2011 American Control Conference
on O'Farrell Street, San Francisco, CA, USA
June 29 - July 01, 2011

Retrospective Cost Model Reference Adaptive Control for Nonminimum-Phase Discrete-Time Systems, Part 2: Stability Analysis

Jesse B. Hoagg[1] and Dennis S. Bernstein[2]

Abstract— **This paper is the second part of a pair of papers, which together present a direct model reference adaptive controller for discrete-time (including sampled-data) systems that are possibly nonminimum phase. The present paper and its companion paper (Part 1) together analyze that stability of the retrospective cost model reference adaptive controller.**

I. Introduction

In this paper and its companion paper [1], we present a model reference adaptive control (MRAC) algorithm for discrete-time systems that are possibly nonminimum phase. This paper is intended to be read in conjunction with [1]. A detailed introduction is provided in [1].

The companion paper [1] develops the retrospective cost model reference adaptive control (RC-MRAC) algorithms, and focuses on the existence and properties of an ideal control law. The results of [1] are used in the present paper to analyze closed-loop stability.

In this second paper, we present a closed-loop error system, which is a system constructed by taking the difference between the closed-loop system with the ideal controller in feedback and the closed-loop system with the adaptive controller in feedback. Then we then examine the closed-loop stability.

II. Review of the Problem Formulation

Consider the discrete-time system

$$y(k) = -\sum_{i=1}^{n} \alpha_i y(k-i) + \sum_{i=d}^{n} \beta_i u(k-i), \tag{1}$$

where $k \geq 0$, $\alpha_1, \ldots, \alpha_n, \beta_d, \ldots, \beta_n \in \mathbb{R}$, $y(k) \in \mathbb{R}$ is the output, $u(k) \in \mathbb{R}$ is the control, and the relative degree is $d > 0$. Furthermore, for all $i < 0$, $u(i) = 0$, and the initial condition is $x_0 = [\ y(-1) \quad \cdots \quad y(-n)\ ]^{\mathrm{T}} \in \mathbb{R}^n$.

Let $\mathbf{q}$ and $\mathbf{q}^{-1}$ denote the forward-shift and backward-shift operators, respectively. For all $k \geq 0$, (1) can be expressed as $\alpha(\mathbf{q})y(k-n) = \beta(\mathbf{q})u(k-n)$, where $\alpha(\mathbf{q}) \stackrel{\triangle}{=} \mathbf{q}^n + \alpha_1\mathbf{q}^{n-1} + \alpha_2\mathbf{q}^{n-2} + \cdots + \alpha_{n-1}\mathbf{q} + \alpha_n$ and $\beta(\mathbf{q}) \stackrel{\triangle}{=} \beta_d\mathbf{q}^{n-d} + \beta_{d+1}\mathbf{q}^{n-d-1} + \cdots + \beta_{n-1}\mathbf{q} + \beta_n$.

Next, consider the reference model

$$y_{\mathrm{m}} = -\sum_{i=1}^{n_{\mathrm{m}}} \alpha_{\mathrm{m},i} y_{\mathrm{m}}(k-i) + \sum_{i=d_{\mathrm{m}}}^{n_{\mathrm{m}}} \beta_{\mathrm{m},i} r(k-i), \tag{2}$$

where $k \geq 0$, $\alpha_{\mathrm{m},1}, \ldots, \alpha_{\mathrm{m},n_{\mathrm{m}}}, \beta_{\mathrm{m},d_{\mathrm{m}}}, \ldots, \beta_{\mathrm{m},n_{\mathrm{m}}} \in \mathbb{R}$, $y_{\mathrm{m}}(k) \in \mathbb{R}$ is the reference model output, $r(k) \in \mathbb{R}$ is the reference model command, and $d_{\mathrm{m}} > 0$ is the relative degree of (2). Furthermore, for all $i < 0$, $r(i) = 0$, and the initial condition is $x_{\mathrm{m},0} = [\ y_{\mathrm{m}}(-1) \quad \cdots \quad y_{\mathrm{m}}(-n_{\mathrm{m}})\ ]^{\mathrm{T}} \in \mathbb{R}^n$. For all $k \geq 0$, (2) can be expressed as $\alpha_{\mathrm{m}}(\mathbf{q})y_{\mathrm{m}}(k-n_{\mathrm{m}}) = \beta_{\mathrm{m}}(\mathbf{q})r(k-n_{\mathrm{m}})$, where $\alpha_{\mathrm{m}}(\mathbf{q}) \stackrel{\triangle}{=} \mathbf{q}^{n_{\mathrm{m}}} + \alpha_{\mathrm{m},1}\mathbf{q}^{n_{\mathrm{m}}-1} + \cdots + \alpha_{\mathrm{m},n_{\mathrm{m}}-1}\mathbf{q} + \alpha_{\mathrm{m},n_{\mathrm{m}}}$ and $\beta_{\mathrm{m}}(\mathbf{q}) \stackrel{\triangle}{=} \beta_{\mathrm{m},d_{\mathrm{m}}}\mathbf{q}^{n_{\mathrm{m}}-d_{\mathrm{m}}} + \cdots + \beta_{\mathrm{m},n_{\mathrm{m}}-1}\mathbf{q} + \beta_{\mathrm{m},n_{\mathrm{m}}}$. Our goal is to drive the tracking error $z(k) \stackrel{\triangle}{=} y(k) - y_{\mathrm{m}}(k)$ to zero asymptotically. We make the following assumptions regarding the open-loop system (1):

(A1) $\alpha(\mathbf{q})$ and $\beta(\mathbf{q})$ are coprime.
(A2) d is known.
(A3) β_d is known.
(A4) If $\lambda \in \mathbb{C}$, $|\lambda| \geq 1$, and $\beta(\lambda) = 0$, then λ is known.
(A5) There exists an integer $\bar{n}$ such that $n \leq \bar{n}$ and $\bar{n}$ is known.
(A6) $\alpha(\mathbf{q})$, $\beta(\mathbf{q})$, n, and x_0 are not known.

In addition, we make the following assumptions regarding the reference model (2):

(A7) $\alpha_{\mathrm{m}}(\mathbf{q})$ and $\beta_{\mathrm{m}}(\mathbf{q})$ are coprime.
(A8) $\alpha_{\mathrm{m}}(\mathbf{q})$ is asymptotically stable.
(A9) If $\lambda \in \mathbb{C}$, $|\lambda| \geq 1$, and $\beta(\lambda) = 0$, then $\beta_{\mathrm{m}}(\lambda) = 0$.
(A10) If $\lambda \in \mathbb{C}$ and $\alpha(\lambda) = 0$, then $\beta_{\mathrm{m}}(\lambda) \neq 0$.
(A11) $d_{\mathrm{m}} \geq d$.
(A12) $r(k)$ is bounded.
(A13) $\alpha_{\mathrm{m}}(\mathbf{q})$, $\beta_{\mathrm{m}}(\mathbf{q})$, d_{m}, and n_{m} are known.

Next, let $\beta_{\mathrm{u}}(\mathbf{q})$ be a monic polynomial whose roots are a subset of the roots of $\beta(\mathbf{q})$ and include all the zeros of $\beta(\mathbf{q})$ that lie on or outside the unit circle. Furthermore, write $\beta_{\mathrm{u}}(\mathbf{q}) = \mathbf{q}^{n_{\mathrm{u}}} + \beta_{\mathrm{u},1}\mathbf{q}^{n_{\mathrm{u}}-1} + \cdots + \beta_{\mathrm{u},n_{\mathrm{u}}-1}\mathbf{q} + \beta_{\mathrm{u},n_{\mathrm{u}}}$, where $\beta_{\mathrm{u},1}, \ldots, \beta_{\mathrm{u},n_{\mathrm{u}}} \in \mathbb{R}$, and $n_{\mathrm{u}} \leq n - d$ is the degree of $\beta_{\mathrm{u}}(\mathbf{q})$, and let $\beta_{\mathrm{u},0} = 1$.

III. Brief Review of [1]

In this section, we briefly review select aspects of [1]. First, let $n_{\mathrm{c}} \geq n$, and [1] shows that, for all $k \geq n_{\mathrm{c}}$, (1) has the $(3n_{\mathrm{c}} + 1)^{\mathrm{th}}$-order nonminimal-state-space realization

$$\phi(k+1) = \mathcal{A}\phi(k) + \mathcal{B}u(k) + \mathcal{D}r(k+1), \tag{3}$$
$$y(k) = \mathcal{C}\phi(k), \tag{4}$$

where $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, and $\mathcal{D}$ are given in [1], and

$$\phi(k) \stackrel{\triangle}{=} \begin{bmatrix} y(k-1) & \cdots & y(k-n_{\mathrm{c}}) \\ u(k-1) & \cdots & u(k-n_{\mathrm{c}}) \\ r(k) & \cdots & r(k-n_{\mathrm{c}}) \end{bmatrix}^{\mathrm{T}} \in \mathbb{R}^{3n_{\mathrm{c}}+1}. \tag{5}$$

[1] Assistant Professor, Department of Mechanical Engineering, University of Kentucky, Lexington, KY 40506-0503, email: jhoagg@engr.uky.edu.

[2] Professor, Department of Aerospace Engineering, University of Michigan, Ann Arbor, MI 48109-2140, email: dsbaero@umich.edu.

Next, for all $k \geq n_c$, consider the time-varying controller

$$u(k) = \sum_{i=1}^{n_c} L_i(k) y(k-i) + \sum_{i=1}^{n_c} M_i(k) u(k-i) + \sum_{i=0}^{n_c} N_i(k) r(k-i), \tag{6}$$

where, for all $i = 1, \ldots, n_c$, $L_i : \mathbb{N} \to \mathbb{R}$ and $M_i : \mathbb{N} \to \mathbb{R}$, and, for all $i = 0, 1, \ldots, n_c$, $N_i : \mathbb{N} \to \mathbb{R}$ are given by either the instantaneous RC-MRAC algorithm [1, Lemma 1] or the cumulative RC-MRAC algorithm [1, Lemma 2]. For all $k \geq n_c$, the controller (6) can be expressed as

$$u(k) = \phi^{\mathrm{T}}(k)\theta(k), \tag{7}$$

where $\theta(k) \triangleq [L_1(k) \ \cdots \ L_{n_c}(k) \ M_1(k) \ \cdots \ M_{n_c}(k) \ N_0(k) \ \cdots \ N_{n_c}(k)]^{\mathrm{T}}$.

For all $k \geq 0$, we define the filtered performance $z_f(k) \triangleq \bar{\alpha}_m(\mathbf{q}^{-1})z(k)$, where $\bar{\alpha}_m(\mathbf{q}^{-1}) \triangleq \mathbf{q}^{-n_m}\alpha_m(\mathbf{q})$. Next, for all $k \geq 0$, define the retrospective performance

$$\hat{z}_f(\hat{\theta}, k) \triangleq z_f(k) + \Phi^{\mathrm{T}}(k)\hat{\theta} - \beta_d \bar{\beta}_u(\mathbf{q}^{-1})u(k), \tag{8}$$

where the filtered regressor is defined by $\Phi(k) \triangleq \beta_d \bar{\beta}_u(\mathbf{q}^{-1})\phi(k)$ and $\bar{\beta}_u(\mathbf{q}^{-1}) \triangleq \mathbf{q}^{-n_u-d}\beta_u(\mathbf{q})$. Finally, for all $k \geq 0$, define retrospective performance measure

$$z_{f,r}(k) \triangleq \hat{z}_f(\theta(k), k). \tag{9}$$

IV. Error System

We now construct an error system using the ideal fixed-gain controller (which is given by [1, Theorem 1]) and RC-MRAC. Since n is unknown, the lower bound for the controller order n_c given by [1, Theorem 1] is unknown. Thus, for the remainder of the paper, let n_c satisfy the lower bound

$$n_c \geq \max(2\bar{n} - n_u - d, n_m - n_u - d), \tag{10}$$

where assumptions (A2), (A4), (A5), and (A13) imply that the lower bound on n_c given by (10) is known. Furthermore, since, by assumption (A5), $n \leq \bar{n}$, it follows that (10) satisfies the conditions of [1, Theorem 1].

Let $\theta_* \in \mathbb{R}^{3n_c+1}$ be the ideal fixed-gain controller given by [1, Theorem 1], and, for all $k \geq n_c$, let $\phi_*(k)$ be the state of the ideal closed-loop system, which according to [1] is given by

$$\phi_*(k+1) = \mathcal{A}_*\phi_*(k) + \mathcal{D}r(k+1), \tag{11}$$

$$y_*(k) = \mathcal{C}\phi_*(k), \tag{12}$$

where $\mathcal{A}_* \triangleq \mathcal{A} + \mathcal{B}\theta_*^{\mathrm{T}}$ is asymptotically stable and the initial condition is $\phi_*(n_c) = \phi(n_c)$. Furthermore, define $k_0 = 2n_c + n_u + d$.

Next, for all $k \geq n_c$, the closed-loop system consisting of (3), (4), and (7) becomes

$$\phi(k+1) = \mathcal{A}_*\phi(k) + \mathcal{B}\phi^{\mathrm{T}}(k)\tilde{\theta}(k) + \mathcal{D}r(k+1), \tag{13}$$

$$y(k) = \mathcal{C}\phi(k), \tag{14}$$

where $\tilde{\theta}(k) \triangleq \theta(k) - \theta_*$.

Now, we construct an error system by combining the ideal closed-loop system (11), (12) with the adaptive closed-loop system (13), (14). For all $k \geq n_c$, define the error state $\tilde{\phi}(k) \triangleq \phi(k) - \phi_*(k)$, and subtract (11), (12) from (13), (14) to obtain, for all $k \geq n_c$,

$$\tilde{\phi}(k+1) = \mathcal{A}_*\tilde{\phi}(k) + \mathcal{B}\phi^{\mathrm{T}}(k)\tilde{\theta}(k), \tag{15}$$

$$\tilde{y}(k) = \mathcal{C}\tilde{\phi}(k), \tag{16}$$

where $\tilde{y}(k) \triangleq y(k) - y_*(k)$.

Lemma 1. *Consider the open-loop system (1) with the feedback (7). Then, for all initial conditions x_0, all sequences $\theta(k)$, and, all $k \geq k_0$,*

$$z_f(k) = \beta_d \bar{\beta}_u(\mathbf{q}^{-1}) \left[\phi^{\mathrm{T}}(k)\tilde{\theta}(k)\right]. \tag{17}$$

Proof. For all $k \geq n_c$, the error system (15), (16) has the solution

$$\tilde{y}(k) = \mathcal{C}\mathcal{A}_*^{k-n_c}\tilde{\phi}(n_c) + \sum_{i=1}^{k-n_c} \mathcal{C}\mathcal{A}_*^{i-1}\mathcal{B}\phi^{\mathrm{T}}(k-i)\tilde{\theta}(k-i).$$

Since $\phi_*(0) = \phi(0)$ it follows that $\tilde{\phi}(0) = 0$, and thus, for all $k \geq n_c$, $\tilde{y}(k) = \sum_{i=1}^{k-n_c} \mathcal{C}\mathcal{A}_*^{i-1}\mathcal{B}\phi^{\mathrm{T}}(k-i)\tilde{\theta}(k-i)$, which implies that, for all $k \geq n_c + n_m$

$$\bar{\alpha}_m(\mathbf{q}^{-1})\tilde{y}(k) = \bar{\alpha}_m(\mathbf{q}^{-1}) \left[\sum_{i=1}^{k-n_c} \mathcal{C}\mathcal{A}_*^{i-1}\mathcal{B}\phi^{\mathrm{T}}(k-i)\tilde{\theta}(k-i)\right].$$

Next, it follows from [1, *(iv)* of Theorem 1] (with $e(k) = \phi^{\mathrm{T}}(k)\tilde{\theta}(k)$) that, for all $k \geq k_0$, $\bar{\alpha}_m(\mathbf{q}^{-1})\tilde{y}(k) = \beta_d\bar{\beta}_u(\mathbf{q}^{-1})[\phi^{\mathrm{T}}(k)\tilde{\theta}(k)]$. Finally, note that $\bar{\alpha}_m(\mathbf{q}^{-1})\tilde{y}(k) = \bar{\alpha}_m(\mathbf{q}^{-1})y(k) - \bar{\alpha}_m(\mathbf{q}^{-1})y_*(k)$ and it follows from [1, *(i)* of Theorem 1] that $\bar{\alpha}_m(\mathbf{q}^{-1})y_*(k) = \bar{\alpha}_m(\mathbf{q}^{-1})y_m(k)$. Therefore, for all $k \geq k_0$, $z_f(k) = \bar{\alpha}_m(\mathbf{q}^{-1})\tilde{y}(k)$, thus verifying (17). □

Lemma 1 relates the filtered performance $z_f(k)$ to the estimation error $\tilde{\theta}(k)$. The relationship (17) is not a linear regression in the estimation error $\tilde{\theta}(k)$; however, the following result expresses $z_{f,r}(k)$ as a linear regression in $\tilde{\theta}(k)$.

Lemma 2. *Consider the open-loop system (1) with the feedback (7). Then, for all initial conditions x_0, all sequences $\theta(k)$, and all $k \geq k_0$,*

$$z_{f,r}(k) = \Phi^{\mathrm{T}}(k)\tilde{\theta}(k). \tag{18}$$

Proof. It follows from (8) and (9) that, for all $k \geq 0$,

$$z_{f,r}(k) = z_f(k) - \beta_d\bar{\beta}_u(\mathbf{q}^{-1})\left[\phi^{\mathrm{T}}(k)\theta(k)\right] + \beta_d\left[\bar{\beta}_u(\mathbf{q}^{-1})\phi(k)\right]^{\mathrm{T}}\theta(k).$$

Next, adding and subtracting $\beta_d\left[\bar{\beta}_u(\mathbf{q}^{-1})\phi(k)\right]^{\mathrm{T}}\theta_*$ to the left-hand side yields, for all $k \geq 0$, $z_{f,r}(k) = z_f(k) - \beta_d\bar{\beta}_u(\mathbf{q}^{-1})[\phi^{\mathrm{T}}(k)\tilde{\theta}(k)] + \beta_d[\bar{\beta}_u(\mathbf{q}^{-1})\phi(k)]^{\mathrm{T}}\tilde{\theta}(k)$. Finally, it follows from Lemma 1 that, for all $k \geq k_0$, $z_f(k) - \beta_d\bar{\beta}_u(\mathbf{q}^{-1})[\phi^{\mathrm{T}}(k)\tilde{\theta}(k)] = 0$, which implies that, for all

$k \geq k_0$, $z_{\mathrm{f,r}}(k) = \beta_d[\bar{\beta}_{\mathrm{u}}(\mathbf{q}^{-1})\phi(k)]^{\mathrm{T}}\tilde{\theta}(k) = \Phi^{\mathrm{T}}(k)\tilde{\theta}(k)$, thus verifying (18). □

Lastly, we develop a filtered error system. For all $k \geq k_0$, we define the ideal filtered regressor $\Phi_*(k) \triangleq \beta_d\bar{\beta}_{\mathrm{u}}(\mathbf{q}^{-1})\phi_*(k)$, and the filtered regressor error $\tilde{\Phi}(k) \triangleq \Phi(k) - \Phi_*(k) = \beta_d\bar{\beta}_{\mathrm{u}}(\mathbf{q}^{-1})\tilde{\phi}(k)$. Next, we apply the operator $\beta_d\bar{\beta}_{\mathrm{u}}(\mathbf{q}^{-1})$ to (15) and use Lemma 1 to obtain the filtered error system

$$\begin{aligned}\tilde{\Phi}(k+1) &= \mathcal{A}_*\tilde{\Phi}(k) + \mathcal{B}\beta_d\bar{\beta}_{\mathrm{u}}(\mathbf{q}^{-1})\left[\phi^{\mathrm{T}}(k)\tilde{\theta}(k)\right]\\ &= \mathcal{A}_*\tilde{\Phi}(k) + \mathcal{B}z_{\mathrm{f}}(k),\end{aligned} \tag{19}$$

which is defined for all $k \geq k_0$.

V. Stability Analysis for Instantaneous RC-MRAC

In this section, we analyze the stability of instantaneous RC-MRAC. For review, instantaneous RC-MRAC (developed in [1, Lemma 1]) is given by (7) and

$$\theta(k+1) = \theta(k) - \eta(k)R^{-1}\Phi(k)z_{\mathrm{f,r}}(k), \tag{20}$$

where

$$\eta(k) \triangleq \frac{1}{\zeta(k) + \Phi^{\mathrm{T}}(k)R^{-1}\Phi(k)}, \tag{21}$$

and $R \in \mathbb{R}^{(3n_{\mathrm{c}}+1)\times(3n_{\mathrm{c}}+1)}$ is positive definite, $\theta(0) \in \mathbb{R}^{3n_{\mathrm{c}}+1}$, and $\zeta : \mathbb{N} \to (0,\infty)$. We assume that $\zeta_{\mathrm{L}} \triangleq \inf_{k\geq 0}\zeta(k) > 0$ and $\zeta_{\mathrm{U}} \triangleq \sup_{k\geq 0}\zeta(k) < \infty$.

Lemma 3. *Consider the open-loop system (1) satisfying assumptions (A1)-(A13), and the instantaneous retrospective cost model reference adaptive controller (7), (20), and (21), where n_{c} satisfies (10). Then, for all initial conditions x_0 and $\theta(0)$, the following properties hold:*

(i) $\theta(k)$ *is bounded.*
(ii) $\lim_{k\to\infty}\sum_{j=0}^{k}\eta(j)z_{\mathrm{f,r}}^2(j)$ *exists.*
(iii) *For all* $N > 0$, $\lim_{k\to\infty}\sum_{j=N}^{k}\|\theta(j) - \theta(j-N)\|^2$ *exists.*

Proof. Subtracting θ_* from both sides of (20) yields the estimator-error update equation

$$\tilde{\theta}(k+1) = \tilde{\theta}(k) - \eta(k)R^{-1}\Phi(k)z_{\mathrm{f,r}}(k). \tag{22}$$

Define the positive-definite, radially unbounded Lyapunov-like function $V_{\tilde{\theta}}(\tilde{\theta}(k)) \triangleq \tilde{\theta}^{\mathrm{T}}(k)R\tilde{\theta}(k)$, and the Lyapunov-like difference

$$\Delta V_{\tilde{\theta}}(k) \triangleq V_{\tilde{\theta}}(\tilde{\theta}(k+1)) - V_{\tilde{\theta}}(\tilde{\theta}(k)). \tag{23}$$

Evaluating $\Delta V_{\tilde{\theta}}(k)$ along the trajectories of the estimator-error system (22) yields $\Delta V_{\tilde{\theta}}(k) = -2\eta(k)z_{\mathrm{f,r}}(k)\Phi^{\mathrm{T}}(k)\tilde{\theta}(k) + \eta^2(k)z_{\mathrm{f,r}}^2(k)\Phi^{\mathrm{T}}(k)R^{-1}\Phi(k)$. Next, it follows from Lemma 2 and (21) that, for all $k \geq k_0$,

$$\begin{aligned}\Delta V_{\tilde{\theta}}(k) &= -2\eta(k)z_{\mathrm{f,r}}^2(k) + \eta^2(k)z_{\mathrm{f,r}}^2(k)\Phi^{\mathrm{T}}(k)R^{-1}\Phi(k)\\ &= -\eta(k)z_{\mathrm{f,r}}^2(k) - \zeta(k)\eta^2(k)z_{\mathrm{f,r}}^2(k)\\ &\leq -\eta(k)z_{\mathrm{f,r}}^2(k).\end{aligned} \tag{24}$$

Since $V_{\tilde{\theta}}$ is a positive-definite radially unbounded function of $\tilde{\theta}(k)$ and, for $k \geq k_0$, $\Delta V_{\tilde{\theta}}(k)$ is non-positive, it follows that $\tilde{\theta}(k)$ is bounded and thus $\theta(k)$ is bounded. Thus, we have verified *(i)*.

To show *(ii)*, first we show that $\lim_{k\to\infty}\sum_{j=k_0}^{k}\Delta V_{\tilde{\theta}}(j)$ exists. Since $V_{\tilde{\theta}}$ is positive definite, and, for all $k \geq k_0$, $\Delta V_{\tilde{\theta}}(k)$ is non-positive, it follows from (23) that

$$\begin{aligned}0 \leq -\lim_{k\to\infty}\sum_{j=k_0}^{k}\Delta V_{\tilde{\theta}}(j) &= V_{\tilde{\theta}}(\tilde{\theta}(k_0)) - \lim_{k\to\infty}V_{\tilde{\theta}}(\tilde{\theta}(k))\\ &\leq V_{\tilde{\theta}}(\tilde{\theta}(k_0)),\end{aligned}$$

where the upper and lower bounds imply that both limits exist. Since $\lim_{k\to\infty}\sum_{j=k_0}^{k}\Delta V_{\tilde{\theta}}(j)$ exists, (24) implies that $\lim_{k\to\infty}\sum_{j=k_0}^{k}\eta(j)z_{\mathrm{f,r}}^2(j)$ exists, and thus $\lim_{k\to\infty}\sum_{j=0}^{k}\eta(j)z_{\mathrm{f,r}}^2(j)$ exists, which verifies *(ii)*.

To show *(iii)*, we first show that $\lim_{k\to\infty}\sum_{j=0}^{k}\|\theta(j+1) - \theta(j)\|^2$ exists. It follows from (20) that

$$\begin{aligned}\sum_{j=0}^{\infty}\|\theta(j+1) - \theta(j)\|^2 &= \sum_{j=0}^{\infty}\eta^2(j)z_{\mathrm{f,r}}^2(j)\Phi^{\mathrm{T}}(j)R^{-2}\Phi(j)\\ &\leq \|R^{-1}\|_{\mathrm{F}}\sum_{j=0}^{\infty}\eta^2(j)z_{\mathrm{f,r}}^2(j)\Phi^{\mathrm{T}}(j)R^{-1}\Phi(j),\end{aligned}$$

where $\|\cdot\|_{\mathrm{F}}$ denotes the Frobenius norm. Next, it follows from (21) that, for all $k \geq 0$, $\eta(k)\Phi^{\mathrm{T}}(k)R^{-1}\Phi(k) \leq 1$, which implies that

$$\lim_{k\to\infty}\sum_{j=0}^{k}\|\theta(j+1) - \theta(j)\|^2 \leq \|R^{-1}\|_{\mathrm{F}}\lim_{k\to\infty}\sum_{j=0}^{k}\eta(j)z_{\mathrm{f,r}}^2(j).$$

Furthermore, since by *(ii)*, $\lim_{k\to\infty}\sum_{j=0}^{k}\eta(j)z_{\mathrm{f,r}}^2(j)$ exists, it follows that $\lim_{k\to\infty}\sum_{j=0}^{k}\|\theta(j+1)-\theta(j)\|^2$ exists. Next, let $N > 0$ and note that

$$\begin{aligned}&\lim_{k\to\infty}\sum_{j=N}^{k}\|\theta(j) - \theta(j-N)\|^2\\ &= \lim_{k\to\infty}\sum_{j=N}^{k}\|\theta(j) - \theta(j-1) + \theta(j-1) - \theta(j-2)\\ &\qquad + \cdots + \theta(j-N+1) - \theta(j-N)\|^2\\ &\leq \lim_{k\to\infty}\sum_{j=N}^{k}(\|\theta(j) - \theta(j-1)\|\\ &\qquad + \cdots + \|\theta(j-N+1) - \theta(j-N)\|)^2\\ &\leq \lim_{k\to\infty}2^{N-1}\sum_{j=N}^{k}\left(\|\theta(j) - \theta(j-1)\|^2\right.\\ &\qquad \left.+ \cdots + \|\theta(j-N+1) - \theta(j-N)\|^2\right).\end{aligned} \tag{25}$$

Since all of the limits on the right hand side of (25) exist, it follows that $\lim_{k\to\infty}\sum_{j=N}^{k}\|\theta(j) - \theta(j-N)\|^2$ exists. This verifies *(iii)*. □

Next, let $\xi_1, \ldots, \xi_{n_{\mathrm{u}}} \in \mathbb{C}$ denote the n_{u} roots of $\beta_{\mathrm{u}}(\mathbf{z})$, and define $M(\mathbf{z},k) \triangleq \mathbf{z}^{n_{\mathrm{c}}} - M_1(k)\mathbf{z}^{n_{\mathrm{c}}-1} - M_2(k)\mathbf{z}^{n_{\mathrm{c}}-2} -$

$\cdots - M_{n_c-1}(k)\mathbf{z} - M_{n_c}(k)$, which can be interpreted as the denominator polynomial of the controller (7) at frozen time k. Before presenting the main result of the paper, we make the following additional assumption:

(A14) There exist $\epsilon > 0$ and $k_1 > 0$ such that, for all $k \geq k_1$ and for all $i = 1, \ldots, n_u$, $|M(\xi_i, k)| \geq \epsilon$.

The following theorem is the main result of the paper regarding instantaneous RC-MRAC.

Theorem 1. *Consider the open-loop system (1) satisfying assumptions (A1)-(A14), and the instantaneous retrospective cost model reference adaptive controller (7), (20), and (21), where n_c satisfies (10). Then, for all initial conditions x_0 and $\theta(0)$, θ is bounded, u is bounded, and $\lim_{k\to\infty} z(k) = 0$.*

Proof. It follows from *(i)* of Lemma 3 that $\theta(k)$ is bounded. To prove the remaining properties, define the quadratic function $J(\tilde{\Phi}(k)) \triangleq \tilde{\Phi}^{\mathrm{T}}(k)\mathcal{P}\tilde{\Phi}(k)$, where $\mathcal{P} > 0$ satisfies the discrete-time Lyapunov equation $\mathcal{P} = \mathcal{A}_*^{\mathrm{T}}\mathcal{P}\mathcal{A}_* + \mathcal{Q} + \alpha I$, where $\mathcal{Q} > 0$ and $\alpha > 0$. Note that $\mathcal{P}$ exists since $\mathcal{A}_*$ is asymptotically stable. Defining $\Delta J(k) \triangleq J(\tilde{\Phi}(k+1)) - J(\tilde{\Phi}(k))$, it follows from (19) that, for all $k \geq k_0$,

$$\begin{aligned}\Delta J(k) = & -\tilde{\Phi}^{\mathrm{T}}(k)\left(\mathcal{Q} + \alpha I\right)\tilde{\Phi}(k) + \tilde{\Phi}^{\mathrm{T}}(k)\mathcal{A}_*^{\mathrm{T}}\mathcal{P}\mathcal{B}z_{\mathrm{f}}(k) \\ & + z_{\mathrm{f}}(k)\mathcal{B}^{\mathrm{T}}\mathcal{P}\tilde{\mathcal{A}}^*\tilde{\Phi}(k) + z_{\mathrm{f}}^2(k)\mathcal{B}^{\mathrm{T}}\mathcal{P}\mathcal{B} \\ \leq & -\tilde{\Phi}^{\mathrm{T}}(k)\left(\mathcal{Q} + \alpha I\right)\tilde{\Phi}(k) + z_{\mathrm{f}}^2(k)\mathcal{B}^{\mathrm{T}}\mathcal{P}\mathcal{B} \\ & + \alpha\tilde{\Phi}^{\mathrm{T}}(k)\tilde{\Phi}(k) + \frac{1}{\alpha}z_{\mathrm{f}}^2(k)\mathcal{B}^{\mathrm{T}}\mathcal{P}\mathcal{A}_*\mathcal{A}_*^{\mathrm{T}}\mathcal{P}\mathcal{B} \\ = & -\tilde{\Phi}^{\mathrm{T}}(k)\mathcal{Q}\tilde{\Phi}(k) + \sigma_1 z_{\mathrm{f}}^2(k), \end{aligned} \tag{26}$$

where $\sigma_1 \triangleq \mathcal{B}^{\mathrm{T}}\mathcal{P}\mathcal{B} + \frac{1}{\alpha}\mathcal{B}^{\mathrm{T}}\mathcal{P}\mathcal{A}_*\mathcal{A}_*^{\mathrm{T}}\mathcal{P}\mathcal{B}$.

Now, consider the positive-definite, radially unbounded Lyapunov-like function $V(\tilde{\Phi}(k)) \triangleq \ln\left(1 + a_1 J(\tilde{\Phi}(k))\right)$, where $a_1 > 0$ is specified below. The Lyapunov-like difference is thus given by $\Delta V(k) \triangleq V(\tilde{\Phi}(k+1)) - V(\tilde{\Phi}(k))$. For all $k \geq k_0$, evaluating $\Delta V(k)$ along the trajectories of (19) yields $\Delta V(k) = \ln\left(1 + \frac{a_1\Delta J(k)}{1 + a_1 J(\tilde{\Phi}(k))}\right)$. Since, for all $x > 0$, $\ln x \leq x - 1$, and using (26) we have

$$\Delta V(k) \leq a_1\frac{\Delta J(k)}{1 + a_1 J(\tilde{\Phi}(k))} \leq -W(\tilde{\Phi}(k)) + a_1\sigma_1\ell^2(k), \tag{27}$$

where

$$W(\tilde{\Phi}(k)) \triangleq a_1\frac{\tilde{\Phi}^{\mathrm{T}}(k)\mathcal{Q}\tilde{\Phi}(k)}{1 + a_1\tilde{\Phi}^{\mathrm{T}}(k)\mathcal{P}\tilde{\Phi}(k)}, \tag{28}$$

$$\ell(k) \triangleq \frac{z_{\mathrm{f}}(k)}{\sqrt{1 + a_1\lambda_{\min}\left(\mathcal{P}\right)\tilde{\Phi}^{\mathrm{T}}(k)\tilde{\Phi}(k)}}. \tag{29}$$

Now, we show that $\lim_{k\to\infty}\sum_{j=0}^{k}\ell^2(j)$ exists. First, it follows from Lemma 1 and Lemma 2 that, for all $k \geq k_0$,

$$z_{\mathrm{f}}(k) = z_{\mathrm{f,r}}(k) - \beta_d\sum_{i=d}^{n_u+d}\beta_{\mathrm{u},i-d}\phi^{\mathrm{T}}(k-i)\left[\theta(k) - \theta(k-i)\right].$$

Therefore, it follows from (29) that, for all $k \geq k_0$,

$$|\ell(k)| \leq \frac{|z_{\mathrm{f,r}}(k)|}{\sqrt{1 + a_1\lambda_{\min}(\mathcal{P})\tilde{\Phi}^{\mathrm{T}}(k)\tilde{\Phi}(k)}} + \ell_2(k), \tag{30}$$

where

$$\ell_2(k) \triangleq \frac{|\beta_d|\sum_{i=d}^{n_u+d}|\beta_{\mathrm{u},i-d}|\|\phi(k-i)\|\|\theta(k) - \theta(k-i)\|}{\sqrt{1 + a_1\lambda_{\min}(\mathcal{P})\tilde{\Phi}^{\mathrm{T}}(k)\tilde{\Phi}(k)}}.$$

It follows from Lemma 3 that $\theta(k)$ is bounded and $\lim_{k\to\infty}\|\theta(k) - \theta(k-1)\| = 0$. Therefore, Lemma 5 implies that there exist $k_2 \geq k_0 > 0$, $c_1 > 0$, and $c_2 > 0$, such that, for all $k \geq k_2$ and all $i = d, \ldots, n_u + d$, $\|\phi(k-i)\| \leq c_1 + c_2\|\Phi(k)\|$. In addition, note that $\|\Phi(k)\| = \|\tilde{\Phi}(k) + \Phi_*(k)\| \leq \|\tilde{\Phi}(k)\| + \|\Phi_*(k)\| \leq \|\tilde{\Phi}(k)\| + \Phi_{*,\max}$, where $\Phi_{*,\max} \triangleq \sup_{k\geq 0}\|\Phi_*(k)\|$ exists because Φ_* is bounded. Therefore, for all $k \geq k_2$, $\|\phi(k-i)\| \leq c_1 + c_2\Phi_{*,\max} + c_2\|\tilde{\Phi}(k)\|$, which implies that

$$\ell_2(k) \leq \frac{\left(c_3 + c_4\|\tilde{\Phi}(k)\|\right)\left(\sum_{i=d}^{n_u+d}\|\theta(k) - \theta(k-i)\|\right)}{\sqrt{1 + a_1\lambda_{\min}(\mathcal{P})\tilde{\Phi}^{\mathrm{T}}(k)\tilde{\Phi}(k)}}, \tag{31}$$

where $c_3 \triangleq (c_1 + c_2\Phi_{*,\max})|\beta_d|(\max_{d\leq i\leq n_u+d}|\beta_{\mathrm{u},i-d}|) > 0$ and $c_4 \triangleq c_2|\beta_d|(\max_{d\leq i\leq n_u+d}|\beta_{\mathrm{u},i-d}|) > 0$. Next, note that $\frac{1}{\sqrt{1+a_1\lambda_{\min}(\mathcal{P})\tilde{\Phi}^{\mathrm{T}}(k)\tilde{\Phi}(k)}} \leq 1$ and $\frac{\|\tilde{\Phi}(k)\|}{\sqrt{1+a_1\lambda_{\min}(\mathcal{P})\tilde{\Phi}^{\mathrm{T}}(k)\tilde{\Phi}(k)}} \leq \max\left(1, 1/\sqrt{a_1\lambda_{\min}(\mathcal{P})}\right)$, which implies that $\ell_2(k) \leq c_5\sum_{i=d}^{n_u+d}\|\theta(k) - \theta(k-i)\|$, where $c_5 \triangleq c_3 + c_4\max\left(1, 1/\sqrt{a_1\lambda_{\min}(\mathcal{P})}\right) > 0$. Thus, (30) becomes

$$\begin{aligned}|\ell(k)| \leq & \frac{|z_{\mathrm{f,r}}(k)|}{\sqrt{1 + a_1\lambda_{\min}(\mathcal{P})\tilde{\Phi}^{\mathrm{T}}(k)\tilde{\Phi}(k)}} \\ & + c_5\sum_{i=d}^{n_u+d}\|\theta(k) - \theta(k-i)\|. \end{aligned} \tag{32}$$

Next, we show that we can choose $a_1 > 0$ such that the first term of (32) is less than a constant times $\sqrt{\eta(k)}|z_{\mathrm{f,r}}(k)|$, which is square summable according to *(ii)* of Lemma 3. Note that $\Phi^{\mathrm{T}}(k)\Phi(k) \leq 2\tilde{\Phi}^{\mathrm{T}}(k)\tilde{\Phi}(k) + 2\Phi_*^{\mathrm{T}}(k)\Phi_*(k)$. Therefore, it follows from (21) that

$$\begin{aligned}\frac{1}{\eta(k)} &= \zeta(k) + \Phi^{\mathrm{T}}(k)R^{-1}\Phi(k) \\ &\leq \zeta_{\mathrm{U}} + \lambda_{\max}(R^{-1})\left[2\tilde{\Phi}^{\mathrm{T}}(k)\tilde{\Phi}(k) + 2\Phi_*^{\mathrm{T}}(k)\Phi_*(k)\right] \\ &\leq \zeta_{\mathrm{U}} + 2\lambda_{\max}(R^{-1})\Phi_{*,\max}^2 + 2\lambda_{\max}(R^{-1})\tilde{\Phi}^{\mathrm{T}}(k)\tilde{\Phi}(k) \\ &= c_6\left[1 + a_1\lambda_{\min}(\mathcal{P})\tilde{\Phi}^{\mathrm{T}}(k)\tilde{\Phi}(k)\right], \end{aligned}$$

where $a_1 \triangleq \frac{2\lambda_{\max}(R^{-1})}{c_6\lambda_{\min}(\mathcal{P})} > 0$ and $c_6 \triangleq \zeta_{\mathrm{U}} + 2\lambda_{\max}(R^{-1})\Phi_{*,\max}^2 > 0$. Therefore,

$$\frac{1}{\sqrt{1 + a_1\lambda_{\min}(\mathcal{P})\tilde{\Phi}^{\mathrm{T}}(k)\tilde{\Phi}(k)}} \leq \sqrt{c_6}\sqrt{\eta(k)},$$

which combining with (32) implies that, for all $k \geq k_2$,

$$|\ell(k)| \leq \sqrt{c_6}\sqrt{\eta(k)}|z_{\mathrm{f,r}}(k)| + c_5 \sum_{i=d}^{n_\mathrm{u}+d} \|\theta(k) - \theta(k-i)\|.$$

Therefore, for all $k \geq k_2$,

$$\begin{aligned}\ell^2(k) &\leq \left[\sqrt{c_6}\sqrt{\eta(k)}|z_{\mathrm{f,r}}(k)| + c_5 \sum_{i=d}^{n_\mathrm{u}+d} \|\theta(k) - \theta(k-i)\|\right]^2 \\ &\leq 2c_6\eta(k)z_{\mathrm{f,r}}^2(k) + 2c_5^2\left[\sum_{i=d}^{n_\mathrm{u}+d} \|\theta(k) - \theta(k-i)\|\right]^2 \\ &\leq 2c_6\eta(k)z_{\mathrm{f,r}}^2(k) + 2^{n_\mathrm{u}+1}c_5^2 \sum_{i=d}^{n_\mathrm{u}+d} \|\theta(k) - \theta(k-i)\|^2. \end{aligned} \tag{33}$$

It follows from *(ii)* of Lemma 3 that $\lim_{k\to\infty}\sum_{j=0}^{k}\eta(j)z_{\mathrm{f,r}}^2(j)$ exists. Furthermore, it follows from *(iii)* of Lemma 3 that, for all $i = d, \ldots, n_\mathrm{u} + d$, $\lim_{k\to\infty}\sum_{j=0}^{k}\|\theta(j) - \theta(j-i)\|^2$ exists. Thus, (33) implies that $\lim_{k\to\infty}\sum_{j=0}^{k}\ell^2(j)$ exists.

Now, we show that $\lim_{k\to\infty} W(\tilde{\Phi}(k)) = 0$. Since W and V are positive definite, it follows from (27) that

$$\begin{aligned}0 \leq \sum_{j=0}^{\infty} W(\tilde{\Phi}(j)) &\leq \sum_{j=0}^{\infty} -\Delta V(j) + a_1\sigma_1 \sum_{j=0}^{\infty}\ell^2(j) \\ &= V(\tilde{\Phi}(0)) - \lim_{k\to\infty} V(\tilde{\Phi}(k)) + a_1\sigma_1\sum_{j=0}^{\infty}\ell^2(j) \\ &\leq V(\tilde{\Phi}(0)) + a_1\sigma_1 \lim_{k\to\infty}\sum_{j=0}^{k}\ell^2(j),\end{aligned}$$

where the upper and lower bound imply that all limits exist. Thus, $\lim_{k\to\infty} W(\tilde{\Phi}(k)) = 0$, which implies that $\lim_{k\to\infty}\|\tilde{\Phi}(k)\| = 0$.

To prove that $u(k)$ is bounded, first note that since $\lim_{k\to\infty}\|\tilde{\Phi}(k)\| = 0$ and $\Phi_*(k)$ is bounded, it follows that $\Phi(k)$ is bounded. Next, since $\Phi(k)$ is bounded, it follows from Lemma 5 that $\phi(k)$ is bounded. Furthermore, since $y(k)$ and $u(k)$ are components of $\phi(k+1)$, it follows that $y(k)$ and $u(k)$ are bounded.

To prove that $\lim_{k\to\infty} z(k) = 0$, note that it follows from (19) and the fact that $\|\mathcal{B}z_\mathrm{f}(k)\| = |z_\mathrm{f}(k)|$ that

$$\lim_{k\to\infty}|z_\mathrm{f}(k)| \leq \lim_{k\to\infty}\|\tilde{\Phi}(k+1)\| + \|\mathcal{A}_*\|_\mathrm{F}\lim_{k\to\infty}\|\tilde{\Phi}(k)\| = 0.$$

Since $\lim_{k\to\infty} z_\mathrm{f}(k) = 0$, $z_\mathrm{f}(k) = \bar{\alpha}_\mathrm{m}(\mathbf{q}^{-1})z(k)$, and $\alpha_\mathrm{m}(\mathbf{q}) = \mathbf{q}^{n_\mathrm{m}}\bar{\alpha}_\mathrm{m}(\mathbf{q}^{-1})$ is an asymptotically stable polynomial, it follows that $\lim_{k\to\infty} z(k) = 0$. □

Theorem 1 invokes assumption (A14), which asymptotically bounds the frozen time controller poles (i.e., the roots of $M(\mathbf{z}, k)$) away from the nonminimum-phase zeros of (1), and thus, asymptotically prevents unstable pole-zero cancellation between the plant zeros and the controller poles.

The assumption $|M(\xi_i, k)| \geq \epsilon$ for some arbitrarily small $\epsilon > 0$ can be verified at each time step since $M(\xi_i, k)$ can be computed from known values (i.e., the roots of $\beta_\mathrm{u}(\mathbf{q})$ and the controller parameter $\theta(k)$). In fact, if, for some arbitrarily small $\epsilon > 0$, the condition $|M(\xi_i, k)| \geq \epsilon$ is violated at a particular time step, then the controller parameter $\theta(k)$ can be perturbed to ensure $|M(\xi_i, k)| \geq \epsilon$. However, the stability of such a perturbation is an open problem.

VI. Stability Analysis for Cumulative RC-MRAC

In this section, we present the analogous results to Lemma 3 and Theorem 1 for the cumulative RC-MRAC. For review, cumulative RC-MRAC (developed in [1, Lemma 2]) is given by (7) and

$$\theta(k+1) = \theta(k) - \frac{P(k)\Phi(k)z_{\mathrm{f,r}}(k)}{\lambda + \Phi^\mathrm{T}(k)P(k)\Phi(k)}, \tag{34}$$

where

$$P(k+1) = \frac{1}{\lambda}\left[P(k) - \frac{P(k)\Phi(k)\Phi^\mathrm{T}(k)P(k)}{\lambda + \Phi^\mathrm{T}(k)P(k)\Phi(k)}\right]. \tag{35}$$

and $P(0) \in \mathbb{R}^{(3n_\mathrm{c}+1)\times(3n_\mathrm{c}+1)}$ is positive definite and $\theta(0) \in \mathbb{R}^{3n_\mathrm{c}+1}$.

Lemma 4. *Consider the open-loop system (1) satisfying assumptions (A1)-(A13), and the cumulative retrospective cost model reference adaptive controller (7), (34), and (35), where n_c satisfies (10). Furthermore, define*

$$\eta_\mathrm{C}(k) \triangleq \frac{1}{1 + \Phi^\mathrm{T}(k)P(0)\Phi(k)}. \tag{36}$$

Then, for all initial conditions x_0 and $\theta(0)$, the following properties hold:

(i) *$\theta(k)$ is bounded.*
(ii) *$\lim_{k\to\infty}\sum_{j=0}^{k}\eta_\mathrm{C}(j)z_{\mathrm{f,r}}^2(j)$ exists.*
(iii) *For all $N > 0$, $\lim_{k\to\infty}\sum_{j=N}^{k}\|\theta(j) - \theta(j-N)\|^2$ exists.*

Proof. Subtracting θ_* from both sides of (34) yields the estimator-error update equation

$$\tilde{\theta}(k+1) = \tilde{\theta}(k) - \frac{P(k)\Phi(k)z_{\mathrm{f,r}}(k)}{\lambda + \Phi^\mathrm{T}(k)P(k)\Phi(k)}. \tag{37}$$

Next, note from (35) that

$$\begin{aligned}P(k+1)\Phi(k) &= \frac{1}{\lambda}\left[P(k) - \frac{P(k)\Phi(k)\Phi^\mathrm{T}(k)P(k)}{\lambda + \Phi^\mathrm{T}(k)P(k)\Phi(k)}\right]\Phi(k) \\ &= \frac{P(k)\Phi(k)}{\lambda + \Phi^\mathrm{T}(k)P(k)\Phi(k)},\end{aligned} \tag{38}$$

and thus,

$$\tilde{\theta}(k+1) = \tilde{\theta}(k) - P(k+1)\Phi(k)z_{\mathrm{f,r}}(k). \tag{39}$$

Furthermore, note the RLS identity [2]

$$P^{-1}(k+1) = \lambda P^{-1}(k) + \Phi(k)\Phi^\mathrm{T}(k). \tag{40}$$

Define $V_P(P(k), k) \triangleq \lambda^{-k}P^{-1}(k)$, and $\Delta V_P(k) \triangleq V_P(P(k+1), k+1) - V_P(P(k), k)$. Evaluating $\Delta V_P(k)$ along the trajectories of (40) yields

$$\Delta V_P(k) = \lambda^{-k-1}\Phi(k)\Phi^\mathrm{T}(k). \tag{41}$$

Since $P(0)$ is positive definite and ΔV_P is positive semidefinite, it follows that, for all $k \geq 0$, $V_P(P(k), k)$ is positive definite and $V_P(P(k), k) \geq V_P(P(k-1), k-1)$. Therefore, for all $k \geq 0$, $V_P(P(0), 0) \leq V_P(P(k), k)$, which implies that $\lambda^k P(k) \leq P(0)$.

Next, define the positive-definite Lyapunov-like function $V_{\tilde{\theta}}(\tilde{\theta}(k), P(k), k) \stackrel{\triangle}{=} \tilde{\theta}^{\mathrm{T}}(k) V_P(P(k), k)\tilde{\theta}(k)$, and define the Lyapunov-like difference

$$\Delta V_{\tilde{\theta}}(k) \stackrel{\triangle}{=} V_{\tilde{\theta}}(\tilde{\theta}(k+1), P(k+1), k+1) - V_{\tilde{\theta}}(\tilde{\theta}(k), P(k), k). \tag{42}$$

Evaluating $\Delta V_{\tilde{\theta}}(k)$ along the trajectories of the estimator-error system (39) and using (41) yields

$$\begin{aligned}\Delta V_{\tilde{\theta}}(k) &= \lambda^{-k-1}\left[\tilde{\theta}(k) - P(k+1)\Phi(k) z_{\mathrm{f,r}}(k)\right]^{\mathrm{T}} \\ &\quad \times P^{-1}(k+1)\left[\tilde{\theta}(k) - P(k+1)\Phi(k) z_{\mathrm{f,r}}(k)\right] \\ &\quad - \lambda^{-k}\tilde{\theta}^{\mathrm{T}}(k) P^{-1}(k)\tilde{\theta}(k) \\ &= \tilde{\theta}^{\mathrm{T}}(k)\Delta V_P(k)\tilde{\theta}(k) - 2\lambda^{-k-1} z_{\mathrm{f,r}}(k)\Phi^{\mathrm{T}}(k)\tilde{\theta}(k) \\ &\quad + \lambda^{-k-1} z_{\mathrm{f,r}}^2(k)\Phi^{\mathrm{T}}(k) P(k+1)\Phi(k) \\ &= \lambda^{-k-1}\left[\tilde{\theta}^{\mathrm{T}}(k)\Phi(k)\Phi^{\mathrm{T}}(k)\tilde{\theta}(k) - 2 z_{\mathrm{f,r}}(k)\Phi^{\mathrm{T}}(k)\tilde{\theta}(k)\right. \\ &\quad \left. + z_{\mathrm{f,r}}^2(k)\Phi^{\mathrm{T}}(k) P(k+1)\Phi(k)\right].\end{aligned}$$

Next, it follows from Lemma 2 and (38) that, for all $k \geq k_0$,

$$\begin{aligned}\Delta V_{\tilde{\theta}}(k) &= -\lambda^{-k-1} z_{\mathrm{f,r}}^2(k)\left(1 - \Phi^{\mathrm{T}}(k) P(k+1)\Phi(k)\right) \\ &= -\lambda^{-k-1} z_{\mathrm{f,r}}^2(k)\frac{\lambda}{\lambda + \Phi^{\mathrm{T}}(k) P(k)\Phi(k)} \\ &= -\bar{\eta}_{\mathrm{C}}(k) z_{\mathrm{f,r}}^2(k),\end{aligned} \tag{43}$$

where $\bar{\eta}_{\mathrm{C}}(k) \stackrel{\triangle}{=} \frac{1}{\lambda^{k+1} + \lambda^k \Phi^{\mathrm{T}}(k) P(k)\Phi(k)}$. Since $V_{\tilde{\theta}}$ is a positive-definite radially unbounded function of $\tilde{\theta}(k)$ and, for $k \geq k_0$, $\Delta V_{\tilde{\theta}}(k)$ is non-positive, it follows that $\tilde{\theta}(k)$ and thus $\theta(k)$ is bounded. Thus, we have verified *(i)*.

To show *(ii)*, first we show that $\lim_{k\to\infty}\sum_{j=k_0}^{k}\Delta V_{\tilde{\theta}}(j)$ exists. Since $V_{\tilde{\theta}}$ is positive definite, and, for all $k \geq k_0$, $\Delta V_{\tilde{\theta}}(k)$ is non-positive, it follows from (42) that

$$0 \leq -\lim_{k\to\infty}\sum_{j=k_0}^{k}\Delta V_{\tilde{\theta}}(j) \leq V_{\tilde{\theta}}(\tilde{\theta}(k_0), P(k_0), k_0),$$

where the upper and lower bounds imply that both limits exist. Since $\lim_{k\to\infty}\sum_{j=k_0}^{k}\Delta V_{\tilde{\theta}}(j)$ exists, (43) implies that $\lim_{k\to\infty}\sum_{j=k_0}^{k}\bar{\eta}_{\mathrm{C}}(j) z_{\mathrm{f,r}}^2(j)$ exists, and thus $\lim_{k\to\infty}\sum_{j=0}^{k}\bar{\eta}_{\mathrm{C}}(j) z_{\mathrm{f,r}}^2(j)$ exists. Since, for all $k \geq 0$, $\lambda^{k+1} \leq 1$ and $\lambda^k P(k) \leq P(0)$, it follows from (36) that, for all $k \geq 0$, $\eta_{\mathrm{C}}(k) \leq \bar{\eta}_{\mathrm{C}}(k)$, which implies that $\lim_{k\to\infty}\sum_{j=0}^{k}\eta_{\mathrm{C}}(j) z_{\mathrm{f,r}}^2(j) \leq \lim_{k\to\infty}\sum_{j=0}^{k}\bar{\eta}_{\mathrm{C}}(j) z_{\mathrm{f,r}}^2(j)$. Thus, $\lim_{k\to\infty}\sum_{j=0}^{k}\eta_{\mathrm{C}}(j) z_{\mathrm{f,r}}^2(j)$ exists, which verifies *(ii)*.

To show *(iii)*, we first show that $\lim_{k\to\infty}\sum_{j=0}^{k}\|\theta(j+1) - \theta(j)\|^2$ exists. Since $\lambda^k P(k) \leq P(0)$, (37) implies that

$$\begin{aligned}&\lim_{k\to\infty}\sum_{j=0}^{k}\|\theta(j+1) - \theta(j)\|^2 \\ &= \sum_{j=0}^{\infty}\bar{\eta}_{\mathrm{C}}(j) z_{\mathrm{f,r}}^2(j)\left(\frac{\lambda^j\Phi^{\mathrm{T}}(j) P^2(j)\Phi(j)}{\lambda + \Phi^{\mathrm{T}}(j) P(j)\Phi(j)}\right) \\ &\leq \sum_{j=0}^{\infty}\bar{\eta}_{\mathrm{C}}(j) z_{\mathrm{f,r}}^2(j)\|\lambda^j P(j)\|_F\left(\frac{\Phi^{\mathrm{T}}(j) P(j)\Phi(j)}{\lambda + \Phi^{\mathrm{T}}(j) P(j)\Phi(j)}\right) \\ &\leq \|P(0)\|_F\sum_{j=0}^{\infty}\bar{\eta}_{\mathrm{C}}(j) z_{\mathrm{f,r}}^2(j).\end{aligned}$$

Since $\lim_{k\to\infty}\sum_{j=0}^{k}\bar{\eta}_{\mathrm{C}}(j) z_{\mathrm{f,r}}^2(j)$ exists, it follows that $\lim_{k\to\infty}\sum_{j=0}^{k}\|\theta(j+1) - \theta(j)\|^2$ exists. The remainder of the proof is identical to the proof of *(iii)* in Lemma 3. □

The following theorem is the main result of the paper regarding cumulative RC-MRAC.

Theorem 2. *Consider the open-loop system (1) satisfying assumptions (A1)-(A14), and the cumulative retrospective cost model reference adaptive controller (7), (34), and (35), where n_{c} satisfies (10). Then, for all initial conditions x_0 and $\theta(0)$, θ is bounded, u is bounded, and $\lim_{k\to\infty} z(k) = 0$.*

The proof of Theorem 2 is identical to the proof of Theorem 1 with $\eta(k)$ replaced by $\eta_{\mathrm{C}}(k)$ and $a_1 \stackrel{\triangle}{=} \frac{2\lambda_{\max}(P(0))}{\lambda_{\min}(\mathcal{P})\left[1 + 2\lambda_{\max}(P(0))\Phi_{*,\max}^2\right]} > 0$.

VII. Conclusions

This paper, in conjunction with its companion paper [1], presented a direct MRAC algorithm for discrete-time (including sampled-data) systems that are possibly nonminimum phase, provided that nonminimum-phase zeros are known. We provided the construction and stability analysis of the RC-MRAC algorithm.

Appendix A:

This appendix presents a lemma that is used in the proofs of Theorem 1 and Theorem 2. The proof has been omitted due to space considerations.

Lemma 5. *Consider the open-loop system (1) satisfying assumptions (A1)-(A13). In addition, consider a feedback controller (7) that satisfies the following assumptions:*

(i) *$\theta(k)$ is bounded.*
(ii) *$\lim_{k\to\infty}\|\theta(k) - \theta(k-1)\| = 0$.*
(iii) *There exist $\epsilon > 0$ and $k_1 > 0$ such that, for all $k \geq k_1$ and for all $i = 1, \ldots, n_{\mathrm{u}}$, $|M(\xi_i, k)| \geq \epsilon$.*

Then, for all initial conditions x_0 and $\theta(0)$, there exist $k_2 > 0$, $c_1 > 0$, and $c_2 > 0$, such that, for all $k \geq k_2$, and, for all $N = 0, \ldots, n_{\mathrm{u}}$, $\|\phi(k - d - N)\| \leq c_1 + c_2\|\Phi(k)\|$.

References

[1] J. B. Hoagg and D. S. Bernstein, "Retrospective cost model reference adaptive control for nonminimum-phase discrete-time systems, Part 1," in *Proc. Amer. Contr. Conf.*, San Francisco, CA, June 2011.

[2] G. C. Goodwin and K. S. Sin, *Adaptive Filtering, Prediction, and Control.* Prentice Hall, 1984.