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Retrospective Cost Model Reference Adaptive Control for Nonminimum-Phase Discrete-Time Systems, Part 1: The Adaptive Controller

Jesse B. Hoagg[1] and Dennis S. Bernstein[2]

***Abstract*— We present a direct model reference adaptive controller for discrete-time systems (and thus sampled-data systems) that are possibly nonminimum phase. The adaptive control algorithm requires knowledge of the nonminimum-phase zeros of the transfer function from the control to the tracking error. This paper and its companion paper (Part 2) together analyze the stability of the instantaneous (gradient-based) retrospective cost model reference adaptive controller and the cumulative (recursive-least-squares-based) retrospective cost model reference adaptive controller. Part 1 develops the adaptive controllers and proves the existence of an ideal control law. Part 2 presents the closed-loop error system and provides a closed-loop stability analysis.**

I. INTRODUCTION

The objective of model reference adaptive control (MRAC) is to cause an uncertain system to behave like a known reference model in response to a family of reference model command signals. MRAC has been studied extensively for continuous-time systems [1]–[4] as well as discrete-time systems [4]–[8]. In addition, the MRAC architecture has been extended to deal with various classes of nonlinear systems [9], [10]. However, the results of [1]–[10] as well as related adaptive control techniques [11] are generally restricted to minimum-phase systems.

Retrospective cost adaptive control (RCAC) is an adaptive control technique for discrete-time systems that are possibly nonminimum-phase [12]–[15]. RCAC uses a retrospective performance measure, in which the performance measurement is modified based on the difference between the actual past control inputs and the recomputed past control inputs, assuming that the current controller had been used in the past. RCAC has been demonstrated on multi-input, multi-output nonminimum-phase systems [12]–[14]. Furthermore, the stability of RCAC for single-input, single-output systems is analyzed in [15] for the combined stabilization, command following, and disturbance rejection problem.

The adaptive laws of [12]–[15] are derived by minimizing a retrospective cost, which is a quadratic function of the retrospective performance. In particular, [12], [13] use an instantaneous retrospective cost, which is a function of the retrospective performance at the current time and is minimized by a gradient-type adaptation algorithm. In contrast, [14] uses a cumulative retrospective cost, which is a function of the retrospective performance at the current time step as well as all previous time steps and is minimized by a recursive-least-squares adaptation algorithm. Stability of instantaneous and cumulative RCAC is analyzed in [15].

This paper is the first part of a pair of papers, which together present the retrospective cost model reference adaptive control (RC-MRAC) algorithm for discrete-time systems that are potentially nonminimum phase (provided that we have knowledge of the nonminimum-phase zeros). This paper is intended to be read in conjunction with [16]. This first paper develops the instantaneous and cumulative RC-MRAC algorithms, and focuses on the existence and properties of an ideal control law. In this paper, we also develop a sufficient model-matching condition, namely, that the numerator polynomial of the reference model contain the nonminimum-phase zeros of the open-loop system. This model matching condition is intuitive because the nonminimum-phase zeros of the open-loop system cannot be moved through feedback or pole-zero cancellation. Thus, an appropriate reference model would need to duplicate those nonminimum-phase zeros. The results in this paper are then used in [16] to construct of a closed-loop error system and analyze the stability of the closed-loop system.

II. PROBLEM FORMULATION

Consider the discrete-time system

$$y(k) = -\sum_{i=1}^{n} \alpha_i y(k-i) + \sum_{i=d}^{n} \beta_i u(k-i), \qquad (1)$$

where $k \geq 0$, $\alpha_1, \ldots, \alpha_n, \beta_d, \ldots, \beta_n \in \mathbb{R}$, $y(k) \in \mathbb{R}$ is the output, $u(k) \in \mathbb{R}$ is the control, and the relative degree is $d > 0$. Furthermore, for all $i < 0$, $u(i) = 0$, and the initial condition is $x_0 = [\ y(-1) \quad \cdots \quad y(-n)\ ]^{\mathrm{T}} \in \mathbb{R}^n$.

Let $\mathbf{q}$ and $\mathbf{q}^{-1}$ denote the forward-shift and backward-shift operators, respectively. For all $k \geq 0$, (1) can be expressed as

$$\alpha(\mathbf{q}) y(k-n) = \beta(\mathbf{q}) u(k-n), \qquad (2)$$

where $\alpha(\mathbf{q}) \triangleq \mathbf{q}^n + \alpha_1 \mathbf{q}^{n-1} + \alpha_2 \mathbf{q}^{n-2} + \cdots + \alpha_{n-1}\mathbf{q} + \alpha_n$ and $\beta(\mathbf{q}) \triangleq \beta_d \mathbf{q}^{n-d} + \beta_{d+1}\mathbf{q}^{n-d-1} + \cdots + \beta_{n-1}\mathbf{q} + \beta_n$. Note that β_d is the first nonzero Markov parameter of (1).

Next, consider the reference model

$$y_{\mathrm{m}} = -\sum_{i=1}^{n_{\mathrm{m}}} \alpha_{\mathrm{m},i} y_{\mathrm{m}}(k-i) + \sum_{i=d_{\mathrm{m}}}^{n_{\mathrm{m}}} \beta_{\mathrm{m},i} r(k-i), \qquad (3)$$

where $k \geq 0$, $\alpha_{\mathrm{m},1}, \ldots, \alpha_{\mathrm{m},n_{\mathrm{m}}}, \beta_{\mathrm{m},d_{\mathrm{m}}}, \ldots, \beta_{\mathrm{m},n_{\mathrm{m}}} \in \mathbb{R}$, $y_{\mathrm{m}}(k) \in \mathbb{R}$ is the reference model output, $r(k) \in \mathbb{R}$ is the reference model command, and $d_{\mathrm{m}} > 0$ is the relative degree of (3). Furthermore, for all $i < 0$, $r(i) = 0$, and the initial

[1]Assistant Professor, Department of Mechanical Engineering, University of Kentucky, Lexington, KY 40506-0503, email: jhoagg@engr.uky.edu.

[2]Professor, Department of Aerospace Engineering, University of Michigan, Ann Arbor, MI 48109-2140, email: dsbaero@umich.edu.

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condition is $x_{\mathrm{m},0} = [\ y_{\mathrm{m}}(-1) \ \cdots \ y_{\mathrm{m}}(-n_{\mathrm{m}})\ ]^{\mathrm{T}} \in \mathbb{R}^n$. For all $k \geq 0$, (3) can be expressed as

$$\alpha_{\mathrm{m}}(\mathbf{q}) y_{\mathrm{m}}(k - n_{\mathrm{m}}) = \beta_{\mathrm{m}}(\mathbf{q}) r(k - n_{\mathrm{m}}), \tag{4}$$

where $\alpha_{\mathrm{m}}(\mathbf{q}) \stackrel{\triangle}{=} \mathbf{q}^{n_{\mathrm{m}}} + \alpha_{\mathrm{m},1}\mathbf{q}^{n_{\mathrm{m}}-1} + \cdots + \alpha_{\mathrm{m},n_{\mathrm{m}}-1}\mathbf{q} + \alpha_{\mathrm{m},n_{\mathrm{m}}}$ and $\beta_{\mathrm{m}}(\mathbf{q}) \stackrel{\triangle}{=} \beta_{\mathrm{m},d_{\mathrm{m}}}\mathbf{q}^{n_{\mathrm{m}}-d_{\mathrm{m}}} + \cdots + \beta_{\mathrm{m},n_{\mathrm{m}}-1}\mathbf{q} + \beta_{\mathrm{m},n_{\mathrm{m}}}$. Our goal is to develop an adaptive output feedback controller that generates a control signal $u(k)$ such that $y(k)$ asymptotically follows $y_{\mathrm{m}}(k)$ for all bounded reference model commands $r(k)$. We make the following assumptions regarding the open-loop system (1):

(A1) $\alpha(\mathbf{q})$ and $\beta(\mathbf{q})$ are coprime.
(A2) d is known.
(A3) β_d is known.
(A4) If $\lambda \in \mathbb{C}$, $|\lambda| \geq 1$, and $\beta(\lambda) = 0$, then λ is known.
(A5) There exists an integer $\bar{n}$ such that $n \leq \bar{n}$ and $\bar{n}$ is known.
(A6) $\alpha(\mathbf{q})$, $\beta(\mathbf{q})$, n, and x_0 are not known.

In addition, we make the following assumptions regarding the reference model (3):

(A7) $\alpha_{\mathrm{m}}(\mathbf{q})$ and $\beta_{\mathrm{m}}(\mathbf{q})$ are coprime.
(A8) $\alpha_{\mathrm{m}}(\mathbf{q})$ is asymptotically stable.
(A9) If $\lambda \in \mathbb{C}$, $|\lambda| \geq 1$, and $\beta(\lambda) = 0$, then $\beta_{\mathrm{m}}(\lambda) = 0$.
(A10) If $\lambda \in \mathbb{C}$ and $\alpha(\lambda) = 0$, then $\beta_{\mathrm{m}}(\lambda) \neq 0$.
(A11) $d_{\mathrm{m}} \geq d$.
(A12) $r(k)$ is bounded.
(A13) $\alpha_{\mathrm{m}}(\mathbf{q})$, $\beta_{\mathrm{m}}(\mathbf{q})$, d_{m}, and n_{m} are known.

Next, consider the factorization of $\beta(\mathbf{q})$ given by

$$\beta(\mathbf{q}) = \beta_d \beta_{\mathrm{u}}(\mathbf{q}) \beta_{\mathrm{s}}(\mathbf{q}), \tag{5}$$

where $\beta_{\mathrm{u}}(\mathbf{q})$ is a monic polynomial with degree $n_{\mathrm{u}} \leq n - d$; $\beta_{\mathrm{s}}(\mathbf{q})$ is a monic polynomial with degree $n_{\mathrm{s}} \stackrel{\triangle}{=} n - n_{\mathrm{u}} - d$; and if $\lambda \in \mathbb{C}$, $|\lambda| \geq 1$, and $\beta(\lambda) = 0$, then $\beta_{\mathrm{u}}(\lambda) = 0$ and $\beta_{\mathrm{s}}(\lambda) \neq 0$.

Assumption (A3) implies that the nonminimum-phase zeros from the control to the tracking error (i.e., the roots of $\beta(\mathbf{q})$ the lie on or outside the unit circle) are known, which is equivalent to the assumption that $\beta_{\mathrm{u}}(\mathbf{q})$ and n_{u} are known.

III. Retrospective Performance and RC-MRAC

In this section, we define the retrospective performance and present two RC-MRAC algorithms, namely, instantaneous RC-MRAC and cumulative RC-MRAC. Let $n_{\mathrm{c}} \geq n$, and for all $k \geq n_{\mathrm{c}}$, consider the time-varying controller

$$u(k) = \sum_{i=1}^{n_{\mathrm{c}}} L_i(k) y(k-i) + \sum_{i=1}^{n_{\mathrm{c}}} M_i(k) u(k-i) + \sum_{i=0}^{n_{\mathrm{c}}} N_i(k) r(k-i), \tag{6}$$

where, for all $i = 1, \ldots, n_{\mathrm{c}}$, $L_i : \mathbb{N} \to \mathbb{R}$ and $M_i : \mathbb{N} \to \mathbb{R}$, and, for all $i = 0, 1, \ldots, n_{\mathrm{c}}$, $N_i : \mathbb{N} \to \mathbb{R}$ are given by either the adaptive law presented in Section III-A or the adaptive law presented in Section III-B. For all $k \geq n_{\mathrm{c}}$, the controller (6) can be expressed as

$$u(k) = \phi^{\mathrm{T}}(k)\theta(k), \tag{7}$$

where

$$\theta(k) \stackrel{\triangle}{=} [\ L_1(k) \ \cdots \ L_{n_{\mathrm{c}}}(k) \ \ M_1(k) \ \cdots \ M_{n_{\mathrm{c}}}(k) \ \ N_0(k) \ \cdots \ N_{n_{\mathrm{c}}}(k)\ ]^{\mathrm{T}}.$$

and, for all $k \geq n_{\mathrm{c}}$

$$\phi(k) \stackrel{\triangle}{=} [\ y(k-1) \ \cdots \ y(k-n_{\mathrm{c}}) \ \ u(k-1) \ \cdots \ u(k-n_{\mathrm{c}}) \ \ r(k) \ \cdots \ r(k-n_{\mathrm{c}})\ ]^{\mathrm{T}} \in \mathbb{R}^{3n_{\mathrm{c}}+1}. \tag{8}$$

Note that (6) cannot be implemented for nonnegative $k < n_{\mathrm{c}}$, because, for nonnegative $k < n_{\mathrm{c}}$, $u(k)$ depends on $y(-1), \ldots, y(-n_{\mathrm{c}})$, that is, the initial condition x_0. Therefore, for all nonnegative integers $k < n_{\mathrm{c}}$, let $u(k)$ be given by (7), where, for all nonnegative integers $k < n_{\mathrm{c}}$, $\phi(k) \in \mathbb{R}^{3n_{\mathrm{c}}+1}$.

Next, for all $k \geq 0$, define the tracking error $z(k) \stackrel{\triangle}{=} y(k) - y_{\mathrm{m}}(k)$. Furthermore, define $\bar{\alpha}_{\mathrm{m}}(\mathbf{q}^{-1}) \stackrel{\triangle}{=} \mathbf{q}^{-n_{\mathrm{m}}}\alpha_{\mathrm{m}}(\mathbf{q})$, $\bar{\beta}_{\mathrm{m}}(\mathbf{q}^{-1}) \stackrel{\triangle}{=} \mathbf{q}^{-n_{\mathrm{m}}}\beta_{\mathrm{m}}(\mathbf{q})$, and $\bar{\beta}_{\mathrm{u}}(\mathbf{q}^{-1}) \stackrel{\triangle}{=} \mathbf{q}^{-n_{\mathrm{u}}-d}\beta_{\mathrm{u}}(\mathbf{q})$. Finally, for all $k \geq 0$, define the filtered performance

$$z_{\mathrm{f}}(k) \stackrel{\triangle}{=} \bar{\alpha}_{\mathrm{m}}(\mathbf{q}^{-1}) z(k). \tag{9}$$

For nonnegative $k < n_{\mathrm{m}}$, $z_{\mathrm{f}}(k)$ depends on $z(-1), \ldots, z(-n_{\mathrm{m}})$ (i.e., the initial condition x_0, which is unknown). Therefore, for nonnegative $k < n_{\mathrm{m}}$, $z_{\mathrm{f}}(k)$ is given by (9), where the values used for $z(-1), \ldots, z(-n_{\mathrm{m}})$ can be chosen arbitrarily.

Now, let $\hat{\theta} \in \mathbb{R}^{3n_{\mathrm{c}}+1}$ be an optimization variable used to develop the controller update equation, and, for all $k \geq 0$, define the retrospective performance

$$\begin{aligned}\hat{z}_{\mathrm{f}}(\hat{\theta}, k) &\stackrel{\triangle}{=} z_{\mathrm{f}}(k) + \beta_d \left[\bar{\beta}_{\mathrm{u}}(\mathbf{q}^{-1})\phi(k)\right]^{\mathrm{T}} \hat{\theta} - \beta_d \bar{\beta}_{\mathrm{u}}(\mathbf{q}^{-1}) u(k) \\ &= z_{\mathrm{f}}(k) + \Phi^{\mathrm{T}}(k)\hat{\theta} - \beta_d \bar{\beta}_{\mathrm{u}}(\mathbf{q}^{-1}) u(k),\end{aligned} \tag{10}$$

where the filtered regressor is defined by

$$\Phi(k) \stackrel{\triangle}{=} \beta_d \bar{\beta}_{\mathrm{u}}(\mathbf{q}^{-1})\phi(k), \tag{11}$$

where, for all $i < 0$, $\phi(i) = 0$. Note that the retrospective performance (10) modifies $z_{\mathrm{f}}(k)$ based on the difference between the actual control $u(k-d), \ldots, u(k-n_{\mathrm{u}}-d)$ and the recomputed control $\hat{u}(\hat{\theta}, k-d) \stackrel{\triangle}{=} \phi^{\mathrm{T}}(k-d)\hat{\theta}, \ldots, \hat{u}(\hat{\theta}, k - n_{\mathrm{u}} - d) \stackrel{\triangle}{=} \phi^{\mathrm{T}}(k - n_{\mathrm{u}} - d)\hat{\theta}$, assuming that the controller parameter vector $\hat{\theta}$ had been used in the past.

For all $k \geq 0$, we also define the retrospective performance measure

$$z_{\mathrm{f,r}}(k) \stackrel{\triangle}{=} \hat{z}_{\mathrm{f}}(\theta(k), k). \tag{12}$$

Although $z_{\mathrm{f,r}}(k)$ is not a measurement, it can be computed from $z_{\mathrm{f}}(k)$, $\theta(k)$, $\theta(k-d), \ldots, \theta(k-n_{\mathrm{u}}-d)$, $\phi(k-d), \ldots, \phi(k-n_{\mathrm{u}}-d)$, and knowledge of $\bar{\beta}_{\mathrm{u}}(\mathbf{q}^{-1})$ by using (10). Now, we develop two adaptive laws using $\hat{z}_{\mathrm{f}}(\hat{\theta}, k)$.

A. Instantaneous RC-MRAC

Define the instantaneous retrospective cost function

$$J_{\rm I}(\hat{\theta},k) \triangleq \hat{z}_{\rm f}^2(\hat{\theta},k) + \zeta(k)\left[\hat{\theta}-\theta(k)\right]^{\rm T} R\left[\hat{\theta}-\theta(k)\right], \quad (13)$$

where $R \in \mathbb{R}^{(3n_{\rm c}+1)\times(3n_{\rm c}+1)}$ is positive definite, $\zeta : \mathbb{N} \to (0,\infty)$, $\zeta_{\rm L} \triangleq \inf_{k\geq 0}\zeta(k)$, and $\zeta_{\rm U} \triangleq \sup_{k\geq 0}\zeta(k)$. We assume that $\zeta_{\rm L} > 0$ and $\zeta_{\rm U} < \infty$.

Lemma 1. *Let $\theta(0) \in \mathbb{R}^{3n_{\rm c}+1}$. Then, for each $k \geq 0$, the unique global minimizer of the instantaneous retrospective cost function (13) is given by*

$$\theta(k+1) = \theta(k) - \eta(k)R^{-1}\Phi(k)z_{\rm f,r}(k), \quad (14)$$

where

$$\eta(k) \triangleq \frac{1}{\zeta(k) + \Phi^{\rm T}(k)R^{-1}\Phi(k)}. \quad (15)$$

Proof. It follows from (10) that $J_{\rm I}(\hat{\theta},k) = \hat{\theta}^{\rm T}\Gamma_1(k)\hat{\theta} + \Gamma_2(k)\hat{\theta} + \Gamma_3(k)$, where

$$\begin{aligned}\Gamma_1(k) &\triangleq \zeta(k)R + \Phi(k)\Phi^{\rm T}(k),\\ \Gamma_2(k) &\triangleq -2\zeta(k)\theta^{\rm T}(k)R\\ &\quad + 2\left(z_{\rm f}(k) - \beta_d\bar{\beta}_{\rm u}(\mathbf{q}^{-1})\left[\phi^{\rm T}(k)\theta(k)\right]\right)\Phi^{\rm T}(k),\\ \Gamma_3(k) &\triangleq \zeta(k)\theta^{\rm T}(k)R\theta(k)\\ &\quad + \left(z_{\rm f}(k) - \beta_d\bar{\beta}_{\rm u}(\mathbf{q}^{-1})\left[\phi^{\rm T}(k)\theta(k)\right]\right)^2.\end{aligned}$$

The cost function $J_{\rm I}$ has the unique global minimizer

$$\begin{aligned}\theta(k+1) &\triangleq -\frac{1}{2}\Gamma_1^{-1}(k)\Gamma_2^{\rm T}(k)\\ &= \Gamma_1^{-1}(k)\left(\zeta(k)R + \Phi(k)\Phi^{\rm T}(k)\right)\theta(k)\\ &\quad - \Gamma_1^{-1}(k)\Phi(k)\left(z_{\rm f}(k) - \beta_d\right.\\ &\quad \left.\times\bar{\beta}_{\rm u}(\mathbf{q}^{-1})\left[\phi^{\rm T}(k)\theta(k)\right] + \Phi^{\rm T}(k)\theta(k)\right)\\ &= \theta(k) - \Gamma_1^{-1}(k)\Phi(k)z_{\rm f,r}(k).\end{aligned}$$

Next, it follows from the matrix inversion lemma [2, Lemma 2.1] that

$$\begin{aligned}\Gamma_1^{-1}(k) &= \frac{1}{\zeta(k)}R^{-1} - \frac{1}{\zeta^2(k)}R^{-1}\Phi(k)\\ &\quad\times\left(1 + \frac{1}{\zeta(k)}\Phi^{\rm T}(k)R^{-1}\Phi(k)\right)^{-1}\Phi^{\rm T}(k)R^{-1}\\ &= \frac{1}{\zeta(k)}\left(R^{-1} - \eta(k)R^{-1}\Phi(k)\Phi^{\rm T}(k)R^{-1}\right),\end{aligned}$$

and thus,

$$\begin{aligned}\theta(k+1) &= \theta(k) - \frac{1}{\zeta(k)}\left[R^{-1}\Phi(k)z_{\rm f,r}(k)\right.\\ &\quad \left.-\eta(k)R^{-1}\Phi(k)\Phi^{\rm T}(k)R^{-1}\Phi(k)z_{\rm f,r}(k)\right]\\ &= \theta(k) - \frac{1}{\zeta(k)}\left[\eta(k)\left(\zeta(k) + \Phi^{\rm T}(k)R^{-1}\Phi(k)\right)\right.\\ &\quad \left.\times R^{-1}\Phi(k)z_{\rm f,r}(k)\right]\\ &\quad + \frac{1}{\zeta(k)}\eta(k)R^{-1}\Phi(k)\Phi^{\rm T}(k)R^{-1}\Phi(k)z_{\rm f,r}(k)\\ &= \theta(k) - \eta(k)R^{-1}\Phi(k)z_{\rm f,r}(k),\end{aligned}$$

which verifies (14). □

In summary, instantaneous RC-MRAC is given by (7), (14), and (15), where $\phi(k)$, $\Phi(k)$, and $z_{\rm f,r}(k)$ are given by (8), (11), and (12), respectively.

B. Cumulative RC-MRAC

As an alternative to (13), define the cumulative retrospective cost function

$$\begin{aligned}J_{\rm C}(\hat{\theta},k) &\triangleq \sum_{i=0}^{k}\lambda^{k-i}\hat{z}_{\rm f}^2(\hat{\theta},i)\\ &\quad + \lambda^k\left[\hat{\theta}-\theta(0)\right]^{\rm T} R\left[\hat{\theta}-\theta(0)\right], \quad (16)\end{aligned}$$

where $\lambda \in (0,1]$ and $R \in \mathbb{R}^{(3n_{\rm c}+1)\times(3n_{\rm c}+1)}$ is positive definite. Note that λ serves as a forgetting factor, which allows more recent data to be weighted more heavily than past data. The next result follows from standard recursive least-squares (RLS) theory [2], [4], [6].

Lemma 2. *Let $P(0) = R^{-1}$ and $\theta(0) \in \mathbb{R}^{3n_{\rm c}+1}$. Then, for each $k \geq 0$, the unique global minimizer of the cumulative retrospective cost function (16) is given by*

$$\theta(k+1) = \theta(k) - \frac{P(k)\Phi(k)z_{\rm f,r}(k)}{\lambda + \Phi^{\rm T}(k)P(k)\Phi(k)}, \quad (17)$$

where

$$P(k+1) = \frac{1}{\lambda}\left[P(k) - \frac{P(k)\Phi(k)\Phi^{\rm T}(k)P(k)}{\lambda + \Phi^{\rm T}(k)P(k)\Phi(k)}\right]. \quad (18)$$

Thus, cumulative RC-MRAC is given by (7), (17), and (18). The remainder of this paper focuses on the existence and properties of an ideal control law. The stability of the instantaneous RC-MRAC and cumulative RC-MRAC is analyzed in [16].

IV. Nonminimal-State-Space Realization

We use a nonminimal-state-space realization of the time-series model (1) whose state consists entirely of measured information, specifically, y, u, and r. We introduce the following notation. For a positive integer p, define the nilpotent matrix

$$\mathcal{N}_p \triangleq \begin{bmatrix} 0 & \cdots & 0 & 0\\ 1 & & 0 & 0\\ \vdots & \ddots & \vdots & \vdots\\ 0 & \cdots & 1 & 0\end{bmatrix} \in \mathbb{R}^{p\times p},$$

and the column vector

$$E_p \triangleq \begin{bmatrix} 1\\ 0_{(p-1)\times 1}\end{bmatrix} \in \mathbb{R}^p.$$

Next, for all $k \geq n_{\rm c}$, consider the $(3n_{\rm c}+1)^{\rm th}$-order nonminimal-state-space realization of (1) given by

$$\phi(k+1) = \mathcal{A}\phi(k) + \mathcal{B}u(k) + \mathcal{D}r(k+1), \quad (19)$$

$$y(k) = \mathcal{C}\phi(k), \quad (20)$$

where

$$\mathcal{A} \triangleq \mathcal{A}_{\rm nil} + E_{3n_{\rm c}+1}\mathcal{C}, \tag{21}$$

$$\mathcal{C} \triangleq \begin{bmatrix} -\alpha_1 & \cdots & -\alpha_n & 0_{1\times(n_{\rm c}-n)} & 0_{1\times(d-1)} \\ \beta_d & \cdots & \beta_n & 0_{1\times(n_{\rm c}-n)} & 0_{1\times(n_{\rm c}+1)} \end{bmatrix}, \tag{22}$$

$$\mathcal{B} \triangleq \begin{bmatrix} 0_{n_{\rm c}\times 1} \\ E_{n_{\rm c}} \\ 0_{(n_{\rm c}+1)\times 1} \end{bmatrix}, \qquad \mathcal{D} \triangleq \begin{bmatrix} 0_{2n_{\rm c}\times 1} \\ E_{n_{\rm c}+1} \end{bmatrix}, \tag{23}$$

$$\mathcal{A}_{\rm nil} \triangleq \begin{bmatrix} \mathcal{N}_{n_{\rm c}} & 0_{n_{\rm c}\times n_{\rm c}} & 0_{n_{\rm c}\times(n_{\rm c}+1)} \\ 0_{n_{\rm c}\times n_{\rm c}} & \mathcal{N}_{n_{\rm c}} & 0_{n_{\rm c}\times(n_{\rm c}+1)} \\ 0_{(n_{\rm c}+1)\times n_{\rm c}} & 0_{(n_{\rm c}+1)\times n_{\rm c}} & \mathcal{N}_{n_{\rm c}+1} \end{bmatrix}. \tag{24}$$

The triple $(\mathcal{A},\mathcal{B},\mathcal{C})$ is stabilizable and detectable but is neither controllable nor observable. In fact, $(\mathcal{A},\mathcal{B},\mathcal{C})$ has n controllable and observable eigenvalues, while $(\mathcal{A},\mathcal{B})$ has $2n_{\rm c}+1-n$ uncontrollable eigenvalues at 0, and $(\mathcal{A},\mathcal{C})$ has $3n_{\rm c}+1-n$ unobservable eigenvalues at 0.

V. Ideal Fixed-Gain Controller

In this section, we prove the existence of an ideal fixed-gain controller for the open-loop system (1). This controller, whose structure is illustrated in Figure 1, is used in the companion paper [16] to construct an error system for analyzing the closed-loop adaptive system. An ideal fixed-gain controller consists of a precompensator, which cancels the stable zeros of the open-loop system, and a feedback-feedforward controller whose inputs are y and r.

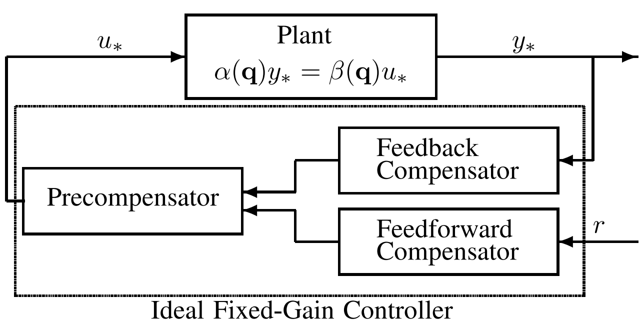

Fig. 1. Closed-loop system with the ideal fixed-gain controller.

For all $k \ge n_{\rm c}$, consider the system (1) with $u(k) = u_*(k)$, where $u_*(k)$ is the ideal control. More precisely, for all $k \ge n_{\rm c}$, consider the system

$$y_*(k) = -\sum_{i=1}^{n} \alpha_i y_*(k-i) + \sum_{i=d}^{n} \beta_i u_*(k-i), \tag{25}$$

where, for all $k \ge n_{\rm c}$, $u_*(k)$ is given by the strictly proper ideal fixed-gain controller

$$u_*(k) = \sum_{i=1}^{n_{\rm c}} L_{*,i} y_*(k-i) + \sum_{i=1}^{n_{\rm c}} M_{*,i} u_*(k-i) + \sum_{i=0}^{n_{\rm c}} N_{*,i} r(k-i), \tag{26}$$

where $L_{*,1},\ldots,L_{*,n_{\rm c}} \in \mathbb{R}$, $M_{*,1},\ldots,M_{*,n_{\rm c}} \in \mathbb{R}$, $N_{*,0},\ldots,N_{*,n_{\rm c}} \in \mathbb{R}$, and the initial condition at $k = n_{\rm c}$ for (25) and (26) is $\phi_{*,0} = [y_*(n_{\rm c}-1) \ \cdots \ y_*(0) \ u_*(n_{\rm c}-1) \ \cdots \ u_*(0) \ r(n_{\rm c}) \ \cdots \ r(0)]^{\rm T} \in \mathbb{R}^{3n_{\rm c}+1}$.

For all $k \ge n_{\rm c}$, the ideal control (26) can be written as

$$u_*(k) = \phi_*^{\rm T}(k)\theta_*, \tag{27}$$

where

$$\theta_* \triangleq \begin{bmatrix} L_{*,1} & \cdots & L_{*,n_{\rm c}} & M_{*,1} & \cdots & M_{*,n_{\rm c}} & N_{*,0} & \cdots & N_{*,n_{\rm c}} \end{bmatrix}^{\rm T},$$

$$\phi_*(k) \triangleq \begin{bmatrix} y_*(k-1) & \cdots & y_*(k-n_{\rm c}) & u_*(k-1) & \cdots & u_*(k-n_{\rm c}) & r(k) & \cdots & r(k-n_{\rm c}) \end{bmatrix}^{\rm T}.$$

Therefore, it follows from (19)-(24) and (27) that, for all $k \ge n_{\rm c}$, the ideal closed-loop system (25), (26) has the $(3n_{\rm c}+1)^{\rm th}$-order nonminimal-state-space realization

$$\phi_*(k+1) = \mathcal{A}_*\phi_*(k) + \mathcal{D}r(k+1), \tag{28}$$

$$y_*(k) = \mathcal{C}\phi_*(k), \tag{29}$$

where

$$\mathcal{A}_* \triangleq \mathcal{A} + \mathcal{B}\theta_*^{\rm T}, \tag{30}$$

and the initial condition is $\phi_*(n_{\rm c}) \triangleq \phi_{*,0}$.

Theorem 1. *Let*

$$n_{\rm c} \ge \max(2n - n_{\rm u} - d, n_{\rm m} - n_{\rm u} - d). \tag{31}$$

Then there exists an ideal fixed-gain controller (26) of order $n_{\rm c}$ such that the following statements hold for the ideal closed-loop system consisting of (25), (26), which has the $(3n_{\rm c}+1)^{\rm th}$-order nonminimal-state-space realization (28), (29), where $\mathcal{A}_*$ is given by (30):

(i) For all initial conditions $\phi_{*,0}$, and, for all $k \ge k_0 \triangleq 2n_{\rm c} + n_{\rm u} + d$,

$$\bar{\alpha}_{\rm m}(\mathbf{q}^{-1})y_*(k) = \bar{\beta}_{\rm m}(\mathbf{q}^{-1})r(k),$$

and thus,

$$\bar{\alpha}_{\rm m}(\mathbf{q}^{-1})y_*(k) = \bar{\alpha}_{\rm m}(\mathbf{q}^{-1})y_{\rm m}(k).$$

(ii) $\mathcal{A}_*$ is asymptotically stable.

(iii) For all initial conditions $\phi_{*,0}$, $u_*(k)$ is bounded.

(iv) For all $k \ge k_0$, and, for all sequences $e(k)$,

$$\beta_d\bar{\beta}_{\rm u}(\mathbf{q}^{-1})e(k) = \bar{\alpha}_{\rm m}(\mathbf{q}^{-1})\left[\sum_{i=1}^{k-n_{\rm c}} \mathcal{C}\mathcal{A}_*^{i-1}\mathcal{B}e(k-i)\right].$$

The proof of Theorem 1 is in Appendix A.

Property *(iv)* of Theorem 1 is a time-domain property. It has the following $\mathbf{z}$-domain interpretation

$$\mathcal{C}\left(\mathbf{z}I - \mathcal{A}_*\right)^{-1}\mathcal{B} = \frac{\beta_d\beta_{\rm u}(\mathbf{z})\mathbf{z}^{n_{\rm m}-n_{\rm u}-d}}{\alpha_{\rm m}(\mathbf{z})}, \tag{32}$$

which implies that the nonminimum-phase zeros of the closed-loop transfer function (32) are exactly the nonminimum-phase zeros of the open-loop system, that is, the roots of $\beta_{\rm u}(\mathbf{q})$. Furthermore, (32) is the closed-loop transfer function from a control input perturbation to the performance. In the companion paper [16], property *(iv)* of Theorem 1 is used to develop a closed-loop error system and to relate the performance of the closed-loop adaptive system to the controller-parameter-estimation error, that is, the distance between $\theta(k)$ and θ_*. Furthermore, the companion paper [16] provides a stability analysis for both the instantaneous RC-MRAC and cumulative RC-MRAC algorithms.

Appendix A: Proof of Theorem 1

Proof of Theorem 1. We construct the ideal fixed-gain controller (26), which is depicted in Figure 1, and show that it satisfies *(i)-(iv)*.

First, for all $k \geq 0$, (26) can be expressed as

$$M_*(\mathbf{q})u_*(k) = L_*(\mathbf{q})y_*(k) + N_*(\mathbf{q})r(k), \qquad (33)$$

where $M_*(\mathbf{q}) = \mathbf{q}^{n_{\rm c}} - M_{*,1}\mathbf{q}^{n_{\rm c}-1} - \cdots - M_{*,n_{\rm c}}$, $L_*(\mathbf{q}) = L_{*,1}\mathbf{q}^{n_{\rm c}-1} + \cdots + L_{*,n_{\rm c}-1}\mathbf{q} + L_{*,n_{\rm c}}$, and $N_*(\mathbf{q}) = N_{*,0}\mathbf{q}^{n_{\rm c}} + \cdots + N_{*,n_{\rm c}-1}\mathbf{q} + N_{*,n_{\rm c}}$. Thus, it suffices to show that there exist polynomials $L_*(\mathbf{q})$, $M_*(\mathbf{q})$, and $N_*(\mathbf{q})$ such that *(i)-(iv)* are satisfied.

Define $n_{\rm f} \stackrel{\triangle}{=} n_{\rm c} - n_{\rm s}$, and consider the exactly proper precompensator

$$\beta_{\rm s}(\mathbf{q})u_*(k) = N_{\rm pc}(\mathbf{q})v(k), \qquad (34)$$

where $N_{\rm pc}(\mathbf{q})$ is a polynomial with degree $n_{\rm s}$ and $v(k)$ is given by the compensator

$$M_{\rm f}(\mathbf{q})v(k) = L_{\rm f}(\mathbf{q})y_*(k) + N_{\rm f}(\mathbf{q})r(k), \qquad (35)$$

where $M_{\rm f}(\mathbf{q})$ is a monic polynomial with degree $n_{\rm f}$, $L_{\rm f}(\mathbf{q})$ is a polynomial with degree $n_{\rm f}-1$, and $N_{\rm f}(\mathbf{q})$ is a polynomial with degree $n_{\rm f} - d_{\rm m} + d$. For all $k \geq 0$, the cascade (34) and (35) can be expressed as (33), where $L_*(\mathbf{q}) \stackrel{\triangle}{=} L_{\rm f}(\mathbf{q})N_{\rm pc}(\mathbf{q})$, $M_*(\mathbf{q}) \stackrel{\triangle}{=} M_{\rm f}(\mathbf{q})\beta_{\rm s}(\mathbf{q})$, and $N_*(\mathbf{q}) \stackrel{\triangle}{=} N_{\rm f}(\mathbf{q})N_{\rm pc}(\mathbf{q})$. Now, it suffices to show that there exist polynomials $L_{\rm f}(\mathbf{q})$, $M_{\rm f}(\mathbf{q})$, $N_{\rm f}(\mathbf{q})$, and $N_{\rm pc}(\mathbf{q})$, such that *(i)-(iv)* are satisfied.

To show *(i)*, we consider the closed-loop system consisting of (25), (34), and (35). First, it follows from (5) and (25) that, for all $k \geq n_{\rm c}$,

$$\alpha(\mathbf{q})y_*(k) = \beta_d\beta_{\rm u}(\mathbf{q})\beta_{\rm s}(\mathbf{q})u_*(k). \qquad (36)$$

Next, multiplying (36) by $M_{\rm f}(\mathbf{q})$ yields $M_{\rm f}(\mathbf{q})\alpha(\mathbf{q})y_*(k) = \beta_d\beta_{\rm u}(\mathbf{q})M_{\rm f}(\mathbf{q})\beta_{\rm s}(\mathbf{q})u_*(k)$. Using (34) and (35) yields, for all $k \geq n_{\rm c}$, $M_{\rm f}(\mathbf{q})\alpha(\mathbf{q})y_*(k) = \beta_d\beta_{\rm u}(\mathbf{q})N_{\rm pc}(\mathbf{q})L_{\rm f}(\mathbf{q})y_*(k) + \beta_d\beta_{\rm u}(\mathbf{q})N_{\rm pc}(\mathbf{q})N_{\rm f}(\mathbf{q})r(k)$, which implies

$$[M_{\rm f}(\mathbf{q})\alpha(\mathbf{q}) - \beta_d\beta_{\rm u}(\mathbf{q})N_{\rm pc}(\mathbf{q})L_{\rm f}(\mathbf{q})]y_*(k) = \beta_d\beta_{\rm u}(\mathbf{q})N_{\rm pc}(\mathbf{q})N_{\rm f}(\mathbf{q})r(k). \qquad (37)$$

Now, we show that there exist $N_{\rm f}(\mathbf{q})$ and $N_{\rm pc}(\mathbf{q})$ such that $\beta_d\beta_{\rm u}(\mathbf{q})N_{\rm pc}(\mathbf{q})N_{\rm f}(\mathbf{q}) = \beta_{\rm m}(\mathbf{q})\mathbf{q}^{n_1}$, where $n_1 \stackrel{\triangle}{=} n_{\rm c} + n_{\rm u} + d - n_{\rm m}$. Note that it follows from (31) that $n_1 \geq 0$. Assumption (A9) implies that $\beta_{\rm u}(\mathbf{q})$ divides $\beta_{\rm m}(\mathbf{q})$. Since, in addition, deg $\beta_{\rm u}(\mathbf{q})N_{\rm pc}(\mathbf{q})N_{\rm f}(\mathbf{q}) = n_{\rm u} + n_{\rm s} + n_{\rm f} + d - d_{\rm m} = n_{\rm m} - d_{\rm m} + n_1 =$ deg $\beta_{\rm m}(\mathbf{q}) + n_1$, it follows that there exist polynomials $N_{\rm f}(\mathbf{q})$ and $N_{\rm pc}(\mathbf{q})$ such that $\beta_d\beta_{\rm u}(\mathbf{q})N_{\rm pc}(\mathbf{q})N_{\rm f}(\mathbf{q}) = \beta_{\rm m}(\mathbf{q})\mathbf{q}^{n_1}$. Thus, for all $k \geq n_{\rm c}$, (37) becomes

$$[M_{\rm f}(\mathbf{q})\alpha(\mathbf{q}) - \beta_d\beta_{\rm u}(\mathbf{q})N_{\rm pc}(\mathbf{q})L_{\rm f}(\mathbf{q})]y_*(k) = \beta_{\rm m}(\mathbf{q})\mathbf{q}^{n_1}r(k). \qquad (38)$$

Next, we show that there exist polynomials $L_{\rm f}(\mathbf{q})$ and $M_{\rm f}(\mathbf{q})$ such that $M_{\rm f}(\mathbf{q})\alpha(\mathbf{q}) - \beta_d\beta_{\rm u}(\mathbf{q})N_{\rm pc}(\mathbf{q})L_{\rm f}(\mathbf{q}) = \alpha_{\rm m}(\mathbf{q})\mathbf{q}^{n_1}$. First, note that deg $M_{\rm f}(\mathbf{q})\alpha(\mathbf{q}) = n_{\rm f} + n = n_{\rm c} + n_{\rm u} + d = n_{\rm m} + n_1 =$ deg $\alpha_{\rm m}(\mathbf{q})\mathbf{q}^{n_1}$. Next, the degree of $M_{\rm f}(\mathbf{q})$ is $n_{\rm f}$ and the degree of $L_{\rm f}(\mathbf{q})$ is $n_{\rm f} - 1$, where $n_{\rm f} = n_{\rm c} - n_{\rm s} = \max(n, n_{\rm m} - n) \geq n =$ deg $\alpha_{\rm m}(\mathbf{q})$. Since, in addition, assumptions (A1) and (A10) imply that $\alpha(\mathbf{q})$ and $\beta_{\rm u}(\mathbf{q})N_{\rm pc}(\mathbf{q})$ are coprime, it follows from the Diophantine equation that the roots of $M_{\rm f}(\mathbf{q})\alpha(\mathbf{q}) - \beta_d\beta_{\rm u}(\mathbf{q})N_{\rm pc}(\mathbf{q})L_{\rm f}(\mathbf{q})$ can be assigned arbitrarily by choice of $L_{\rm f}(\mathbf{q})$ and $M_{\rm f}(\mathbf{q})$. Therefore, there exist polynomials $L_{\rm f}(\mathbf{q})$ and $M_{\rm f}(\mathbf{q})$ such that $M_{\rm f}(\mathbf{q})\alpha(\mathbf{q}) - \beta_d\beta_{\rm u}(\mathbf{q})N_{\rm pc}(\mathbf{q})L_{\rm f}(\mathbf{q}) = \alpha_{\rm m}(\mathbf{q})\mathbf{q}^{n_1}$. Thus, for all $k \geq n_{\rm c}$, (38) becomes $\alpha_{\rm m}(\mathbf{q})\mathbf{q}^{n_1}y_*(k) = \beta_{\rm m}(\mathbf{q})\mathbf{q}^{n_1}r(k)$, which implies that, for all $k \geq n_{\rm c} + n_1$,

$$\alpha_{\rm m}(\mathbf{q})y_*(k) = \beta_{\rm m}(\mathbf{q})r(k). \qquad (39)$$

Thus, for all $k \geq k_0 \stackrel{\triangle}{=} n_{\rm c} + n_1 + n_{\rm m} = 2n_{\rm c} + n_{\rm u} + d$, $\bar{\alpha}_{\rm m}(\mathbf{q}^{-1})y_*(k) = \bar{\beta}_{\rm m}(\mathbf{q}^{-1})r(k)$, thus, confirming *(i)*.

To show *(ii)*, note that, for all $k \geq n_{\rm c}$, the closed-loop system (28), (29) is a $(3n_{\rm c}+1)^{\rm th}$-order nonminal-state-space realization of the closed-loop system (25), (26), which has the closed-loop characteristic polynomial $M_*(\mathbf{q})\alpha(\mathbf{q}) - \beta(\mathbf{q})L_*(\mathbf{q}) = \beta_{\rm s}(\mathbf{q})\alpha_{\rm m}(\mathbf{q})\mathbf{q}^{n_1}$. Thus, the spectrum of $\mathcal{A}_*$ consists of the $n_{\rm c} + n$ roots of $\beta_{\rm s}(\mathbf{q})\alpha_{\rm m}(\mathbf{q})\mathbf{q}^{n_1}$ along with $2n_{\rm c} + 1 - n$ eigenvalues located at 0, which are exactly the uncontrollable eigenvalues of $(\mathcal{A}, \mathcal{B})$. Therefore, since $\alpha_{\rm m}(\mathbf{q})$ and $\beta_{\rm s}(\mathbf{q})$ are asymptotically stable, it follows that $\mathcal{A}_*$ is asymptotically stable. Thus, we have verified *(ii)*.

To show *(iii)*, it follows from Assumption (A12) that $r(k)$ is bounded. Since, in addition, $\mathcal{A}_*$ is asymptotically stable, it follows from (28) that $\phi_*(k)$ is the state of an asymptotically stable linear system with the bounded input $r(k)$. Thus, $\phi_*(k)$ is bounded. Finally, since $u_*(k)$ is a component of $\phi_*(k+1)$, it follows that $u_*(k)$ is bounded.

To show *(iv)*, consider the $(3n_{\rm c}+1)^{\rm th}$-order nonminimal state-space realization (28), (29), which, for all $k \geq n_{\rm c}$, has the solution

$$y_*(k) = \mathcal{C}\mathcal{A}_*^{k-n_{\rm c}}\phi_*(n_{\rm c}) + \sum_{i=1}^{k-n_{\rm c}} \mathcal{C}\mathcal{A}_*^{i-1}\mathcal{D}_1 r(k-i+1),$$

which implies that

$$\alpha_{\rm m}(\mathbf{q})y_*(k) = \alpha_{\rm m}(\mathbf{q})\left[\sum_{i=1}^{k-n_{\rm c}} \mathcal{C}\mathcal{A}_*^{i-1}\mathcal{D}_1 r(k-i+1)\right] + \alpha_{\rm m}(\mathbf{q})\left[\mathcal{C}\mathcal{A}_*^{k-n_{\rm c}}\phi_*(n_{\rm c})\right]. \qquad (40)$$

Comparing (39) and (40) yields, for all $k \geq n_c + n_1$,

$$\beta_{\rm m}(\mathbf{q})r(k) = \alpha_{\rm m}(\mathbf{q}) \left[\sum_{i=1}^{k-n_{\rm c}} \mathcal{C}\mathcal{A}_*^{i-1}\mathcal{D}_1 r(k-i+1) \right] + \alpha_{\rm m}(\mathbf{q}) \left[\mathcal{C}\mathcal{A}_*^{k-n_{\rm c}}\phi_*(n_{\rm c}) \right]. \quad (41)$$

Next, for all $k \geq n_{\rm c}$, consider the system (1), where $u(k)$ consists of two components: one that is generated from the ideal controller, and one that is an arbitrary sequence $e(k)$. More precisely, for all $k \geq n_{\rm c}$, consider the system

$$y_e(k) = -\sum_{i=1}^{n} \alpha_i y_e(k-i) + \sum_{i=d}^{n} \beta_i u_e(k-i), \quad (42)$$

where, for all $k \geq n_{\rm c}$, $u_e(k)$ is given by

$$u_e(k) = \sum_{i=1}^{n_{\rm c}} L_{*,i} y_e(k-i) + \sum_{i=1}^{n_{\rm c}} M_{*,i} u_e(k-i) + \sum_{i=0}^{n_{\rm c}} N_{*,i} r(k-i) + e(k), \quad (43)$$

where the initial condition at $k = n_{\rm c}$ for (42), (43) is $\phi_{e,0} = [y_e(n_{\rm c}-1) \ \cdots \ y_e(0) \ u_e(n_{\rm c}-1) \ \cdots \ u_e(0) \ r(n_{\rm c}) \ \cdots \ r(0)]^{\rm T} \in \mathbb{R}^{3n_{\rm c}+1}$. Furthermore, let (42), (43) have the same initial condition as the ideal closed-loop system (25), (26), that is, let $\phi_{e,0} = \phi_{*,0}$. For all $k \geq n_{\rm c}$, (42) implies

$$\alpha(\mathbf{q})y_e(k) = \beta(\mathbf{q})u_e(k), \quad (44)$$

and (43) implies

$$M_{\rm f}(\mathbf{q})\beta_{\rm s}(\mathbf{q})u_e(k) = L_*(\mathbf{q})y_e(k) + N_*(\mathbf{q})r(k) + \mathbf{q}^{n_{\rm c}}e(k). \quad (45)$$

Next, closing the feedback loop between (44) and (45) yields, for all $k \geq n_{\rm c}$, $[M_{\rm f}(\mathbf{q})\alpha(\mathbf{q}) - \beta_d\beta_{\rm u}(\mathbf{q})L_*(\mathbf{q})]\, y_e(k) = \beta_d\beta_{\rm u}(\mathbf{q})N_*(\mathbf{q})r(k) + \beta_d\beta_{\rm u}(\mathbf{q})\mathbf{q}^{n_{\rm c}}e(k)$. Since (43) is constructed with the ideal controller parameters, it follows from earlier in this proof that $M_{\rm f}(\mathbf{q})\alpha(\mathbf{q}) - \beta_d\beta_{\rm u}(\mathbf{q})L_*(\mathbf{q}) = \alpha_{\rm m}(\mathbf{q})\mathbf{q}^{n_1}$ and $\beta_d\beta_{\rm u}(\mathbf{q})N_*(\mathbf{q}) = \beta_{\rm m}(\mathbf{q})\mathbf{q}^{n_1}$, which implies that, for all $k \geq n_{\rm c}$, $\alpha_{\rm m}(\mathbf{q})\mathbf{q}^{n_1}y_e(k) = \beta_{\rm m}(\mathbf{q})\mathbf{q}^{n_1}r(k) + \beta_d\beta_{\rm u}(\mathbf{q})\mathbf{q}^{n_{\rm c}}e(k)$. Therefore, for all $k \geq n_{\rm c} + n_1$,

$$\alpha_{\rm m}(\mathbf{q})y_e(k) = \beta_{\rm m}(\mathbf{q})r(k) + \beta_d\beta_{\rm u}(\mathbf{q})\mathbf{q}^{n_{\rm m}-n_{\rm u}-d}e(k). \quad (46)$$

Next, for all $k \geq n_{\rm c}$, consider the $(3n_{\rm c}+1)^{\rm th}$-order nonminimal state-space realization (19)-(24) with the feedback (43), which has the closed-loop representation

$$\phi_e(k+1) = \mathcal{A}_*\phi_e(k) + \mathcal{B}e(k) + \mathcal{D}r(k+1),$$
$$y_e(k) = \mathcal{C}\phi_e(k),$$

where the $\phi_e(k)$ has the same form as $\phi(k)$ with $y_e(k)$ and $u_e(k)$ replacing $y(k)$ and $u(k)$, respectively. For all $k \geq n_{\rm c}$, this system has the solution

$$y_e(k) = \mathcal{C}\mathcal{A}_*^{k-n_{\rm c}}\phi_e(n_{\rm c}) + \sum_{i=1}^{k-n_{\rm c}} \mathcal{C}\mathcal{A}_*^{i-1}\mathcal{B}e(k-i) + \sum_{i=1}^{k-n_{\rm c}} \mathcal{C}\mathcal{A}_*^{i-1}\mathcal{D}r(k-i+1).$$

Multiplying both sides by $\alpha_{\rm m}(\mathbf{q})$ yields, for all $k \geq n_{\rm c}$,

$$\alpha_{\rm m}(\mathbf{q})y_e(k) = \alpha_{\rm m}(\mathbf{q}) \left[\mathcal{C}\mathcal{A}_*^{k-n_{\rm c}}\phi_e(n_{\rm c}) \right] + \alpha_{\rm m}(\mathbf{q}) \left[\sum_{i=1}^{k-n_{\rm c}} \mathcal{C}\mathcal{A}_*^{i-1}\mathcal{D}r(k-i+1) \right] + \alpha_{\rm m}(\mathbf{q}) \left[\sum_{i=1}^{k-n_{\rm c}} \mathcal{C}\mathcal{A}_*^{i-1}\mathcal{B}e(k-i) \right]. \quad (47)$$

Since $\phi_e(n_{\rm c}) = \phi_{e,0} = \phi_{*,0} = \phi_*(n_{\rm c})$, it follows from (41) and (47) that, for all $k \geq n_{\rm c} + n_1$,

$$\alpha_{\rm m}(\mathbf{q})y_e(k) = \alpha_{\rm m}(\mathbf{q}) \left[\sum_{i=1}^{k-n_{\rm c}} \mathcal{C}\mathcal{A}_*^{i-1}\mathcal{B}e(k-i) \right] + \beta_{\rm m}(\mathbf{q})r(k). \quad (48)$$

Finally, comparing (46) and (48) yields, for all $k \geq n_{\rm c} + n_1$,

$$\beta_d\beta_{\rm u}(\mathbf{q})\mathbf{q}^{n_{\rm m}-n_{\rm u}-d}e(k) = \alpha_{\rm m}(\mathbf{q}) \left[\sum_{i=1}^{k-n_{\rm c}} \mathcal{C}\mathcal{A}_*^{i-1}\mathcal{B}e(k-i) \right],$$

and multiplying both sides by $\mathbf{q}^{-n_{\rm m}}$, yields, for all $k \geq k_0$,

$$\beta_d\bar{\beta}_{\rm u}(\mathbf{q}^{-1})e(k) = \bar{\alpha}_{\rm m}(\mathbf{q}^{-1}) \left[\sum_{i=1}^{k-n_{\rm c}} \mathcal{C}\mathcal{A}_*^{i-1}\mathcal{B}e(k-i) \right],$$

thus verifying *(iv)*. □

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