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On Using LQG Performance Metrics for Sensor Placement

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***Abstract*— We discuss four metrics for determining sensor placement in energy efficient building design. These include the norm of the observer gain, the trace of the observer Riccati solution, the distance to the nearest unobservable system, and the linear quadratic Gaussian (LQG) cost from given initial state and state estimates. These metrics have different computational complexity, but all lead to the same optimal sensor location in this study where a single room model is considered.**

I. INTRODUCTION

In this paper, we investigate methodologies for determining optimal sensor placement in the design of HVAC systems in buildings. The current state of the science in whole building simulation uses lumped models that are not amenable to addressing these control questions. Studies that involve more sophisticated modeling approaches are appearing in the literature, but are limited to addressing the model/simulation problem, e.g. [1], [2], [3]. Our present study is limited to the flow in one room and considers the use of distributed parameter control theory to design the physical configuration of the control system. The model in this preliminary study leads to an advection diffusion equation where the advection velocity is given by the Navier-Stokes equations. We build upon related efforts in the actuator/sensor placement problem over the past decade, e.g. [4], [5], [6], [7], [8]. In particular, we consider four metrics that can be used: the norm of the observer gain, the trace of the observer Riccati solution, the distance to the nearest unobservable system, and the LQG cost from given initial state and state estimates. Numerical experiments are given that compare the efficiency and quality of these metrics.

The remainder of this paper is organized as follows: in the next section we present a model for the transport of thermal energy, moisture, etc. in a single room. We then make a simplifying assumption for this study as well as present the distributed control problem. The objective is to control the average temperature in the center region of the room by specifying the incoming air temperature and measure the temperature at one location on a side wall. We present four metrics in the next section that can be used to evaluate the quality of the sensor location. This is followed by a numerical study that considers an entire range of sensor locations on one wall of a room. Finally, we present our conclusions as well as outline our plans for future work.

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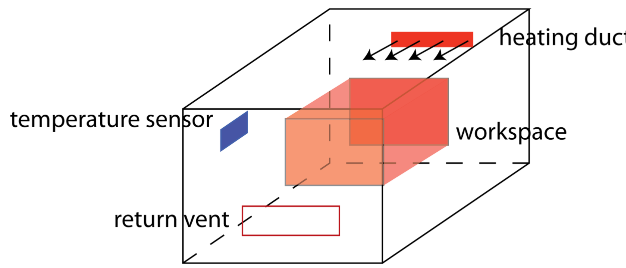

Fig. 1. Simplified room: the region Ω_w is shaded

II. MODEL AND CONTROL DESIGN PROBLEM

The transport of air, thermal energy, and chemical species in a room can be modeled using the Boussinesq equations along with passive scalar equations for each component of interest (water vapor, CO_2, etc.). We ignore the buoyancy forces that gradients in chemical species could impart on the flow. The PDE system is

$$\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v}\cdot\nabla\mathbf{v} = -\nabla p + \frac{1}{\mathrm{Re}}\Delta\mathbf{v} + \frac{\mathrm{Gr}}{\mathrm{Re}^2}T\hat{k} + \mathbf{B}_{\mathbf{v}}u_{\mathbf{v}} \quad (1)$$

$$\nabla\cdot\mathbf{v} = 0 \quad (2)$$

$$\frac{\partial T}{\partial t} + \mathbf{v}\cdot\nabla T = \frac{1}{\mathrm{RePr}}\Delta T + B_T u \quad (3)$$

$$\frac{\partial s_i}{\partial t} + \mathbf{v}\cdot\nabla s_i = \varepsilon_i\Delta s_i + B_{s_i}u_{s_i} \quad (4)$$

where for each time $t > 0$, $\mathbf{x} \in \Omega \subset \mathbb{R}^3$, $\mathbf{v}(t,\mathbf{x})$ is the velocity vector, $p(t,\mathbf{x})$ is the pressure, $T(t,\mathbf{x})$ is the temperature, and $s_i(t,\mathbf{x})$, $i = 1,\ldots,S$, are additional scalar fields of interest. Nondimensionalization has been carried out where Re is the Reynolds number, Gr is the Grashof number, Pr is the Prandtl number, and ε_i is a diffusion coefficient associated with scalar s_i (cf. [9]). The flow is considered in a unit room Ω depicted in Fig. 1 with an inlet vent on one wall and an outlet vent on the opposing wall.

The model includes control input terms $\mathbf{B}_{\mathbf{v}}u_{\mathbf{v}}$, $B_T u$ and $B_s u_{s_i}$ that affect, respectively, the velocity, temperature and species at the heating vent. A uniformly heated, uniform species concentration, parabolic inflow velocity profile is assumed at the heating duct with stress-free fluid boundary conditions at the return vent. No slip boundary conditions for the velocity are imposed on the remainder of the walls, while homogeneous Neumann conditions are assumed for the remaining boundary conditions on temperature and concentrations.

Our preliminary study, which emphasizes performance metrics for studying sensor location, makes a number of simplifying assumptions. The main assumption is that we ignore the buoyancy term due to the temperature (ignoring the term $\mathrm{Gr}\, T/\mathrm{Re}^2$ in (1)). This completely decouples equations (1) and (2) from equations (3) and (4). If we assume that the fan is always on, then the distributed parameter control problem can be built from equation (3) with $\mathbf{v}$ computed by solving the steady-state Navier-Stokes equations (also ignoring the time derivative term in (1)). The treatment of scalar variables and the temperature are similar, so we only consider the control problem for the temperature.

Using Linear Quadratic Regulator (LQR) control, system (3) takes the form of a differential equation on a Hilbert space Z,

$$\dot{z}(t) = Az(t) + Bu(t), \; z(0) = z_0 \tag{5}$$

where $[z(t)](\mathbf{x}) = T(t, \mathbf{x})$. The objective is to find the control that minimizes

$$J(u) = \int_0^\infty \left[\langle Qz(t), z(t) \rangle_Z + \langle Ru(t), u(t) \rangle \right] dt, \tag{6}$$

subject to (5), where Q corresponds to a characteristic function in a workspace (see Fig. 1). Under reasonable conditions, see [10], an optimal control exists and has the form

$$u_*(t) = -\mathcal{K}z(t), \tag{7}$$

where $\mathcal{K} : Z \to \mathbb{R}$ is a bounded linear "gain" operator. In addition, $\mathcal{K} = R^{-1}B^*\Pi$ where $\Pi : Z \to Z$ is a bounded linear operator, $\Pi = \Pi^*$ and Π satisfies the Riccati equation

$$A^*\Pi + \Pi A - \Pi B R^{-1} B^* \Pi + Q = 0. \tag{8}$$

Under the assumption that the flow is fixed and only the temperature of the air is controlled, the Riesz Representation Theorem implies that there exists a function $k_T(\mathbf{x})$ such that

$$\mathcal{K}z(t) = \int_\Omega k_T(\mathbf{x}) T(t, \mathbf{x}) d\mathbf{x}. \tag{9}$$

The kernel $k_T(\mathbf{x})$ is called a functional gain. The functional gains define the optimal LQR controller and can be used to place sensors and design low order controllers (see [11], [12], [13], [14], [15]).

Assume we place one sensor on a wall with support in the region $\Omega_1 \subset \Omega$, then the sensed output is given by

$$y(t) = \int_{\Omega_1} c(\mathbf{x}) T(t, \mathbf{x}) d\mathbf{x} = Cz(t). \tag{10}$$

It is often not possible to locate sensors in the "optimal" places and/or to have full state information even with optimal sensor location. Hence, one must construct a state estimator (observer) and use sensed information to construct (partial) state information needed for the control. Therefore, we consider a standard linear state estimator (observer) of the form

$$\dot{z}_e(t) = \mathcal{A}_e z_e(t) + \mathcal{F} y(t), \tag{11}$$

where $\mathcal{F} : \mathbb{R} \to Z$ has the representation

$$[\mathcal{F}y](\mathbf{x}) = f_T(\mathbf{x}) y$$

where the function f_T is the observer functional gain.

III. SENSOR PLACEMENT METRICS

A. *Optimal LQG Cost*

A Linear Quadratic Gaussian (LQG) control would have the form of (5) with an additional noise term,

$$\dot{z}(t) = \mathcal{A} z_e(t) + Bu(t) + G\eta(t), \; z(0) = z_0. \tag{12}$$

The additional term $G(\eta)$ accounts for model errors or unmodeled dynamics (additional measurement error models could be added to (10) but have not been included here). The control is given by $u(t) = -\mathcal{K} z_e(t)$ and rather than computing the optimal LQR cost, which does not have any information about the sensor, we consider the LQG cost functional

$$\mathcal{J}_{LQG}(\Omega_1) \equiv \int_0^{t_f} \left[\langle Qz(t), z(t) \rangle_Z + \langle R\mathcal{K} z_e(t), \mathcal{K} z_e(t) \rangle \right] dt, \tag{13}$$

for a sufficiently large t_f. Finding the sensor location (Ω_1) that produces the best state estimate for the control problem, will produce a minimum LQG cost. Note that there are a number of parameters in this metric: the structure of the operator G, the value of t_f, and the initial state and state estimates, $z(0)$ and $z_e(0)$. We note here that this leads to a very expensive computational cost since it includes the cost of solving two Riccati equations (for the control gain and for the observer gain) as well as the cost of numerically integrating the coupled $[z; \; z_e]^T$ system.

B. *Norm of the Observer Gain*

Our second proposed metric is the norm of the observer gain. Specifically, for a given Ω_1, we use finite element methods to compute approximations A_n, C_n and G_n to A, $C = C(\Omega_1)$, G, respectively. An approximation to the observer gain f_T can be found by solving the observer Riccati equation

$$PA_n^* + A_n P - PC_n^* C_n P + G_n G_n^* = 0,$$

whence $f_n = PC_n^*$. The observer gain provides a mapping from observations y to their contribution to the observer equation (11). Thus, if this gain is large, it indicates that observations are "better felt" by the observer. Therefore, we propose using the norm of the observer gain as a metric for sensor placement. This metric has the distinct advantage of being the most computationally feasible for complex problems as there are methods available for computing the action of the matrix P on a vector without having to fully compute P first, cf. [16], [17], [18]. The previous metric also shares this advantage in that only $\mathcal{K}$ and $\mathcal{F}$ are needed, and Π and P don't have to be formed explicitly.

C. *Minimum Error Variance*

A well known result [19], [20] of observers is that the *trace* of P gives the variance of $e(t) \equiv z(t) - z_e(t)$. Therefore, finding the $P = P(\Omega_1)$ with *minimum* trace is a good metric for determining sensor placement. Unfortunately, this requires the full solution to the observer Riccati equation and is expensive for large problems.

D. Distance to Nearest Unobservable System

Traditional methods used to determine a yes/no result about the observabilty (alternatively controllability) of a system, may lead to incorrect conclusions, see [21]. Therefore a continuous metric, the distance between the original observable (controllable) system and the nearest unobservable (uncontrollable) system is used.

This measure of the distance to unobservability is defined by

$$\begin{aligned}\tau(A_n, C_n) \quad \equiv \quad & \min\{\|\Delta A_n \ \Delta C_n\| \text{ such that the system} \\ & (A_n + \Delta A_n,\ C_n + \Delta C_n) \text{ is unobservable}\}\end{aligned}$$

where $\|\cdot\|$ denotes the spectral or Frobenius norms.

This measure is equivalent to the singular value minimization problem, see [22], [23],

$$\tau(A_n, C_n) = \inf_{\lambda \in \mathbb{C}} \sigma_{\min}[A_n^T - \lambda I \quad C_n^T]$$

where $\sigma_{\min}[X]$ denotes the smallest singlar value of $X \in \mathbb{C}^{n\times n}$.

There exist a vast literature on algorithms designed to compute $\tau(A_n, C_n)$, see for example [24], [25], [26], [27], [28], [29], [30], [31], [32], [33], [34], [35], [36], [37]. Each of these algorithms has significant limitations. For example, the local minimum found is not necessarily the global minimum, or the computing time is inversely proportional to $\tau^2(A_n, C_n)$ which is extremely expensive for systems that are nearly uncontrollable.

The recent breakthrough by Gu [38], lead to polynomial time algorithms by Gu *et al.*, [39]. Mengi [40], proposed an algorithm that reduces the cost, on average, from $O(n^6)$ to $O(n^4)$.

IV. NUMERICAL RESULTS

A finite element method with 33 points in each direction (producing a mesh with 143,360 tetrahedral Taylor-Hood elements) is used for the flow simulation. The inlet heating duct is located at $\{(0, y, z) \mid 0.375 < y < 0.625,\ 0.75 < z < 0.875\}$, and the return vent is located at the $x = 1$ wall where y ranges from 0.375 to 0.625 and z ranges from 0.125 to 0.375. A single sensor is now placed at different locations on the wall $0 < x < 1$, $y = 0$, $0 < z < 1$. Each sensor location corresponds to an interior vertex of the tetrahedra. This results in 225 different locations and the different measures are computed for each of these locations.

The feedback functional gain as described in (9) is presented in Fig. 2. This gain has most support near the center area, $\Omega_Q = (1/3, 2/3)^3$, where the temperature is being controlled, see Fig. 1. This suggests that the sensor should ideally be placed in that region Ω_Q, but is not possible since we only consider wall sensors.

A. Optimal LQG Cost

In this preliminary study, we computed the optimal LQG cost (13) where we selected $G = 1$. Due to the enormous computational time required for this metric, we selected only the four corners of the $y = 0$ wall as four sensor locations. For each of these sensor locations we ran the simulations for $t_f = 1$. The results differed in the 13th significant digit due to the fact that the control had yet to produce a significant difference between the estimated solutions. However, this study favored the sensor location at $(x, z) = (.8125, .1875)$ over the locations at $(.1875, .1875)$, $(.1875, .8125)$, and $(.8125, .8125)$. We are currently in the process of accurately integrating the solutions over substantially long simulation times so that differences in this metric are greater than numerical integration errors. Based on the enormous computational time required to integrate these stiff equations, we do not expect this metric to be of practical use, but to be used in this study to validate the other proposed metrics.

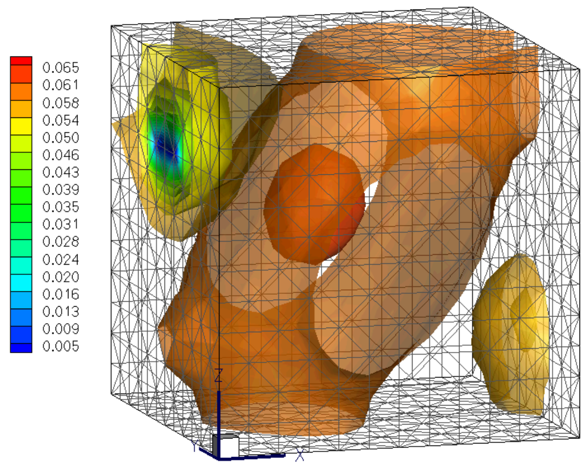

Fig. 2. Feedback functional gain.

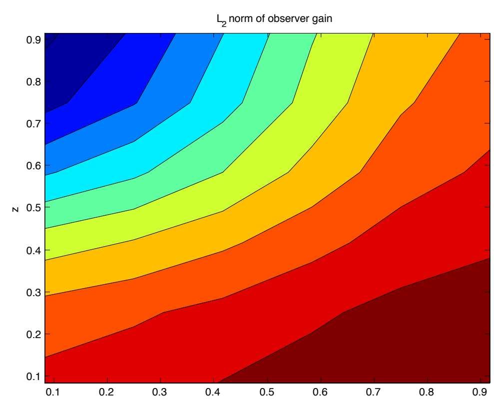

Fig. 3. L_2-norm of observer gain.

B. Norm of the Observer Gain

In Fig. 3 the L_2-norm of the observer functional gain computed for each of the 225 sensor locations associated with the interior vertices of the tetrahedra on the wall

$(x, 0, z), 0 < x < 1,\ 0 < z < 1$, is presented. The support of $f_T(\mathbf{x})$ is the smallest if the sensor is placed at the top left corner of the wall, see Fig. 4, and largest if the sensor is placed at the bottom right corner of the wall, see Fig. 5. This indicates that the sensor should ideally be placed near the bottom right corner of the wall.

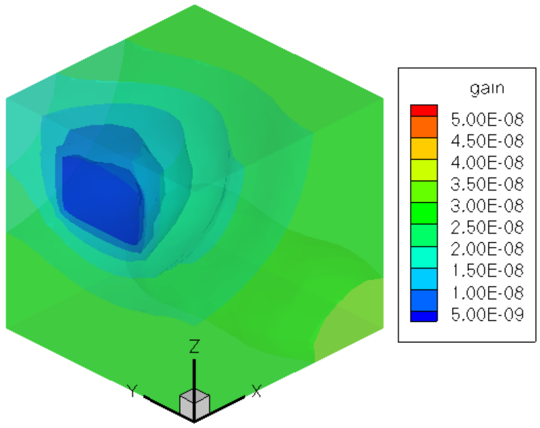

Fig. 4. Observer functional gain, f_T, for sensor in top left corner.

C. Minimum Error Variance

This optimal sensor location in the bottom right hand corner is also confirmed by computing the trace of the solution to the observer Riccati equation which is a measure of the expected value of $\|z(t) - z_e(t)\|^2$. The trace is again computed for each of the 225 sensor locations on the wall $(x, 0, z),\ 0 < x < 1,\ 0 < z < 1$, and is presented in Fig. 6. The trace is smallest in the bottom right hand corner. This indicates that the best location to place the observer is nearest the return vent.

D. Distance to Nearest Unobservable System

These results are also confirmed by the observability radius for different sensor locations on the wall. In Fig. 7 the observability radius for the different sensor locations on the wall $(x, 0, z), 0 < x < 1,\ 0 < z < 1$, is presented.

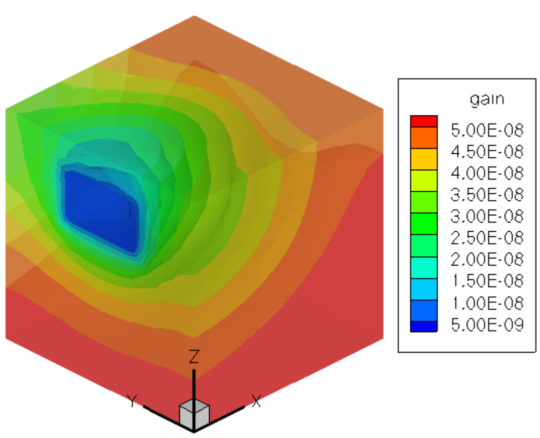

Fig. 5. Observer functional gain, f_T, for sensor in bottom right corner.

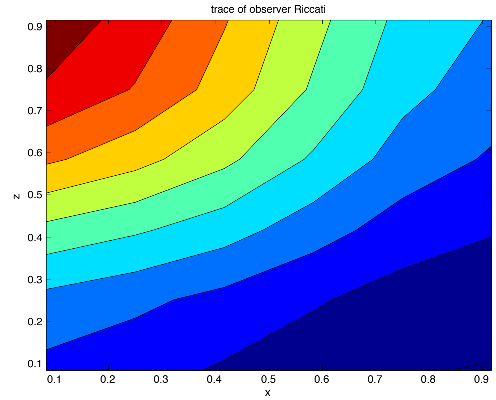

Fig. 6. Trace of solution to observer Riccati equation.

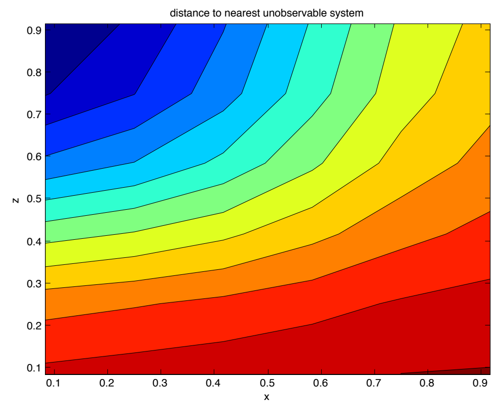

Fig. 7. Observability radius for different sensor locations on the wall.

Again, the distance to the nearest unobservable system is largest near the bottom right corner. This implies that the sensor should be placed in that region close to the return vent.

V. CONCLUSIONS AND FUTURE WORK

In this study, we proposed a number of metrics for evaluating the location of sensors in a room. Each metric indicated the well-known result in HVAC design of placing temperature sensors as close to the return vent as possible. We note that the use of the feedback functional gain would indicate the obvious choice of locating the sensor in the workspace (where the air quality needs to be assured). However, simply using the value of the functional gain along walls would lead to misleading results (see Fig. 2).

The metrics presented here are all computationally challenging to implement for HVAC design problems. This is due to the complex dynamics and discretization sizes that are used to approximate the distributed parameter control problem. Both B.) the norm of the observer gain and C.) the minimum error variance produced adequate sensor placement for the least computational cost. The distance to the nearest unobservable system (D.) is clearly a metric that one would like to maximize, but theoretical underpinnings need to be addressed before this is offered as a production tool. For example, the meaning of this quantity (τ) under mesh refinement and the difficulty to compute this quantity when the system is nearly unobservable. Finally, we would like to have a more complete picture of the optimal LQG cost (A.), however simulation times dramatically increase when the fully coupled state and state estimator equations are integrated. This will require much longer simulation times.

To date, most studies of sensor placement questions have been concerned with one and two dimensional problems. However, the appearance of new algorithms make some of these metrics practicable for complex systems, such as the original flow model given in Section II with realistic parameter values. The ability to use model reduction methods to compute functional gains for systems with low numbers of control inputs and sensors allow us to avoid explicit calculation of the Riccati solution regardless of whether we have low-rank matrices [16] or high-rank matrices [17]. The optimal LQG cost and the norm of the observer gain can then be feasibly computed for complex cases: i.e., the fully coupled (but linearized) equations (1)-(4). More development of these algorithms is needed to enable the practical use of CFD along with distributed parameter control theory in the design of high-performance, energy efficient buildings.

VI. ACKNOWLEDGMENTS

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