[X]$ denotes the vector formed by the column of the matrix X, that is, if $X = [x_1, x_2, \cdots, x_n] \in \mathbb{R}^{m\times n}$, then $\mathrm{col}[X] = [x_1^{\mathrm{T}}, x_2^{\mathrm{T}}, \cdots, x_n^{\mathrm{T}}]^{\mathrm{T}} \in \mathbb{R}^{mn}$. $f(k) = O(g(k))$ means that if there exist finite positive constants δ_1 and k_0 such that $|f(k)| \leqslant \delta_1 g(k)$ for $k \geqslant k_0$.

The identification model in (7) contains a parameter vector $\alpha \in \mathbb{R}^n$ and a parameter matrix $\theta \in \mathbb{R}^{n_0\times r}$. In order to identify α and θ, the model in (7) needs to be transformed into a new form. Let

$$\vartheta := \begin{bmatrix} \alpha \\ \mathrm{col}[\theta] \end{bmatrix} \in \mathbb{R}^{n+n_0 r},$$
$$\Phi(kT) := [-\psi(kT), I_r \otimes \varphi^{\mathrm{T}}(kT)] \in \mathbb{R}^{r\times(n+n_0 r)}. \tag{8}$$

Then we have

$$\underline{y}(kT) = \Phi(kT)\vartheta + v(kT). \tag{9}$$

Minimizing the following least squares criterion function:

$$J(\vartheta) = \sum_{i=1}^{k} \|\underline{y}(iT) - \Phi(iT)\vartheta\|^2$$

leads to the following recursive least squares (RLS) algorithm for estimating the parameter vector ϑ:

$$\begin{aligned}\hat{\vartheta}(kT) &= \hat{\vartheta}(kT-T) + P(kT)\Phi^{\mathrm{T}}(kT) \\ &\quad \times[\underline{y}(kT) - \Phi(kT)\hat{\vartheta}(kT-T)], \quad (10)\\ P^{-1}(kT) &= P^{-1}(kT-T) \\ &\quad + \Phi^{\mathrm{T}}(kT)\Phi(kT),\ P(0) = p_0 I_{n+n_0 r}. \quad (11)\end{aligned}$$

In order to avoid computing the matrix inverse $P^{-1}(kT)$ in (11), defining the gain matrix:

$$L(kT) := P(kT)\Phi^{\mathrm{T}}(kT) \in \mathbb{R}^{(n+n_0 r)\times r}$$

and applying the matrix inversion lemma:

$$(A+BC)^{-1} = A^{-1} - A^{-1}B(I + CA^{-1}B)^{-1}CA^{-1} \tag{12}$$

to (11), we can obtain the equivalent expression of the RLS algorithm in (10)–(11) as follows:

$$\begin{aligned}\hat{\vartheta}(kT) &= \hat{\vartheta}(kT-T) \\ &\quad + L(kT)[\underline{y}(kT) - \Phi(kT)\hat{\vartheta}(kT-T)], \quad (13)\\ L(kT) &= P(kT-T)\Phi^{\mathrm{T}}(kT) \\ &\quad \times[I_r + \Phi(kT)P(kT-T)\Phi^{\mathrm{T}}(kT)]^{-1}, \quad (14)\\ P(kT) &= [I_{n+n_0 r} - L(kT)\Phi(kT)]P(kT-T). \quad (15)\end{aligned}$$

The drawback of the RLS algorithm in (13)–(15) is that it requires computing the matrix inversion: $[I_r + \Phi(kT)P(kT-T)\Phi^{\mathrm{T}}(kT)]^{-1} \in \mathbb{R}^{r\times r}$ for each step. This causes a heavy computational load, especially for a large r. In order to avoid computing the matrix inversion, the coupled least squares algorithm for estimating ϑ is derived and presented in next section.

IV. The coupled estimation algorithm

Let $\underline{y}_i(kT) := y(kT + t_{i-1})$ and $\Phi_i(kT) \in \mathbb{R}^{1\times(n+n_0 r)}$ be the ith row of $\Phi(kT)$. From (9), we have

$$\underline{y}_i(kT) = \Phi_i(kT)\vartheta + v_i(kT),\ i = 1,2,\cdots,r. \tag{16}$$

Thus the stacked identification model in (7) is decomposed into r subsystems. From (16), we can see that each subsystem has the same parameter vector ϑ and the parameter estimates $\hat{\vartheta}(kT)$ of the subsystems are different and mutually independent. The least squares algorithm for the subsystems is as follows

$$\begin{aligned}\hat{\vartheta}_i(kT) &= \hat{\vartheta}_i(kT-T) \\ &\quad + L_i(kT)[\underline{y}_i(kT) - \Phi_i(kT)\hat{\vartheta}_i(kT-T)], \quad (17)\\ L_i(kT) &= P_i(kT)\Phi_i^{\mathrm{T}}(kT), \quad (18)\\ P_i^{-1}(kT) &= P_i^{-1}(kT-T) + \Phi_i^{\mathrm{T}}(kT)\Phi_i(kT), i = 1,\cdots,r. \quad (19)\end{aligned}$$

$\hat{\vartheta}_i(kT)$ is the parameter estimates for the ith subsystem. The question arises: which $\hat{\vartheta}_i(kT)$ can be seen as the best estimate of ϑ or how to get the estimate for ϑ from all of the $\hat{\vartheta}_i(kT)$s? It is worth mentioning that one can not take the average of all the estimates $\hat{\vartheta}_i(kT)$s as the estimate of ϑ as mentioned in [24], because only a small part of the parameters can be estimated from each subsystem identification. Here, we propose a coupled least squares algorithm to effectively estimate the parameters, without computing the matrix inversion.

By means of the idea of the Jacobi and Gauss-Seidel iterations [26], replacing $\hat{\vartheta}_i(kT-T)$ in (17) with $\hat{\vartheta}_{i-1}(kT)$ for $i = 2,3,\cdots,r$, and replacing $\hat{\vartheta}_1(kT-T)$ with $\hat{\vartheta}_r(kT-T)$

for $i=1$ in the recursive equations give the following coupled least squares (C-LS) algorithm:

$$\hat{\vartheta}_i(kT)=\hat{\vartheta}_{i-1}(kT)+L_i(kT)[\underline{y}_i(kT)-\Phi_i(kT)\hat{\vartheta}_{i-1}(kT)], \quad (20)$$

$$L_i(kT)=P_i(kT)\Phi_i^{\mathrm{T}}(kT), \quad (21)$$

$$P_i^{-1}(kT)=P_{i-1}^{-1}(kT)+\Phi_i^{\mathrm{T}}(kT)\Phi_i(kT),\ i=2,3,\cdots,r \quad (22)$$

and

$$\hat{\vartheta}_1(kT)=\hat{\vartheta}_r(kT-T)+L_1(kT)[\underline{y}_1(kT)-\Phi_1(kT)\hat{\vartheta}_r(kT-T)], \quad (23)$$

$$L_1(kT)=P_1(kT)\Phi_1^{\mathrm{T}}(kT), \quad (24)$$

$$P_1^{-1}(kT)=P_r^{-1}(kT-T)+\Phi_1^{\mathrm{T}}(kT)\Phi_1(kT). \quad (25)$$

Applying the formula in (12) to (22) and (25), the C-LS algorithm can be equivalently expressed as

$$\hat{\vartheta}_i(kT)=\hat{\vartheta}_{i-1}(kT)+L_i(kT)[\underline{y}_i(kT)-\Phi_i(kT)\hat{\vartheta}_{i-1}(kT)], \quad (26)$$

$$L_i(kT)=\frac{P_{i-1}(kT)\Phi_i^{\mathrm{T}}(kT)}{1+\Phi_i(kT)P_{i-1}(kT)\Phi_i^{\mathrm{T}}(kT)}, \quad (27)$$

$$P_i(kT)=[I-L_i(kT)\Phi_i(kT)]P_{i-1}(kT),\ i=2,3,\cdots,r \quad (28)$$

and

$$\hat{\vartheta}_1(kT)=\hat{\vartheta}_r(kT-T)+L_1(kT)[\underline{y}_1(kT)-\Phi_1(kT)\hat{\vartheta}_r(kT-T)], \quad (29)$$

$$L_1(kT)=\frac{P_r(kT-T)\Phi_1^{\mathrm{T}}(kT)}{1+\Phi_1(kT)P_r(kT-T)\Phi_1^{\mathrm{T}}(kT)}, \quad (30)$$

$$P_1(kT)=[I-L_1(kT)\Phi_1(kT)]P_r(kT-T), \quad (31)$$

where $\hat{\vartheta}_i(kT)$, $L_i(kT)$ and $P_i(kT)$ are the parameter estimation vector, gain matrix and covariance matrix of the ith subsystem at time $t=kT$, respectively; $\hat{\vartheta}_{i-1}(kT)$ and $P_{i-1}(kT)$ are the parameter estimation vector and covariance matrix of the $(i-1)$th subsystem at time $t=kT$, respectively; $\hat{\vartheta}_r(kT-T)$ and $P_r(kT-T)$ are the parameter estimation vector and covariance matrix of the rth subsystem at time $t=kT-T$, respectively.

The schematic diagram of the C-LS algorithm in (26)–(31) is shown in Fig.2. In Fig. 2, the parameter estimate $\hat{\vartheta}_1(kT)$ of subsystem 1 is equal to the estimate $\hat{\vartheta}_r(kT-T)$ of subsystem r at the preceding time $t=kT-T$ plus the modified term $L_1(kT)[\underline{y}_1(kT)-\Phi_1(kT)\hat{\vartheta}_r(kT-T)]$ – see (29), and the covariance matrix $P_1(kT)$ of subsystem 1 at time $t=kT$ is computed through the covariance matrix $P_r(kT-T)$ of subsystem r at the preceding time $t=kT-T$ and the gain vector $L_1(kT)$ and information vector $\Phi_1(kT)$ of subsystem 1 – see (31). Similarly, the parameter estimate $\hat{\vartheta}_2(kT)$ of subsystem 2 is equal to the estimate $\hat{\vartheta}_1(kT)$ of subsystem 1 plus the modified term $L_2(kT)[\underline{y}_2(kT)-\Phi_2(kT)\hat{\vartheta}_1(kT)]$ – see (26) with $i=2$, and the covariance matrix $P_2(kT)$ of subsystem 2 is computed through the covariance matrix $P_1(kT)$ of subsystem 1 and the gain vector $L_2(kT)$ and information vector $\Phi_2(kT)$ of subsystem 2 – see (28) with $i=2$. Similar procedure will be conducted as i increases.

The steps of computing the estimates $\hat{\vartheta}_r(kT)$ by the C-LS algorithm in (26)–(31) are listed in the following:

1) Set the initial values: Let $k=1$, $\hat{\vartheta}_r(0)=\mathbf{1}_n/p_0$, $P_r(0)=p_0I_{n+n_0r}$, $p_0=10^6$.
2) Collect the input-output data $\underline{u}(kT)$ and $\underline{y}(kT)$, and form information vectors $\psi(kT)$ by (7), $\varphi(kT)$ by (7) and $\Phi(kT)$ by (8).
3) Compute the gain vector $L_1(kT)$ by (30) and covariance matrix $P_1(kT)$ by (31) and update the estimate $\hat{\vartheta}_1(kT)$ by (29).
4) for $i=2:r$
   Compute the gain vector $L_i(kT)$ by (27) and covariance matrix $P_i(kT)$ by (28) and update the estimate $\hat{\vartheta}_r(kT)$ by (26).
   end
5) Increase k by 1 and go to Step 2.

About the parameter estimate $\hat{\vartheta}_r(kT)$ and covariance matrix $P_r(kT)$ of subsystem r, we have the following remarks.

Remark 1 The parameter estimate $\hat{\vartheta}_r(kT)$ and covariance matrix $P_r(kT)$ of subsystem r in (26)–(28) with $i=r$ are equivalent with the estimate $\hat{\vartheta}(kT)$ and covariance matrix $P(kT)$ in (13)–(15), i.e.,

$$\begin{cases}\hat{\vartheta}(kT)=\hat{\vartheta}_r(kT),\\ P(kT)=P_r(kT).\end{cases} \quad (32)$$

Remark 2 For the C-LS algorithm in (20)–(25), assume that $\{v_i(kT),\mathscr{F}_{kT}\}$ $(i=1,2,\cdots,r)$ is a martingale difference sequence defined on the a probability space $\{\Omega,\mathscr{F},P\}$, where $\{\mathscr{F}_{kT}\}$ is the σ algebra sequence generated by $\{v_i(kT)\}$, i.e., $\mathscr{F}_{kT}=\sigma(v_i(kT),v_i(kT-T),v_i(kT-2T),\cdots)$. The noise sequence $\{v_i(kT)\}$ satisfies the following assumptions [25]:

$$\begin{aligned}&(A1)\quad \mathrm{E}[v_i(kT)|\mathscr{F}_{kT-T}]=0,\ \text{a.s.},\\ &(A2)\quad \mathrm{E}[v_i^2(kT)|\mathscr{F}_{kT-T}]\leqslant\sigma^2<\infty,\ \text{a.s.}\end{aligned}$$

the following inequality holds:

$$\sum_{k=1}^{\infty}\sum_{i=1}^{r}\frac{\Phi_i(kT)P_i(kT)\Phi_i^{\mathrm{T}}(kT)}{[\ln|P_i^{-1}(kT)|]^c}<\infty,\ \text{a.s.},\ c>1.$$

V. EXAMPLES

Example Assume that the process model P_c has the following transfer function:

$$P_c(s)=\frac{2s+0.8}{s^2+0.8s+0.8}.$$

This is a second-order system and its corresponding state-space realization is given by

$$\begin{cases}\dot{x}(t)=\begin{bmatrix}-0.8 & -0.8\\ 1 & 0\end{bmatrix}x(t)+\begin{bmatrix}1\\ 0\end{bmatrix}u(t),\\ y(t)=[2,\ 0.8]x(t).\end{cases}$$

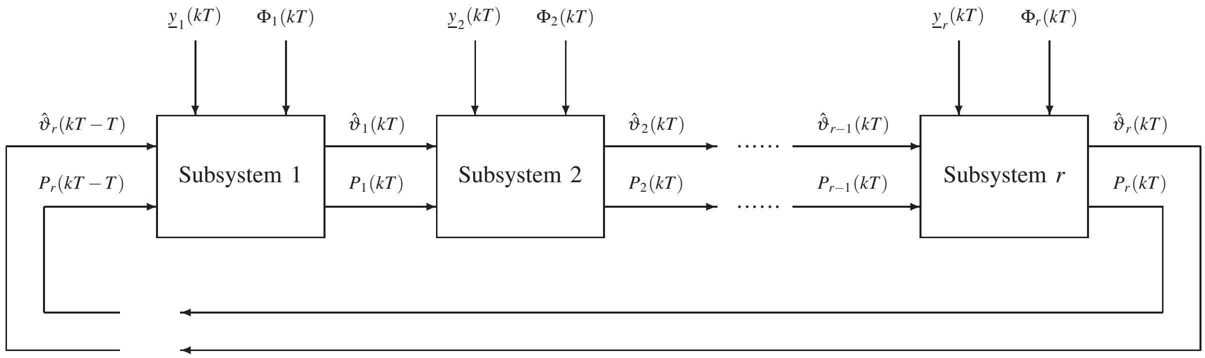

Fig. 2. The schematic diagram of the coupled least squares algorithm

Let $r=2$, $t_0=0$, $t_1=\sqrt{2}-1$ s, $t_2=T=1$ s, i.e., $\tau_1=t_1$ s, $\tau_2=2-\sqrt{2}$ s. Discretizing this example system gets

$$\begin{aligned} x(kT+T) &= \begin{bmatrix} 0.22659 & -0.48086 \\ 0.60107 & 0.70745 \end{bmatrix} x(kT) \\ &\quad + \begin{bmatrix} 0.15443 & 0.44665 \\ 0.22129 & 0.14440 \end{bmatrix} \begin{bmatrix} u(kT) \\ u(kT+t_1) \end{bmatrix}, \\ \begin{bmatrix} y(kT) \\ y(kT+t_1) \end{bmatrix} &= \begin{bmatrix} 2 & 0.80 \\ 1.60243 & 0.19991 \end{bmatrix} x(kT) \\ &\quad + \begin{bmatrix} 0 & 0 \\ 0.75011 & 0 \end{bmatrix} \begin{bmatrix} u(kT) \\ u(kT+t_1) \end{bmatrix}. \end{aligned}$$

Assume that the corresponding non-uniform discrete-time system to be identified has the following form:

$$\begin{aligned} &\alpha(z)y(kT+t_{i-1}) = \beta_i(z) \begin{bmatrix} u(kT) \\ u(kT+t_1) \end{bmatrix} + v(kT), \ i=1,2, \\ &\alpha(z) = 1+\alpha_1 z^{-1}+\alpha_2 z^{-2} \in \mathbb{R}^1, \\ &\beta_i(z) = \beta_{i0}+\beta_{i1}z^{-1}+\beta_{i2}z^{-2} \in \mathbb{R}^{1\times 2}, \ i=1,2. \end{aligned}$$

In the simulation, the inputs $\{u(kT+t_i),\ i=0,1\}$ are taken as persistent excitation signal sequences with zero mean and unit variance, and $\{v(kT)\}$ as a white noise sequence with zero mean and variance σ^2. Consider two cases with the noise variances $\sigma=0.10^2$ and $\sigma^2=0.50^2$, the corresponding noise-to-signal ratios are $\delta_{\rm ns}=11.59\%$ and $\delta_{\rm ns}=57.94\%$, respectively. Applying the C-LS algorithm to estimate the parameters of this non-uniform multirate system, the parameter estimates and their errors with different data lengths k are shown in Tables I – II and the parameter estimation errors δ versus $t=kT$ are shown in Fig.3.

From Tables I – II and Fig. 3, we can see that the parameter estimation error δ is becoming smaller (in general) with k increasing and a lower noise level leads to more accurate parameter estimates.

VI. Conclusions

This paper presents a C-LS algorithms for non-uniformly sampled multirate systems. The proposed algorithm is simple and easy to implement because it is not required to calculate the matrix inversion at each step. The estimates given by the C-LS algorithm are equivalent to those from the standard recursive least squares algorithm, thus a good performance

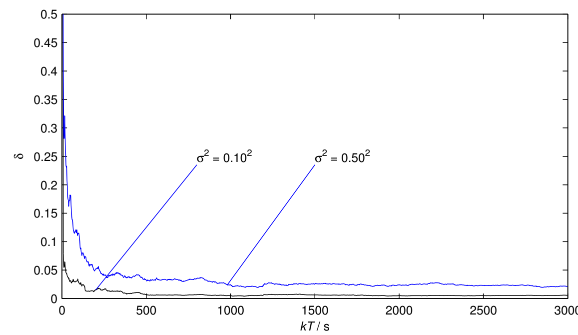

Fig. 3. The C-LS estimation errors δ versus $t=kT$ with $\sigma^2=0.10^2$ and $\sigma^2=0.50^2$

of the proposed algorithm can be guaranteed. Since the converted model of the non-uniformly sampled system is in a multi-input multi-output form, the gradient based algorithm [27], [28] and the multi-innovation technique [29]–[33] can be further extended to the coupled parameter estimation algorithm.

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TABLE I

THE C-LS ESTIMATES AND ERRORS ($\sigma^2 = 0.10^2$)

k	α_1	α_2	$\beta_{11}(1)$	$\beta_{11}(2)$	$\beta_{12}(1)$	$\beta_{12}(2)$	$\beta_{20}(1)$	$\beta_{21}(1)$	$\beta_{21}(2)$	$\beta_{22}(2)$	δ (%)
100	-0.91396	0.44754	0.49105	0.99295	-0.37737	-0.57507	0.75018	-0.41135	0.74442	-0.53936	2.78422
200	-0.91812	0.44983	0.48663	0.99701	-0.38139	-0.57455	0.75352	-0.40050	0.74057	-0.56303	1.50028
500	-0.92560	0.44948	0.48301	1.00745	-0.39003	-0.57830	0.75227	-0.40552	0.74261	-0.56663	0.75654
1000	-0.93525	0.45076	0.48207	1.00980	-0.39207	-0.58466	0.75328	-0.41411	0.74434	-0.57471	0.52058
2000	-0.93776	0.45151	0.48494	1.00857	-0.39877	-0.58602	0.75235	-0.41580	0.74352	-0.57234	0.50544
3000	-0.93924	0.45079	0.48635	1.00933	-0.40065	-0.58862	0.75039	-0.41547	0.74349	-0.57300	0.57639
True values	-0.93403	0.44933	0.48589	1.00881	-0.39717	-0.58223	0.75011	-0.40893	0.74459	-0.57047	

TABLE II

THE C-LS ESTIMATES AND ERRORS ($\sigma^2 = 0.50^2$)

k	α_1	α_2	$\beta_{11}(1)$	$\beta_{11}(2)$	$\beta_{12}(1)$	$\beta_{12}(2)$	$\beta_{20}(1)$	$\beta_{21}(1)$	$\beta_{21}(2)$	$\beta_{22}(2)$	δ (%)
100	-0.93326	0.46491	0.50873	0.93189	-0.34808	-0.64375	0.74668	-0.49748	0.74725	-0.48829	11.47943
200	-0.90776	0.44819	0.48473	0.94903	-0.34478	-0.59787	0.76565	-0.41033	0.72409	-0.57285	5.16447
500	-0.91310	0.44399	0.47067	1.00215	-0.37005	-0.58384	0.76058	-0.40796	0.73512	-0.56689	3.13628
1000	-0.93154	0.44888	0.46637	1.01364	-0.36760	-0.58578	0.76563	-0.42872	0.74318	-0.58535	2.43135
2000	-0.95101	0.45936	0.48125	1.00745	-0.40415	-0.59966	0.76142	-0.44218	0.73947	-0.57848	2.41501
3000	-0.95041	0.45334	0.48813	1.01132	-0.40996	-0.60463	0.75146	-0.43442	0.73915	-0.57585	2.13697
True values	-0.93403	0.44933	0.48589	1.00881	-0.39717	-0.58223	0.75011	-0.40893	0.74459	-0.57047	

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