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# $\mathcal{L}_1$ Adaptive Controller for Quantized Systems

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*Abstract*— **This paper studies the tracking problem of an uncertain LTI system where both the control input and system state are quantized. The $\mathcal{L}_1$ adaptive controller is designed for the quantized system. Two common types of quantization, logarithmic and uniform, quantization are considered. In both cases, the analysis of the closed-loop system provides a uniform transient performance bound, which depends on the adaptation rate and the quantization densities of the state and the input. By increasing the adaptation rate and improving the state and the input quantization, the closed-loop system response can be rendered arbitrarily close to the reference system output. Finally, the simulations illustrate the theoretical results.**

## I. Introduction

Real world systems are usually described by continuous-value continuous-time models. The variables used in the models take values in finite-dimensional Euclidean spaces. However, the values can be obtained only with finite precision. Quantization is a mapping from a larger set (such as a finite-dimensional Euclidean space) to a smaller set of finite or countably many symbols. It describes both hardware and software limitations. For instance, for hardware, it can describe the imprecise measurement, where only finite digits can be read from a meter, or the constrained control, where only selected values of control are allowed. For software limitation, such as in communication, it provides an approximation to a continuous-value variable, and thus reduces the transmission bits for a single value from infinite to finite.

Several papers studied quantization in LTI systems [1]–[4]. In quantized systems, two quantizers are typical. In [2], [5], the problem of stabilizing an unstable LTI system has been studied. The logarithmic quantizer has been shown to be the coarsest quantizer to stabilize the system. The idea of logarithmic quantizer is to maintain a small relative error. So it gets finer around the origin and coarser away from the origin. In [4], the problem of state estimation has been considered. Using information theoretic criteria, such as monotonic boundedness of entropy of the estimation error, it has been shown that the uniform quantization is the one that achieved the minimum rate.

Consider the tracking problem of an uncertain quantized system in the aforementioned two typical cases: logarithmic quantization and uniform quantization. To deal with the system uncertainty, we use an adaptive controller to estimate the uncertainty and adjust the control according to the estimate. We refer to the $\mathcal{L}_1$ adaptive controller due to its ability of fast adaptation with guaranteed robustness (bounded away from zero time-delay margin) [6], [7]. With $\mathcal{L}_1$ adaptive controller the closed-loop system has a predictable response, i.e., uniform scaled response with all the changes in initial conditions, reference inputs and unknown parameters [7]–[10]. In this paper we show that in the presence of quantization, the $\mathcal{L}_1$ architecture leads to uniform performance bounds, which can be decoupled into three terms, highlighting the trade-off between adaptation, robustness and quantization. Compared to the adaptive control of regular (non-quantized) systems, quantization introduces nonlinear and non-differentiable quantization errors.

We notice that event-triggering with $\mathcal{L}_1$ adaptation in networked systems was reported in [11]. Input quantization was considered in [12] for linear systems, and in [13] for nonlinear multi-input multi-output systems. This paper extends the results of [12] to accommodate also state quantization. The focus of the paper is on the quantization effect for system performance in the absence of an event detector.

The paper is organized as follows. The problem formulation is introduced in Section II, and the controller is designed in Section III. Sections IV and V introduce the two typical quantization schemes: logarithmic and uniform quantization. It is followed by the analysis of the closed-loop system in Section VI. Finally, simulations in Section VII illustrate the results.

Throughout the paper, $\|\cdot\|_1$,$\|\cdot\|_2$ and $\|\cdot\|_\infty$ denote the 1-norm, 2-norm and infinity norm of a vector, respectively. Notations $\|\xi_t\|_{\mathcal{L}_\infty}$ and $\|\xi\|_{\mathcal{L}_\infty}$ denote, respectively, the truncated (to $[0,t]$) $\mathcal{L}_\infty$-norm and the (untruncated) $\mathcal{L}_\infty$-norm of the time-varying signal $\xi(t)$. For a stable proper transfer matrix $G(s)$, $\|G(s)\|_{\mathcal{L}_1}$ denotes its $\mathcal{L}_1$-norm.

## II. Problem Formulation

Consider a networked control system, where the plant and the controller are connected by a communication network. Specifically, we analyze the quantization effect of the system. To communicate over the network, the state is quantized and sent over the network to the controller. At the other end, the generated control is preprocessed, quantized, and sent to control the plant. The system diagram is shown in Figure 1.

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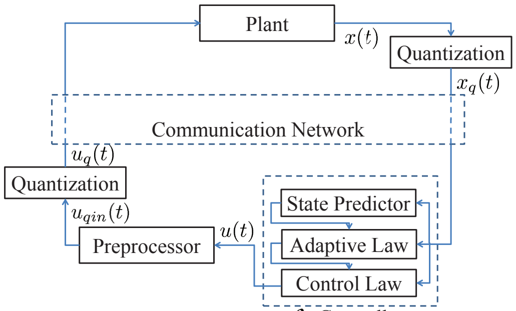



Fig. 1: Quantized System with $\mathcal{L}_1$ adaptive controller

The plant dynamics are given by

$$\begin{aligned}\dot{x}(t) =& A_m x(t) + b\left(-\theta^\top x(t) + u_{\rm q}(t)\right) ,\\ y(t) =& c^\top x(t) , \quad x(0) = x_0 ,\\ x_{\rm q}(t) =& Q_x(x(t)) , \quad u_{\rm q}(t) = Q_u(u_{\rm qin}(t)) ,\end{aligned} \tag{1}$$

where $A_m$ is a known $n \times n$ Hurwitz matrix, $b, c \in \mathbb{R}^n$ are known constant vectors, $\theta \in \mathbb{R}^n$ is an unknown constant vector, $x(t) \in \mathbb{R}^n$ is the system state vector (measured), $y(t) \in \mathbb{R}$ is the regulated output, $u_{\rm qin}(t)$ is the designed control signal, and $Q_x(\cdot)$ and $Q_u(\cdot)$ are the quantization functions for state and input respectively.

*Assumption 1:* The unknown parameter $\theta$ belongs to a given compact convex set $\Theta_B$, $\theta \in \Theta_B$. Let $\theta_{1\max} \triangleq \max_{\theta\in\Theta_B} \|\theta\|_1$.

The objective is to design an adaptive controller that would compensate for the uncertainties in the system and ensure analytically quantifiable uniform transient and steady-state performance bounds in the presence of both input and state quantization.

## III. $\mathcal{L}_1$ Adaptive Controller

In this section we present the $\mathcal{L}_1$ adaptive controller for the system in (1). The $\mathcal{L}_1$ adaptive controller consists of a state predictor, an adaptive law and a control law, Fig. 1.

We consider the following state predictor

$$\begin{aligned}\dot{\hat{x}}(t) =& A_m \hat{x}(t) + b(-\hat{\theta}^\top(t) x_{\rm q}(t) + u_{\rm q}(t))\\ \hat{y}(t) =& c^\top \hat{x}(t) , \quad \hat{x}(0) = x_0 ,\end{aligned} \tag{2}$$

where $\hat{x}(t) \in \mathbb{R}^n$, $\hat{y}(t) \in \mathbb{R}$ are the state and the output of the state predictor and $\hat{\theta}(t) \in \mathbb{R}^n$ is an estimate of the parameter $\theta$. The projection-type adaptive law for $\hat{\theta}(t)$ is given by

$$\dot{\hat{\theta}}(t) = \Gamma \mathrm{Proj}(\hat{\theta}(t), x_{\rm q}(t)\tilde{x}_{\rm q}^\top(t)Pb),\ \hat{\theta}(0) = \hat{\theta}_0, \tag{3}$$

where $\tilde{x}_{\rm q}(t) \triangleq \hat{x}(t) - x_{\rm q}(t)$ is the error between prediction and quantized state, $\Gamma > 0$ is the adaptation rate, $\mathrm{Proj}(\cdot,\cdot)$ denotes the projection operator [14], which ensures that $\hat{\theta}(t) \in \Theta_B$ for all $t \geq 0$, and $P = P^\top > 0$ solves the algebraic Lyapunov equation $A_m^\top P + PA_m = -Q$ for some symmetric $Q > 0$. The control signal is defined by

$$u(s) = C(s)\left(\hat{\eta}_{\rm q}(s) + k_g r(s)\right) ,\ u_{\rm qin}(t) = f(u(t)), \tag{4}$$

where $k_g \triangleq 1/(c^\top H(0))$, $H(s) \triangleq (s\mathbb{I} - A_m)^{-1}b$, $\hat{\eta}_{\rm q}(t) \triangleq \hat{\theta}^\top(t)x_{\rm q}(t)$, $u_{\rm qin}(t)$ is a modified control signal, and the function $f$ is selected according to an appropriate quantization method, while $C(s)$ is a BIBO stable and strictly proper transfer function with DC gain $C(0) = 1$, and its state-space realization assumes zero initialization. Let

$$G(s) \triangleq H(s)(C(s) - 1) . \tag{5}$$

The $\mathcal{L}_1$ adaptive controller consists of (2) - (4), with $C(s)$ verifying the following upper bound for the $\mathcal{L}_1$ norm of $G(s)$

$$\lambda \triangleq \|G(s)\|_{\mathcal{L}_1}\theta_{1\max} < 1. \tag{6}$$

## IV. Logarithmic Quantization

Let $Q_{\log}(\cdot)$ be the quantization function of the logarithmic quantizer. If the input signal is $v(t)$, the quantization density is $\rho$, and the parameters for the quantization intervals are $\alpha_0$ and $v_0$, the quantization function is defined by [15]

$$v_{\rm q}(t) = Q_{\log}(v(t)) = \begin{cases} v_i, & \alpha_i \leq v(t) < \alpha_{i+1},\\ 0, & v(t) = 0,\\ -v_i, & -\alpha_{i+1} \leq v(t) < -\alpha_i,\end{cases} \tag{7}$$

where $\alpha_0 > 0$, $0 < \rho < 1$, $\alpha_0 < v_0 < \alpha_1$, $\alpha_0 \leq v_0 \leq \frac{1}{\rho}\alpha_0$, $\alpha_{i+1} = \frac{1}{\rho}\alpha_i$, $v_{i+1} = \frac{1}{\rho}v_i$, $i \in \mathcal{I}$. The logarithmic quantization function $Q_{\log}$ is shown in Figure 2.

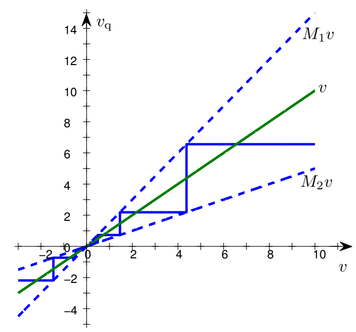



Fig. 2: Logarithmic Quantization Function, $\rho = \frac{1}{3}$.

Let $M_1 \triangleq \frac{v_i}{\alpha_i} = \frac{v_0}{\alpha_0}$ and $M_2 \triangleq \frac{v_i}{\alpha_{i+1}} = \rho\frac{v_0}{\alpha_0} = \rho M_1$. We know from (7) that

$$M_2|v| \leq |v_{\rm q}| \leq M_1|v|. \tag{8}$$

### A. State Quantization Error

If the state quantization is logarithmic, then $Q_x(x(t)) = Q_{\log}(x(t))$, $x(t) \in \mathbb{R}$. For state quantization, let the quantization density be $\rho_{{\rm q}x}$, and the parameters for quantization intervals be $\alpha_{0x}$ and $v_{0x}$. Let $M_{1x} = \frac{v_{0x}}{\alpha_{0x}}$, and $M_{2x} = \rho_{{\rm q}x} M_{1x}$. The sector bounds become $M_{2x}|x| \leq |x_{\rm q}| \leq M_{1x}|x|$. If the state quantization error is defined by $x_{\rm qe}(t) = x(t) - x_{\rm q}(t)$, as shown in the Figure 2, it is bounded by

$$|x_{\rm qe}(t)| \leq \delta_x|x(t)|, \quad \delta_x \triangleq \max\{(M_{1x} - 1), (1 - M_{2x})\}.$$

*Remark 1:* The constant $\delta_x$ represents the relative error for the quantization. It is independent of the value $x(t)$, and depends only on $M_{1x}$ and $M_{2x}$, and thus depends on the quantization parameters $\alpha_{0x}$, $v_{0x}$, and $\rho_{{\rm q}x}$. When these quantization parameters are fixed, $\delta_x$ is fixed.

When $\alpha_{0x}$ and $v_{0x}$ are fixed, $\delta_x$ depends only on the quantization density $\rho_{{\rm q}x}$. When the quantization is finer, i.e. $\rho_{{\rm q}x}$ is larger, $\delta_x$ is smaller. □

The quantization error bound carries to the vector case $x(t) \in \mathbb{R}^n$. For each entry, assign a logarithmic quantizer as described before. Let the constants be $\delta_{xj}$, $j = 1, \ldots, n$. Then $\delta_x = \max_{j=1,\ldots,n} \delta_{xj}$.

### B. Input Quantization Error

If the input quantization is logarithmic, then $Q_u(u_{\rm qin}(t)) = Q_{\log}(u_{\rm qin}(t))$. For input quantization, let the quantization density be $\rho_{\rm qu}$, and the parameters for quantization intervals be $\alpha_{0u}$ and $v_{0u}$. Let $M_{1u} = \frac{v_{0u}}{\alpha_{0u}}$ and $M_{2u} = \rho_{\rm qu} M_{1u}$. By (8), the sector bounds of $u_{\rm q}(t)$ are given by $M_{2u}|u_{\rm qin}| \le |u_{\rm q}| \le M_{1u}|u_{\rm qin}|$ .Thus,

$$\left||u_{\rm q}| - \tfrac{M_{1u}+M_{2u}}{2}|u_{\rm qin}|\right| \le \tfrac{M_{1u}-M_{2u}}{2}|u_{\rm qin}| . \tag{9}$$

Since $u_q$ and $u_{\rm qin}$ always have the same sign, we have $\left|u_{\rm q} - \frac{M_{1u}+M_{2u}}{2}u_{\rm qin}\right| \le \frac{M_{1u}-M_{2u}}{2}|u_{\rm qin}|$. Let $f_{\log}$ be the modification function as in (4) for the logarithmic quantizer. Following [15], we choose

$$u_{\rm qin} = f_{\log}(u) = \tfrac{2}{M_{1u}+M_{2u}} u . \tag{10}$$

Let the input quantization error be $u_{\rm qe}(t) \triangleq u_{\rm q}(t) - u(t)$. Since $u_{\rm q} = u + u_{\rm qe} = \frac{M_{1u}+M_{2u}}{2}u_{\rm qin} + u_{\rm qe}$, (9) implies

$$|u_{\rm qe}(t)| \le \tfrac{M_{1u}-M_{2u}}{2}|u_{\rm qin}(t)| = \delta_u |u(t)| , \tag{11}$$

where $\delta_u \triangleq \frac{M_{1u}-M_{2u}}{M_{1u}+M_{2u}}$. Since the inequality in (11) holds for all $t \in (0, \infty)$, we have

$$\|u_{qe}\|_{\mathcal{L}_\infty} \le \delta_u \|u\|_{\mathcal{L}_\infty} . \tag{12}$$

Note that $\delta_u = \frac{M_{1u}-M_{2u}}{M_{1u}+M_{2u}} = \frac{1-\rho_{\rm qu}}{1+\rho_{\rm qu}}$ is also a constant representing the coarseness of the quantizer as $\rho_{\rm qu}$ in (7). When the quantizer is finer, $\rho$ increases, $\delta_u$ decreases.

## V. Uniform Quantization

Let $Q_{\rm unif}(\cdot)$ be the quantization function of the uniform quantization. If the input signal is $v(t)$, the quantization function $v_{\rm q}(t) = Q_{\rm unif}(v(t))$ is defined by

$$v_{\rm q}(t) = \begin{cases} v_i, & \alpha_i \le v(t) < \alpha_{i+1}, \\ 0, & \alpha_0 \le v(t) < \alpha_0, \\ -v_i, & -\alpha_{i+1} \le v(t) < -\alpha_i, \end{cases}$$

where $\alpha_0 = \frac{1}{2}l$, $\alpha_0 < v_0 < \alpha_1$, $\alpha_{i+1} = \alpha_i + l$, $i \in \mathcal{I}$, and $l$ is the length of the quantization interval. The uniform quantization function $Q_{\rm unif}$ is shown in Fig. 3. The quantization error in this case is bounded by $|v_q - v| \le \frac{1}{2}l$.

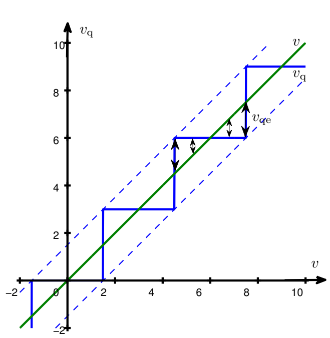



Fig. 3: Uniform Quantization Function, $l = 3$.

Similar to Section IV-A, the quantization error bound carries to the vector case $v(t) \in \mathbb{R}^n$. Likewise, assign a uniform quantizer for each entry as described before. Let the constants be $l_{xj}$, $j = 1, \ldots, n$. Then $l_x = \max_{j=1,\ldots,n} l_{xj}$.

## VI. Analysis of $\mathcal{L}_1$ Adaptive Controller

### A. Stability of Reference System

Consider the reference system

$$\begin{aligned} \dot{x}_{\rm ref}(t) &= A_m x_{\rm ref}(t) + b\left(-\theta^\top x_{\rm ref}(t) + u_{\rm ref}(t)\right) \\ y_{\rm ref}(t) &= c^\top x_{\rm ref}(t) , \quad x_{\rm ref}(0) = x_0 , \end{aligned} \tag{13}$$
$$u_{\rm ref}(s) = C(s)\left(\theta^\top x_{\rm ref}(s) + k_g r(s)\right) , \tag{14}$$

which can be rewritten as

$$x_{\rm ref}(s) = H(s)k_g C(s) r(s) + G(s)\theta^\top x_{\rm ref}(s) + (s\mathbb{I} - A_m)^{-1}x_0,$$
$$x_{\rm ref}(s) = (\mathbb{I} - G(s)\theta^\top)^{-1}[H(s)C(s)k_g r(s) + (s\mathbb{I} - A_m)^{-1}x_0] \tag{15}$$

*Lemma 1 ([8]):* If the condition in (6) holds, then $(\mathbb{I} - G(s)\theta^\top)^{-1}$ and $(\mathbb{I} - G(s)\theta^\top)^{-1}G(s)$ are BIBO stable. □

If the two transfer functions are BIBO stable, the relations in (15) and (14) lead to the following bounds

$$\|x_{\rm ref}\|_{\mathcal{L}_\infty} \le \rho_{x_r}, \quad \|u_{\rm ref}\|_{\mathcal{L}_\infty} \le \rho_{u_r}, \tag{16}$$
$$\begin{aligned} \rho_{x_r} &\triangleq k_g \|(\mathbb{I} - G(s)\theta^\top)^{-1}H(s)C(s)\|_{\mathcal{L}_1}\|r\|_{\mathcal{L}_\infty}, \\ &\quad + \|(\mathbb{I} - G(s)\theta^\top)^{-1}\|_{\mathcal{L}_1}\|(s\mathbb{I} - A_m)^{-1}x_0\|_{\mathcal{L}_\infty} \\ \rho_{u_r} &\triangleq \|C(s)\theta^\top\|_{\mathcal{L}_1}\rho_{x_r} + k_g\|r\|_{\mathcal{L}_\infty}. \end{aligned}$$

### B. Prediction Error

Let $\tilde{x}(t) \triangleq \hat{x}(t) - x(t)$, and $\tilde{\theta}(t) \triangleq \hat{\theta}(t) - \theta$. From (1) and (2), we have the prediction error dynamics

$$\dot{\tilde{x}}(t) = A_m\tilde{x}(t) + b\left(\theta^\top x(t) - \hat{\theta}^\top(t)x_{\rm q}(t)\right), \tilde{x}(0) = 0 . \tag{17}$$

*Lemma 2:* For the system in (1) and the controller defined by (4), if $\|x_t\|_{\mathcal{L}_\infty} \le \rho_x$ and the state quantization is the logarithmic quantization as defined in Section IV-A, we have the following bound

$$\|\tilde{x}_t\|_{\mathcal{L}_\infty} \le \max\left\{\sqrt{\tfrac{\theta_{2\max}}{\lambda_{\min}(P)\Gamma}}, c_{\tilde{x}\log}(\delta_x)\rho_x\right\} , \tag{18}$$
$$\theta_{2\max} \triangleq 4\max_{\theta_a\in\Theta_B}\|\theta\|_2^2 , \; \tilde{\theta}_{1\max} \triangleq \max_{\theta_a\in\Theta_B}\|\theta_a - \theta\|_1 , \tag{19}$$
$$c_{\tilde{x}\log}(\delta_x) \triangleq \frac{\|Pb\|_2\theta_{1\max}\delta_x + \sqrt{c_{\Delta\log}(\delta_x)}}{\lambda_{\min}(Q)} , \tag{20}$$
$$c_{\Delta\log}(\delta_x) \triangleq (\|Pb\|_2\theta_{1\max}\delta_x)^2 + 2\lambda_{\min}(Q)\delta_x\|Pb\|_1\tilde{\theta}_{1\max},$$

where $\theta_{1\max}$ is defined in Assumption 1 and $\lambda_{\min}(\cdot)$ is the smallest eigenvalue of a matrix.

**Proof.** Consider the Lyapunov function $V(t) = \tilde{x}^\top(t)P\tilde{x}(t) + \tilde{\theta}(t)^\top\Gamma^{-1}\tilde{\theta}(t)$, where $P$ and $\Gamma$ are defined in (3). Note that $\tilde{x}(t) = \hat{x}(t) - x(t) = \hat{x}(t) - x_{\rm q}(t) + x_{\rm q}(t) - x(t) = \tilde{x}_{\rm q}(t) + x_{\rm qe}(t)$, where $x_{\rm qe}(t) = x_{\rm q}(t) - x(t)$ is the quantization error of the system state defined in Section IV-A and $\tilde{x}_{\rm q}(t)$ is defined in (3). Then the derivative of $V(t)$ can be written as

$$\begin{aligned} \dot{V}(t) &= \tilde{x}(t)^\top A_m^\top P\tilde{x}(t) + \tilde{x}(t)^\top PA_m\tilde{x}(t) - 2\tilde{x}_{\rm q}(t)^\top Pb\tilde{\theta}(t)^\top x_{\rm q}(t) \\ &\quad + 2\tilde{\theta}^\top\Gamma^{-1}\dot{\tilde{\theta}} - 2x_{\rm qe}(t)^\top Pb\tilde{\theta}(t)^\top x_{\rm q}(t) - 2\tilde{x}(t)^\top Pb\theta^\top x_{\rm qe}(t) . \end{aligned}$$

The design of adaptive law in (3) ensures that

$$\dot{V}(t) \leq -\tilde{x}(t)^\top Q\tilde{x}(t) - 2x_{\text{qe}}(t)^\top Pb\tilde{\theta}(t)^\top x_{\text{q}}(t) - 2\tilde{x}(t)^\top Pb\theta^\top x_{\text{qe}}(t)\,, \tag{21}$$

where $\theta_{1\max}$ is defined in Assumption 1, $\tilde{\theta}_{1\max}$ is defined in (19), and $\lambda_{\min}(\cdot)$ is defined in Lemma 2. If $\|x_t\|_{\mathcal{L}_\infty} < \rho_x$, the inequality becomes

$$\dot{V}(t) \leq -\lambda_{\min}(Q)\|\tilde{x}(t)\|_2^2 + 2\|Pb\|_2\theta_{1\max}\delta_x\rho_x\|\tilde{x}(t)\|_2 + 2\delta_x\|Pb\|_1\tilde{\theta}_{1\max}\rho_x^2\,.$$

Then either the prediction error

$$\|\tilde{x}(t)\|_2 \leq \tfrac{\|Pb\|_2\theta_{1\max}\delta_x\rho_x + \sqrt{c_{\Delta\log}(\delta_x)}\rho_x}{\lambda_{\min}(Q)}\,,$$

or the derivative $\dot{V}(t) < 0$. Hence, either $\|\tilde{x}(t)\|_2 \leq c_{\tilde{x}\log}(\delta_x)\rho_x$, or $\|\tilde{x}(t)\|_2 \leq \sqrt{\frac{\theta_{2\max}}{\lambda_{\min}(P)\Gamma}}$, where $c_{\tilde{x}\log}(\delta_x)$ and $c_{\Delta\log}(\delta_x)$ are defined in (20).

Since $\|\tilde{x}(t)\|_\infty \leq \|\tilde{x}(t)\|_2$, the inequality in (18) holds. □

*Lemma 3:* For the system in (1) and the controller defined by (4), if $\|x_t\|_{\mathcal{L}_\infty} \leq \rho_x$ and the state quantization is the uniform quantization with quantization interval length $l_x$, we have the following bound

$$\|\tilde{x}_t\|_{\mathcal{L}_\infty} \leq \max\left\{\sqrt{\frac{\theta_{2\max}}{\lambda_{\min}(P)\Gamma}}, c_{\tilde{x}\text{unif}}(l_x,\rho_x)\right\}\,, \tag{22}$$

$$c_{\tilde{x}\text{unif}}(l_x,\rho_x) \triangleq \frac{\|Pb\|_2\theta_{1\max}l_x + \sqrt{c_{\Delta\text{unif}}(l_x,\rho_x)}}{2\lambda_{\min}(Q)}\,, \tag{23}$$

$$c_{\Delta\text{unif}}(l_x,\rho_x) \triangleq (\|Pb\|_2\theta_{1\max}l_x)^2 + 4\lambda_{\min}(Q)l_x\|Pb\|_1\tilde{\theta}_{1\max}\left(\rho_x + \tfrac{1}{2}l_x\right)\,, \tag{24}$$

where $\theta_{1\max}$ is defined in Assumption 1, $\theta_{2\max}$ and $\tilde{\theta}_{1\max}$ are defined in (19).
The proof is similar to the one of Lemma 2 and is thus omitted. □

### *C. Performance Bounds for Logarithmic Quantized System*

Let the error signals be defined by

$$e_x(t) = x(t) - x_{\text{ref}}(t)\,,\ \ e_u(t) = u_{\text{q}}(t) - u_{\text{ref}}(t)\,. \tag{25}$$

Let $\bar{\gamma}_{x\log} > 0$ be an arbitrary positive number, and let

$$\rho_{x\log} = \rho_{x_r} + \bar{\gamma}_{x\log}\,,\ \ \gamma_{x\log} = \gamma_{x\text{o}\log} + \gamma_{x\text{q}\log} + \epsilon\,, \tag{26}$$

$$\gamma_{x\text{o}\log} = \|(\mathbb{I} - G(s)\theta^\top)^{-1}[G(s)\theta^\top + (C(s)-1)\mathbb{I} + \mathbb{I}]\|_{\mathcal{L}_1} \max\left\{\sqrt{\tfrac{\theta_{2\max}}{\lambda_{\min}(P)\Gamma}}, c_{\tilde{x}\log}(\delta_x)\rho_{x\log}\right\}\,,$$

$$\gamma_{x\text{q}\log} = \|(\mathbb{I} - G(s)\theta^\top)^{-1}H(s)\|_{\mathcal{L}_1}\delta_u\rho_{u\log}\,,$$

and $\epsilon > 0$ is a small positive number. Let $\Gamma$ be sufficiently large, $\delta_x$ and $\delta_u$ be sufficiently small, such that $\gamma_{x\log} = \gamma_{x\text{o}\log} + \gamma_{x\text{q}\log} + \epsilon < \bar{\gamma}_{x\log}$. Let

$$\rho_{u\log} = \rho_{u_r} + \gamma_{u\log}\,, \tag{27}$$

$$\gamma_{u\log} = \left\|C(s)\frac{1}{c_o^\top H(s)}c_o^\top\right\|_{\mathcal{L}_1} c_{\tilde{x}\log}(\delta_x)\rho_{x\log} + \|C(s)\theta^\top\|_{\mathcal{L}_1}\gamma_{x\log}.$$

*Theorem 1:* Consider the system in (1) and the controller in (4). The tracking errors are upper bounded by

$$\|x - x_{\text{ref}}\|_{\mathcal{L}_\infty} \leq \gamma_{x\log}\,,\ \ \|u - u_{\text{ref}}\|_{\mathcal{L}_\infty} \leq \gamma_{u\log}\,, \quad \|x\|_{\mathcal{L}_\infty} \leq \rho_{x\log},\ \ \|u\|_{\mathcal{L}_\infty} \leq \rho_{u\log}\,. \tag{28}$$

**Proof.** (By contradiction)

Assume the bounds in (28) do not hold. Since $\|e_x(0)\|_\infty = 0 < \gamma_{x\log}$, $\|e_u(0)\|_\infty = 0 < \gamma_{u\log}$, and $x(t)$, $x_{\text{ref}}(t)$, $u(t)$, and $u_{\text{ref}}(t)$ are continuous, there exists $t'$ such that $e_x(\tau)$ and $e_u(\tau)$ are within the bounds before $t'$, $\|e_x(\tau)\|_\infty < \gamma_{x\log}$, $\|e_u(\tau)\|_\infty < \gamma_{u\log}$, $\forall\tau \in [0,t')$, and hit the bound at $t'$, i.e.,

$$\text{either}\ \ \|e_x(t')\|_\infty = \gamma_{x\log},\ \ \text{or}\ \ \|e_u(t')\|_\infty = \gamma_{u\log}. \tag{29}$$

When $t \leq t'$, $\|e_x(t)\|_\infty \leq \gamma_{x\log}$, we have $\|e_{x_{t'}}\|_{\mathcal{L}_\infty} \leq \gamma_{x\log}$, $\|x_{t'}\|_{\mathcal{L}_\infty} < \rho_{x\log}$, and the inequality (18) in Lemma 2 holds. To use the upper bound on $\tilde{x}(t)$, $e_x(t)$ can be written as $e_x(t) = x(t) - x_{\text{ref}}(t) = \hat{x}(t) - x_{\text{ref}}(t) - \tilde{x}(t)$.

On one hand, from (2), $\hat{x}(s)$ is given by

$$\hat{x}(s) = -H(s)\hat{\eta}_{\text{q}}(s) + H(s)u(s) + H(s)u_{\text{qe}}(s) + (s\mathbb{I} - A_m)^{-1}x_0,$$

where $\hat{\eta}_{\text{q}}(t) = \hat{\theta}^\top(t)x_{\text{q}}(t)$ is defined in (4). By the control law in (4) and the definition of $\tilde{x}(t)$, we further write

$$\hat{x}(s) = G(s)\theta^\top\hat{x}(s) - G(s)\theta^\top\tilde{x}(s) + H(s)(C(s)-1)\tilde{\eta}_{\text{q}}(s) + H(s)C(s)k_g r(s) + H(s)u_{\text{qe}}(s) + (s\mathbb{I} - A_m)^{-1}x_0, \tag{30}$$

where $\tilde{\eta}_{\text{q}}(t) \triangleq \hat{\eta}_{\text{q}}(t) - \theta^\top x(t) = \hat{\theta}^\top(t)x_{\text{q}}(t) - \theta^\top x(t)$. For the third term, note that the prediction error dynamics, given by (17), can be further written as

$$\tilde{x}(s) = H(s)(\theta^\top x(s) - \hat{\eta}_{\text{q}}(s)) = -H(s)\tilde{\eta}_{\text{q}}(s)\,. \tag{31}$$

Substitute (31) into (30) to obtain

$$\begin{aligned}\hat{x}(s) =& (\mathbb{I} - G(s)\theta^\top)^{-1}[-G(s)\theta^\top - (C(s)-1)\mathbb{I}]\tilde{x}(s)\\ &+ (\mathbb{I} - G(s)\theta^\top)^{-1}H(s)u_{\text{qe}}(s)\\ &+ (\mathbb{I} - G(s)\theta^\top)^{-1}H(s)C(s)k_g r(s)\\ &+ (\mathbb{I} - G(s)\theta^\top)^{-1}(s\mathbb{I} - A_m)^{-1}x_0.\end{aligned}$$

On the other hand, note that $x_{\text{ref}}(s)$ is given in (15). Subtracting $x_{\text{ref}}(s)$ from $x(s)$ gives

$$e_x(s) = (\mathbb{I} - G(s)\theta^\top)^{-1}[-G(s)\theta^\top - (C(s)-1)\mathbb{I}]\tilde{x}(s) - \tilde{x}(s) + (\mathbb{I} - G(s)\theta^\top)^{-1}H(s)u_{\text{qe}}(s)\,,$$

$$\|e_{x_{t'}}\|_{\mathcal{L}_\infty} \leq \|(\mathbb{I} - G(s)\theta^\top)^{-1}H(s)\|_{\mathcal{L}_1}\delta_u\|u_{t'}\|_{\mathcal{L}_\infty} + \|(\mathbb{I} - G(s)\theta^\top)^{-1}[G(s)\theta^\top + (C(s)-1)\mathbb{I} + \mathbb{I}]\|_{\mathcal{L}_1}\|\tilde{x}_{t'}\|_{\mathcal{L}_\infty}. \tag{32}$$

Now we examine the error between the control signal $u(t)$ and the desired reference control signal $u_{\text{ref}}(t)$. By (4) and (14), we have $e_u(s) = C(s)\tilde{\eta}_q(s) + C(s)\theta^\top(x(s) - x_{\text{ref}}(s))$, where $\tilde{\eta}_q(t)$ is defined in (30). Since $(A_m, b)$ is controllable, and $H(s)$ is strictly proper and stable, there exists $c_o \in \mathbb{R}^n$ such that $c_o^\top H(s)$ is minimum phase with relative degree one (by Lemma 4 in [8]). Then

$$e_u(s) = C(s)\tfrac{1}{c_o^\top H(s)}c_o^\top\tilde{x}(s) + C(s)\theta^\top e_x(s)\,.$$

Since $C(s)$ is BIBO stable and strictly proper, the system $C(s)\frac{1}{c_o^\top H(s)}$ is proper and BIBO stable, which implies that its $\mathcal{L}_1$ norm is bounded. Hence,

$$\|e_{u_{t'}}\|_{\mathcal{L}_\infty} \leq \|C(s)\tfrac{1}{c_o^\top H(s)}c_o^\top\|_{\mathcal{L}_1}\|\tilde{x}_\tau\|_{\mathcal{L}_\infty} + \|C(s)\theta^\top\|_{\mathcal{L}_1}\|e_{x_{t'}}\|_{\mathcal{L}_\infty}. \tag{33}$$

If $\Gamma$ is sufficiently large, $\delta_x$ and $\delta_u$ are sufficiently small, such that $\gamma_{x\log} = \gamma_{xo\log} + \gamma_{xq\log} + \epsilon < \bar{\gamma}_{x\log}$, the error $e_x(t)$ in (32) is strictly upper bounded by

$$\|e_{x'_t}\|_{\mathcal{L}_\infty} \le \gamma_{xo\log} + \gamma_{xq\log} < \gamma_{x\log} < \bar{\gamma}_{x\log}\,, \quad (34)$$

where $\gamma_{x\log}$ is defined in (26). Similarly, the error $e_u(t)$ in (33) is strictly upper bounded by

$$\|e_{u'_t}\|_{\mathcal{L}_\infty} < \gamma_{u\log}\,, \quad (35)$$

where $\gamma_{u\log}$ is defined in (27).

The strict inequalities (34) and (35) contradict the assumption in (29). The proof is complete. □

*Remark 2:* In Theorem 1, the error bound $\gamma_{x\log}$ depends on three terms: $\sqrt{\frac{\theta_{2\max}}{\lambda_{\min}(P)\Gamma}}$, which is inverse proportional to $\sqrt{\Gamma}$, $c_{\tilde{x}\log}(\delta_x)\rho_{x\log}$, which is proportional to the state quantization parameter $\delta_x$, and $\delta_u\rho_{u\log}$, which is proportional to the input quantization parameter $\delta_u$. By increasing the adaptation rate $\Gamma$, the first term can be made arbitrarily small. Note that after taking the maximum, only the larger one of the first two terms affects $\gamma_{x\log}$. When the first one is small enough, $\gamma_{x\log}$ depends only on the second and third terms, i.e., the bottleneck of improving the performance bounds is the quantization. Besides, there is a trade-off between the state and the input quantization. To maintain a certain performance bound, if the state quantization is coarser ($\delta_x$ is larger), the input quantization has to be finer ($\delta_u$ has to be smaller), which agrees with the intuition. The second and the third terms decrease as the quantization parameters $\delta_x$ and $\delta_u$ decrease, i.e., as the quantization gets finer. As $\delta_x$ goes to zero, the second term goes to zero, and the result reduces to the case of input quantization as in [12]. Further, if $\delta_x$ and $\delta_u$ both go to zero, the second and the third terms go to zero, and the error bounds reduce to the case without quantization [8]. □

*Remark 3:* In (33), $\left\|C(s)\frac{1}{c_o^\top H(s)}c_o^\top\right\|_{\mathcal{L}_1}$ needs to be bounded for $e_u(t)$ to be bounded. This is ensured by the strictly proper and stable low-pass filter $C(s)$. In the absence of $C(s) = 1$, the term $\left\|C(s)\frac{1}{c_o^\top H(s)}c_o^\top\right\|_{\mathcal{L}_1}$ is unbounded, since $c_o^\top H(s)$ in the denominator is strictly proper. In this case, $\gamma_{u\log}$ is unbounded, implying that one cannot obtain a similar uniform bound for the control signal of MRAC. Likewise, $C(s)$ is crucial for the uniform quantization. □

### *D. Performance Bounds for Uniform Quantized System*

Similar to the case of logarithmic quantization, use the notations $e_x(t)$ and $e_u(t)$ defined in (25).

Let $\bar{\gamma}_{x\mathrm{unif}} > 0$ be an arbitrary positive number, and let

$$\rho_{x\mathrm{unif}} = \rho_{x_r} + \bar{\gamma}_{x\mathrm{unif}}\,, \quad (36)$$

$$\begin{aligned}
\gamma_{x\mathrm{unif}} &= \gamma_{xo\mathrm{unif}} + \gamma_{xq\mathrm{unif}} + \epsilon\,, \quad (37)\\
\gamma_{xo\mathrm{unif}} &= \|(\mathbb{I} - G(s)\theta^\top)^{-1}[G(s)\theta^\top + (C(s)-1)\mathbb{I} + \mathbb{I}]\|_{\mathcal{L}_1}\\
&\qquad \max\left\{\sqrt{\frac{\theta_{2\max}}{\lambda_{\min}(P)\Gamma}}, c_{\tilde{x}\mathrm{unif}}(l_x, \rho_{x\mathrm{unif}})\right\}\,,\\
\gamma_{xq\mathrm{unif}} &= \frac{1}{2}\|(\mathbb{I} - G(s)\theta^\top)^{-1}H(s)\|_{\mathcal{L}_1} l_u\,,
\end{aligned}$$

where $\epsilon > 0$ is a small positive number, $c_{\tilde{x}\mathrm{unif}}(l_x, \rho_{x\mathrm{unif}})$ is defined in (23), and $l_x$ and $l_u$ are the quantization interval lengths of the uniform quantization on $x$ and $u$, respectively. If $\Gamma$ is sufficiently large, and $l_x$ and $l_u$ are sufficiently small, such that $\gamma_{x\mathrm{unif}} = \gamma_{xo\mathrm{unif}} + \gamma_{xq\mathrm{unif}} + \epsilon < \bar{\gamma}_{x\mathrm{unif}}$, the error $e_x(t)$ is strictly upper bounded by

$$\|e_{x'_t}\|_{\mathcal{L}_\infty} < \gamma_{x\mathrm{unif}} < \bar{\gamma}_{x\mathrm{unif}}\,.$$

Similarly, let

$$\rho_{u\mathrm{unif}} = \rho_{u_r} + \gamma_{u\mathrm{unif}}\,, \quad (38)$$

where

$$\begin{aligned}
\gamma_{u\mathrm{unif}} &= \|C(s)\theta^\top\|_{\mathcal{L}_1}\gamma_{x\mathrm{unif}} \quad (39)\\
&+ \left\|C(s)\frac{1}{c_o^\top H(s)}c_o^\top\right\|_{\mathcal{L}_1}\max\left\{\sqrt{\frac{\theta_{2\max}}{\lambda_{\min}(P)\Gamma}}, c_{\tilde{x}\mathrm{unif}}(l_x, \rho_{x\mathrm{unif}})\right\}.
\end{aligned}$$

*Theorem 2:* Consider the system in (1) and the controller in (4). In the case of uniform quantization, if the quantization interval lengths are $l_x$ and $l_u$ for the state and the input, respectively, the tracking errors are upper bounded by

$$\begin{aligned}
&\|x - x_{\mathrm{ref}}\|_{\mathcal{L}_\infty} \le \gamma_{x\mathrm{unif}}\,,\ \|u - u_{\mathrm{ref}}\|_{\mathcal{L}_\infty} \le \gamma_{u\mathrm{unif}}\,, \quad (40)\\
&\|x\|_{\mathcal{L}_\infty} \le \rho_{x\mathrm{unif}},\ \|u\|_{\mathcal{L}_\infty} \le \rho_{u\mathrm{unif}}\,,
\end{aligned}$$

where $\rho_{x\mathrm{unif}}$, $\gamma_{x\mathrm{unif}}$, $\rho_{u\mathrm{unif}}$, $\gamma_{u\mathrm{unif}}$ are given by (36), (37), (38) and (39).

**Proof.** The proof is similar to the one of Theorem 1. First assume that the bounds in (40) do not hold, and either $e_x(t)$ or $e_u(t)$ hits the bound. The same derivations can be done up to (32). In the case of uniform quantization, the quantization error is bounded by the constant $\frac{1}{2}l$. Thus, $e_x(t)$ is bounded by

$$\begin{aligned}
&\|e_{x_{t'}}\|_{\mathcal{L}_\infty} \le \|(\mathbb{I} - G(s)\theta^\top)^{-1}H(s)\|_{\mathcal{L}_1}\frac{1}{2}l_u\\
&+ \|(\mathbb{I} - G(s)\theta^\top)^{-1}[G(s)\theta^\top + (C(s)-1)\mathbb{I} + \mathbb{I}]\|_{\mathcal{L}_1}\|\tilde{x}_{t'}\|_{\mathcal{L}_\infty}.
\end{aligned}$$

By Lemma 3, the bound can be written as

$$\|e_{x_{t'}}\|_{\mathcal{L}_\infty} \le \gamma_{xo\mathrm{unif}} + \gamma_{xq\mathrm{unif}} < \gamma_{x\mathrm{unif}}\,.$$

Following (33) and Lemma 3, $\|e_{u_{t'}}\|_{\mathcal{L}_\infty} < \gamma_{u\mathrm{unif}}$. These two strict inequalities contradict the assumption that the bound $\gamma_{x\mathrm{unif}}$ or $\gamma_{u\mathrm{unif}}$ is hit at time $t$. Thus, the assumption does not hold, and the proof is complete. □

## VII. Simulation

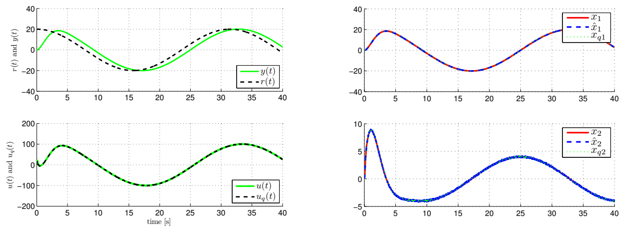
Consider the system in (1) with

$$A_m = \begin{bmatrix} 0 & 1 \\ -1 & -1.4 \end{bmatrix},\ b = \begin{bmatrix} 0 \\ 1 \end{bmatrix},\ c = \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

$$x_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix},\ \theta = \begin{bmatrix} 4 \\ -4.5 \end{bmatrix},$$

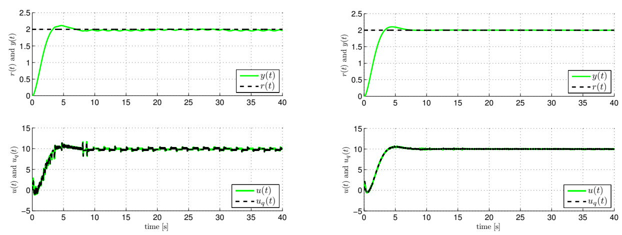
and let $\Theta_B = \{\theta_1 \in [-10, 10], \theta_2 \in [-10, 10]\}$, which gives $\theta_{1\max} = 20$. Let $C(s) = \frac{\omega}{s+\omega}$, where $\omega = 50$, and let the adaptation rate be $\Gamma = 10^6$.

We now show the performance of $\mathcal{L}_1$ adaptive controller for both types of quantizers under different reference signals without any retuning of the controller.



(a) $y(t)$ and $u(t)$ (b) $x(t)$

Fig. 4: System performance for logarithmic quantizer with $\delta_x = 0.0102$, $\delta_u = 0.0204$ and sinusoidal references $r(t) = 20\sin(0.2t + \frac{\pi}{2})$.

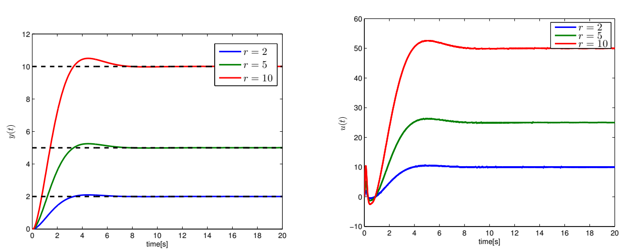


(a) $l_x = 0.1$, $l_u = 0.2$ (b) $l_x = 0.01$, $l_u = 0.02$

Fig. 5: $y(t)$ and $u(t)$ for uniform quantizer with step references $r(t) = 2$.

First, we see that for both quantizers the system output $y(t)$ tracks the reference signal $r(t)$. Two instances are shown in Figures 4a, 4b, and 5a. Among the three, Figures 4a and 4b show the output and the state of the quantized system with logarithmic quantization of both state and input, tracking a sinusoidal reference signal. Figure 5a shows the output of the system with uniform quantization, tracking a step reference signal. In both cases, the designed $\mathcal{L}_1$ controller leads to desired performance in the whole time span, as guaranteed by the uniform transient performance bounds derived in Section VI.

Second, the comparison of Figures 5a and 5b shows the effect of quantization density on the system performance. In Figure 5a, a small tracking error is visible between $y(t)$ and $r(t)$. In the latter case in Figure 5b, where the quantization interval lengths are reduced to $\frac{1}{10}$ of the former values, the quantization becomes finer. The output $y(t)$ almost coincides with $r(t)$. This shows that the performance bound decreases as the quantization density increases.



(a) $y(t)$ (b) $u(t)$

Fig. 6: $y(t)$ and $u(t)$ for uniform quantizer with $l_x = 0.01$, $l_u = 0.02$ and step references $r(t) = 2\,,5\,,10$.

Next, we show the scaled response of the closed-loop system to different reference signals. Figures 6a and 6b show the performance of the system output $y(t)$ and the input $u(t)$ in the case of uniform quantization with quantization intervals $l_x = 0.01$, $l_u = 0.02$ and step references $r = 2\,,5\,,10$. We note that it leads to scaled control inputs and system outputs for scaled reference inputs.

We finally notice that we do not redesign or retune the $\mathcal{L}_1$ controller in these simulations, from one reference input to another, or from one quantizer to the other.

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