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Optimal Controller Synthesis for a Decentralized Two-Player System with Partial Output Feedback

John Swigart[1] Sanjay Lall[2]

Abstract

In this paper, we derive the optimal control policy for a decentralized control problem. The system considered here consists of two interconnected subsystems with communication allowed in only one direction. In addition, full state feedback is not assumed, as in previous instances of this problem. We construct the optimal controllers via a spectral factorization approach. Explicit state-space formulae are provided, and the orders of the optimal controllers are established.

1 Introduction

This paper addresses optimal controller synthesis for a decentralized two-player system. This problem is an example of a larger class of distributed control problems, consisting of multiple subsystems interacting over a network with limited communication. Many important practical problems fall into this category. Examples include formation flight, coordination of teams of vehicles, or large spatially distributed systems such as the internet or the power grid.

In general, decentralized control problems are currently intractable [2]. A classic example illustrates that, even for linear, time-invariant systems, linear control policies can be strictly suboptimal [20]. As a result, much of the research in decentralized control has been aimed at characterizing which problems are tractable [3, 6, 8, 1]. The most general of these results is based on the concept of quadratic invariance [11]. For systems connected over networks, recent results have determined which class of networked systems are currently tractable [10, 15, 13].

Tractability, as it is used here, means that the underlying optimization problem has a convex representation. Though convex, the formulations remain infinite-dimensional. In particular, the approach in [11] requires a change of variables via the Youla parametrization, and optimization over this parameter. Since the parameter itself is a linear stable system, a standard approximation would be via a finite basis for the impulse response function. This is in contrast to the centralized case, for which explicit state-space formulae can be constructed.

This paper focuses on a specific information structure, consisting of two interconnected systems with dynamics such that player 1's state affects the state of player 2. Our objective is to find a pair of controllers such that player 1 can measure only the first state, whereas player 2 has access to the first state and a noisy measurement of the second state. The controller is chosen to minimize the $\mathcal{H}_2$ norm of the closed-loop transfer function.

This work follows a series of papers on this problem. In [17], explicit formulae for the optimal controllers were constructed for the finite-horizon, time-varying, state feedback version of this problem via a spectral factorization approach. These results were reproduced in [16], using a dynamic programming method. The infinite-horizon, state feedback problem was solved in [18]. This paper is the first to solve this problem in which full state feedback is not required. In addition to explicit state-space formulae, we provide intuition to the structure of the optimal controllers. Moreover, we establish the order of the optimal controller for this system, which is an open problem for general decentralized systems, even in the simplest cases. In particular, it differs from the results obtained in [18] for the full state feedback case.

Our approach makes use of spectral factorization. The methods used here extend naturally to more general networks. The two-player problem considered here provides the fundamental understanding for the general case.

Many different approaches have been taken to find numerical solutions to some of these problems. Some methods were suggested, though not implemented, in [19]. Semidefinite programming approaches have been presented in [12, 9, 21] for similar problems. For the quadratic case, vectorization [11] provides a finite-dimensional approach, but loses the intrinsic structure and results in high-order controllers.

However, in none of these previous approaches have explicit state-space formulae been derived. Such formulae offer the practical advantages of computational reliability and simplicity, as well as provide understanding and interpretation of the controller structure. Moreover, we establish the order of the optimal controller which previous approaches do not provide.

[1] J. Swigart is with the Department of Aeronautics and Astronautics, Stanford University, Stanford, CA 94305, USA.
jswigart@stanford.edu
[2] S. Lall is with the Department of Electrical Engineering and Department of Aeronautics and Astronautics, Stanford University, Stanford, CA 94305, USA.
lall@stanford.edu

2 Problem Formulation

The following notation is used in this paper. The real and complex numbers are denoted by $\mathbb{R}$ and $\mathbb{C}$, respectively. The complex unit disc is $\mathbb{D}$, and its boundary, the unit circle, is $\mathbb{T}$. The set $\mathcal{L}_2(\mathbb{T})$ is the Hilbert space of Lebesgue measurable functions which are square integrable on $\mathbb{T}$. As is standard, $\mathcal{H}_2$ denotes the Hardy space of functions analytic outside the closed unit disc, and at infinity, with square-summable power series. The set $\mathcal{H}_2^\perp$ is the orthogonal complement of $\mathcal{H}_2$ in $\mathcal{L}_2$.

Also, $\mathcal{L}_\infty(\mathbb{T})$ denotes the set of Lebesgue measurable functions bounded on $\mathbb{T}$. Similarly, $\mathcal{H}_\infty$ is the subspace of $\mathcal{L}_\infty$ with functions analytic outside of $\mathbb{T}$, and $\mathcal{H}_\infty^-$ is the subspace of $\mathcal{L}_\infty$ with functions analytic inside $\mathbb{T}$.

The prefix $\mathcal{R}$ indicates the subsets of proper real rational functions. That is, $\mathcal{RL}_2$ is the set of transfer functions with no poles on $\mathbb{T}$, and $\mathcal{RH}_2$ is the set of transfer functions with no poles outside $\mathbb{T}$. Note that, in this case, $\mathcal{RH}_2 = \mathcal{RH}_\infty$; we will use these spaces interchangeably.

Some useful facts about these sets which we will make use of in this paper are [22]:

- if $G(z) \in \mathcal{L}_\infty$, then $G(z)\mathcal{L}_2 \subset \mathcal{L}_2$
- if $G(z) \in \mathcal{H}_\infty$, then $G(z)\mathcal{H}_2 \subset \mathcal{H}_2$
- if $G(z) \in \mathcal{H}_\infty^-$, then $G(z)\mathcal{H}_2^\perp \subset \mathcal{H}_2^\perp$

For transfer functions $F \in \mathcal{RL}_2$, we use the notation

$$F(z) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] = C(zI - A)^{-1}B + D$$

We are interested in the following state-space system

$$\begin{bmatrix} x_1(t+1) \\ x_2(t+1) \end{bmatrix} = \begin{bmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} B_{11} & 0 \\ B_{21} & B_{22} \end{bmatrix} \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix} + \begin{bmatrix} H_1 & 0 \\ 0 & H_2 \end{bmatrix} \begin{bmatrix} w_1(t) \\ w_2(t) \end{bmatrix}$$

This corresponds to the two-player system, in which player 1's state can influence player 2's state. In general, noisy measurements y_1 and y_2 of each state are made. For this system, we consider a *partial output feedback* system. That is, player 1's state is measured directly, and player 2's measured output is

$$y_2(t) = C_{21}x_1(t) + C_{22}x_2(t) + H_v v(t)$$

Note that $w_1(t)$, $w_2(t)$, and $v(t)$ are independent, exogenous noise. We are interested in finding transfer functions $\mathcal{K}_{11}, \mathcal{K}_{21}, \mathcal{K}_{22} \in \mathcal{RL}_\infty$ so that our controller is of the form

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} \mathcal{K}_{11} & 0 \\ \mathcal{K}_{21} & \mathcal{K}_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ y_2 \end{bmatrix}$$

That is, player 1 makes decision u_1 based only on the history of his own state x_1, while player 2 makes decision u_2 based on the history of both outputs, x_1 and y_2.

For a set T, we define lower(T) to be the set of 2×2 block lower triangular matrices with elements in T. In particular, our desired controllers are in the set $\mathcal{K} \in \text{lower}(\mathcal{RL}_\infty)$. It will also be convenient to define $E_1 = \begin{bmatrix} I & 0 \end{bmatrix}^T$ and $E_2 = \begin{bmatrix} 0 & I \end{bmatrix}^T$, where the dimensions are defined by the context.

We define $\mathcal{S} = \text{lower}(\mathcal{RH}_2) \subset \mathcal{L}_2$, and note that $\mathcal{S}$ has an orthogonal complement, such that $G \in \mathcal{S}^\perp$ if and only if $G_{11}, G_{21}, G_{22} \in \mathcal{H}_2^\perp$ and $G_{12} \in \mathcal{L}_2$. We will also define $\mathrm{P}_{\mathcal{H}_2} : \mathcal{L}_2 \to \mathcal{H}_2$ as the orthogonal projection onto $\mathcal{H}_2$.

Our cost is the vector

$$z(t) = \begin{bmatrix} C_{11} & C_{12} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} D_1 & D_2 \end{bmatrix} \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix}$$

where, for simplicity, we will assume that $D^T D > 0$. Notice that this formulation allows for coupling of the states and actions in the cost. Consequently, our plant can be expressed as the matrix $P \in \mathcal{RL}_\infty$, which is the mapping $(w, v, u) \mapsto (z, x_1, y_2)$ given by

$$\left[\begin{array}{c|c} P_{11} & P_{12} \\ \hline P_{21} & P_{22} \end{array}\right] = \left[\begin{array}{c|cc|c} A & H & 0 & B \\ \hline C_1 & 0 & 0 & D \\ \hline E_1^T & 0 & 0 & 0 \\ C_2 & 0 & H_v & 0 \end{array}\right] \tag{1}$$

where H_1 and H_2 are invertible. Note that H being invertible simply implies that no component of the state evolves noise-free. This assumption merely simplifies our presentation, while not fundamentally affecting our results. Lastly, we define $\mathcal{F}(P, \mathcal{K})$ as the linear fractional transformation

$$\mathcal{F}(P, \mathcal{K}) = P_{11} + P_{12}\mathcal{K}(I - P_{22}\mathcal{K})^{-1}P_{21}$$

The objective function is the $\mathcal{H}_2$ norm of the closed-loop transfer function from (w, v) to z. In other words, we have the following optimization problem.

$$\begin{aligned} \text{minimize} \quad & \|\mathcal{F}(P, \mathcal{K})\|_2 \\ \text{subject to} \quad & \mathcal{K} \text{ is stabilizing} \\ & \mathcal{K} \in \text{lower}(\mathcal{RL}_\infty) \end{aligned} \tag{2}$$

3 Main Results

Using the notation of the previous section, we can now solve the optimization problem in (2). To this end, the following assumptions will be made throughout the paper.

A1) A is stable

A2) $D^T D > 0$ and $H_v H_v^T > 0$

A3) $\begin{bmatrix} A - \lambda I & B \\ C_1 & D \end{bmatrix}$ has full column rank for all $\lambda \in \mathbb{T}$

A4) $\begin{bmatrix} A_{22} - \lambda I & H_2 & 0 \\ C_{22} & 0 & H_v \end{bmatrix}$ has full row rank for all $\lambda \in \mathbb{T}$

We will discuss the need for these assumptions in Section 4. We state the main results here. The remaining sections will develop the proof of these results.

Theorem 1. *For the system in* (1)*, suppose assumptions A1–A4 hold. Let* X*,* Y*, and* S *be the stabilizing solutions to the algebraic Riccati equations*

$$X = C_1^T C_1 + A^T X A - (A^T X B + C_1^T D) \times (D^T D + B^T X B)^{-1} (B^T X A + D^T C_1) \quad (3)$$

$$Y = C_{12}^T C_{12} + A_{22}^T Y A_{22} - (A_{22}^T Y B_{22} + C_{12}^T D_2) \times (D_2^T D_2 + B_{22}^T Y B_{22})^{-1} (B_{22}^T Y A_{22} + D_2^T C_{12}) \quad (4)$$

$$S = H_2 H_2^T + A_{22} S A_{22}^T - A_{22} S C_{22}^T (H_v H_v^T + C_{22} S C_{22}^T)^{-1} C_{22} S A_{22}^T \quad (5)$$

Define

$$K = (D^T D + B^T X B)^{-1} (B^T X A + D^T C_1) \quad (6)$$

$$J = (D_2^T D_2 + B_{22}^T Y B_{22})^{-1} (B_{22}^T Y A_{22} + D_2^T C_{12}) \quad (7)$$

$$N = S C_{22}^T (H_v H_v^T + C_{22} S C_{22}^T)^{-1} \quad (8)$$

and let

$$A_K = A - BK, \ A_J = A_{22} - B_{22} J, \ A_N = (I - N C_{22}) A_J$$

Then, there exists a unique optimal $\mathcal{K} \in \text{lower}(\mathcal{RL}_\infty)$ *for* (2) *given by:*

- *Controller* 1 *has realization*

$$\begin{aligned} q_1(t+1) &= (A_K)_{22} q_1(t) + (A_K)_{21} x_1(t) \\ u_1(t) &= -K_{12} q_1(t) - K_{11} x_1(t) \end{aligned}$$

- *Controller* 2 *has realization*

$$\begin{aligned} q_2(t+1) &= \begin{bmatrix} (A_K)_{22} & 0 \\ A_N N C_{22} & A_N \end{bmatrix} q_2(t) \\ &\quad + \begin{bmatrix} (A_K)_{21} & 0 \\ A_N N C_{21} & -A_N N \end{bmatrix} \begin{bmatrix} x_1(t) \\ y_2(t) \end{bmatrix} \\ u_2(t) &= \begin{bmatrix} -K_{22} + J N C_{22} & J \end{bmatrix} q_2(t) \\ &\quad + \begin{bmatrix} -K_{21} + J N C_{21} & -J N \end{bmatrix} \begin{bmatrix} x_1(t) \\ y_2(t) \end{bmatrix} \end{aligned}$$

Note that Assumptions A3 and A4 guarantee the existence of stabilizing solutions to the algebraic Riccati equations (3–5). This will be discussed in the following section. Having established the form of the optimal controller, a number of remarks are in order.

We see that both controllers have dynamics; player 1's controller has order equal to the state dimension of x_2, and player 2's controller has order of twice that.

While Theorem 1 provides the mathematical form of the optimal controller, additional insight can be gained by considering the estimation structure of the policies. It will be shown in Section 6 that there are two estimation processes occurring in the optimal policies. In a manner to be defined therein, by letting $\hat{x}_{2|1}(t)$ be the estimate of $x_2(t)$ given the history of x_1, and letting $\hat{x}_{2|12}(t)$ be the estimate of $x_2(t)$ given the histories of x_1 and y_2, the optimal control policy can be written as

$$\begin{aligned} u_1(t) &= -K_{11} x_1(t) - K_{12} \hat{x}_{2|1}(t) \\ u_2(t) &= -K_{21} x_1(t) - K_{22} \hat{x}_{2|1}(t) + J\big(\hat{x}_{2|1}(t) - \hat{x}_{2|12}(t)\big) \end{aligned} \quad (9)$$

Thus, the optimal policy is, in fact, attempting to perform the optimal centralized policy, though using $\hat{x}_{2|1}$ instead of x_2. However, there is an additional term in u_2 which represents player 2's estimate of the error that player 1 makes in estimating x_2. We also see that in the case where $y_2 = x_2$, so that $\hat{x}_{2|12} = x_2$, then the optimal distributed controller reduces to the optimal state feedback solution obtained in [18].

4 Analysis

Before proving the results of the previous section, it is worth making a few remarks regarding the assumptions A1–A4.

The stability of the plant (A1) is not necessary in general. In the case where A is unstable, Assumption A1 is replaced by the assumption that (A_{11}, B_{11}) and (A_{22}, B_{22}) are stabilizable and (A_{22}, C_{22}) is detectable. While the results are not fundamentally affected by this assumption, the proofs in the unstable case are significantly more complicated and omitted here for the sake of clarity. See [14, Chapter 6] for a full treatment of the unstable case.

Assumptions A2–A4 are standard assumptions, which guarantee existence and uniqueness of solutions. In particular, the approach used here is known as *spectral factorization* and requires stabilizing solutions of algebraic Riccati equations. A well-known result to this end is as follows.

Lemma 2. *Suppose* $D^T D > 0$*. Then, there exists a unique* $X \in \mathbb{R}^{n \times n}$ *satisfying*

$$X = C^T C + A^T X A - (A^T X B + C^T D) \times (D^T D + B^T X B)^{-1} (B^T X A + D^T C)$$

such that $A - B(D^T D + B^T X B)^{-1}(B^T X A + D^T C)$ *is stable, if and only if* (A, B) *is stabilizable and*

$$\begin{bmatrix} A - \lambda I & B \\ C & D \end{bmatrix} \quad \text{has full column rank for all } \lambda \in \mathbb{T}$$

Proof. See [22] and [7] for proofs. ■

With Lemma 2, Assumptions A1– A4 immediately guarantee the existence of stabilizing solutions to the algebraic Riccati equations (3) and (5). The existence of (4) similarly follows since A1 implies the stability of A_{22}, and the requisite rank condition of Lemma 2 follows from the rank condition in A3.

Having established the motivation for the Assumptions A1–A4, we turn now to solving (2). For the general decentralized control problem, the primary difficulty in finding optimal solutions is that the optimization problem is not convex. In the classical, centralized problem, this difficulty is avoided by introducing a Youla parametrization which results in a convex formulation of the problem. Fortunately, the problem considered here also admits a convex parametrization.

Lemma 3. *Let $\mathcal{S} = \text{lower}(\mathcal{RH}_\infty)$. For the system in (1), suppose A is stable. Then, the set of all stabilizing controllers $\mathcal{K} \in \text{lower}(\mathcal{RL}_\infty)$ is parametrized by*

$$\mathcal{K} = Q(I + P_{22}Q)^{-1} \qquad Q \in \mathcal{S}$$

Moreover, the set of stable closed-loop transfer functions satisfies

$$\begin{aligned}\{\mathcal{F}(P, \mathcal{K}) \mid \mathcal{K} \in \text{lower}(\mathcal{RL}_\infty),\ \mathcal{K} \text{ stabilizing}\}\\ = \{P_{11} + P_{12}QP_{21} \mid Q \in \mathcal{S}\}\end{aligned}$$

Proof. Note that the set $\text{lower}(\mathcal{RL}_\infty)$ is closed under addition, multiplication, and inversion, so that $Q \in \text{lower}(\mathcal{RL}_\infty)$ if and only if $\mathcal{K} \in \text{lower}(\mathcal{RL}_\infty)$. The result then follows from the standard Youla parametrization for the problem. See, for example [22]. ■

As a result of this Youla parametrization, the optimization problem in (2) is equivalent to

$$\begin{aligned} &\text{minimize} && \|P_{11} + P_{12}QP_{21}\|_2 \\ &\text{subject to} && Q \in \mathcal{S} \end{aligned} \tag{10}$$

In order to solve this new optimization problem, it is convenient to find an equivalent optimality condition, provided by the following lemma.

Lemma 4. *Let $\mathcal{S} = \text{lower}(\mathcal{RH}_2)$, and suppose that $P_{11}, P_{12}, P_{21} \in \mathcal{RH}_\infty$. Then, $Q \in \mathcal{S}$ minimizes (10) if and only if*

$$P_{12}^* P_{11} P_{21}^* + P_{12}^* P_{12} Q P_{21} P_{21}^* \in \mathcal{S}^\perp \tag{11}$$

Proof. The proof follows from the classical projection theorem. Since the construction is standard, we omit the proof; for the general idea, see for example [4]. ■

We will solve this optimality condition in the next section.

5 Spectral Factorization

Our goal is now to find a solution $Q \in \mathcal{S}$ which satisfies the optimality condition (11). Before solving this problem, let us first compare it to the full state feedback case considered in [18].

The difference here is that player 2 has output feedback instead of state feedback. Mathematically, this means that P_{21} is no longer invertible in $\mathcal{S}$, as it was for the full state feedback case. In that case, invertibility of P_{21} meant that we could group QP_{21} as a new variable $\hat{Q} \in \mathcal{S}$. This additional parametrization decoupled the columns of the optimality condition and allowed us to solve for $\hat{Q}$ block by block.

Unfortunately, when output feedback is introduced into the problem, as it is here, this parametrization is not possible. More specifically, an additional step is required before such a parametrization is possible. To this end, we have the following result.

Lemma 5. *For the system in (1), suppose Assumptions A1– A4 hold. Then, there exists $G \in \mathcal{S}$, such that*

$$G^{-1} \in \mathcal{S} \qquad GG^* = P_{21}P_{21}^* \qquad \mathcal{S}^\perp G^{-*} \subset \mathcal{S}^\perp$$

Moreover, such a $G \in \mathcal{S}$ is given by

$$G = \left[\begin{array}{c|cc} A & AHE_1 & E_2A_{22}NV^{\frac{1}{2}} \\ \hline E_1^T & H_1 & 0 \\ C_2 & C_2HE_1 & V^{\frac{1}{2}} \end{array}\right]$$

where $V = H_vH_v^T + C_{22}SC_{22}^T$, with S and N satisfying (5) and (8), respectively.

Proof. First, notice that G_{11} and G_{22} satisfy

$$\begin{aligned} G_{11}^{-1} &= H_1^{-1} - z^{-1}H_1^{-1}A_{11} \\ G_{22}^{-1} &= V^{-\frac{1}{2}}(C_{22}(zI - A_{22}(I - NC_{22}))^{-1}A_{22}N + I) \end{aligned}$$

Thus, $G_{11}^{-1}, G_{22}^{-1} \in \mathcal{RH}_\infty$, so $G^{-1} \in \mathcal{S}$.

As a consequence of this, we see that $G_{11}^{-*}, G_{21}^*, G_{22}^{-*} \in \mathcal{H}_\infty^-$. Thus, for any $\Lambda \in \mathcal{S}^\perp$, it is straightforward to show that the product of $\Lambda G^{-*} \in \mathcal{S}^\perp$.

Our last step is to show that $GG^* = P_{21}P_{21}^*$. To this end, note that $G_{11} = z(P_{21})_{11}$ and $G_{21} = z(P_{21})_{21}$. Lastly, algebraic manipulations of the Riccati equation (5) shows that $G_{22}G_{22}^* = (P_{21})_{22}(P_{21})_{22}^*$. This is a standard spectral factorization result and is the dual of Lemma 7, to follow. See [5] for a simple proof. As a result, we have

$$\begin{aligned} P_{21}P_{21}^* &= \begin{bmatrix} z^{-1}G_{11} & 0 \\ z^{-1}G_{21} & (P_{21})_{22} \end{bmatrix} \begin{bmatrix} zG_{11}^* & zG_{21}^* \\ 0 & (P_{21})_{22} \end{bmatrix} \\ &= \begin{bmatrix} G_{11}G_{11}^* & G_{11}G_{21}^* \\ G_{21}G_{11}^* & G_{21}G_{21}^* + (P_{21})_{22}(P_{21})_{22}^* \end{bmatrix} \\ &= \begin{bmatrix} G_{11}G_{11}^* & G_{11}G_{21}^* \\ G_{21}G_{11}^* & G_{21}G_{21}^* + G_{22}G_{22}^* \end{bmatrix} = GG^* \end{aligned}$$

■

With Lemma 5, the optimality condition (11) is equivalent to

$$P_{12}^* P_{11} P_{21}^* G^{-*} + P_{12}^* P_{12} \hat{Q} \in \mathcal{S}^\perp \tag{12}$$

where, since G is now invertible $\mathcal{S}$, we have grouped QG as the new variable $\hat{Q} \in \mathcal{S}$. As a result, we have once again decoupled the columns of the optimality condition. This allows for us to solve for $\hat{Q}$, as follows.

Lemma 6. *Let $\mathcal{S} = \text{lower}(\mathcal{RH}_2)$, and suppose $F, G \in \mathcal{RH}_\infty$. Then, $Q \in \mathcal{S}$ satisfies*

$$G^*F + G^*GQ \in \mathcal{S}^\perp$$

if and only if the following two conditions both hold:

i) $\begin{bmatrix}(G^*F)_{11}\\(G^*F)_{21}\end{bmatrix} + G^*G\begin{bmatrix}Q_{11}\\Q_{21}\end{bmatrix} \in \mathcal{H}_2^\perp$

ii) $(G^*F)_{22} + (G^*G)_{22}Q_{22} \in \mathcal{H}_2^\perp$

Proof. This result was proved in [18]. In short, (i) comes from solving for the first column of Q and (ii) from the second column of Q. ■

The importance of Lemma 6 is that it reduces (12) over $\mathcal{S}^\perp$ into two separate conditions over $\mathcal{H}_2^\perp$. Each of these conditions can be solved via a spectral factorization approach, described by the following lemmas.

Lemma 7. *Suppose $R_1, R_2 \in \mathcal{RH}_\infty$ have the realizations*

$$R_1 = C(zI - A)^{-1}H$$
$$R_2 = C(zI - A)^{-1}B + D$$

Suppose there exists a stabilizing solution X to the algebraic Riccati equation

$$X = C^TC + A^TXA - (A^TXB + C^TD) \times (D^TD + B^TXB)^{-1}(B^TXA + D^TC)$$

Let $W = D^TD + B^TXB$ and $K = W^{-1}(B^TXA + D^TC)$ and $L \in \mathcal{RH}_\infty$ given by

$$L = W^{\frac{1}{2}}K(zI - A)^{-1}B + W^{\frac{1}{2}}$$

Then, $L^{-1} \in \mathcal{RH}_\infty$, $L^{-*} \in \mathcal{RH}_\infty^-$, and $L^*L = R_2^*R_2$. Moreover,

$$L^{-*}R_2^*R_1 = z^{-1}W^{-\frac{1}{2}}B^T(z^{-1}I - (A - BK)^T)^{-1}XH + W^{\frac{1}{2}}K(zI - A)^{-1}H$$

Proof. This is a standard spectral factorization result. A simple proof follows the approach in [5]. ■

Lemma 8. *For the system in (1), let $G \in \mathcal{S}$ be defined as in Lemma 5. Then,*

$$P_{21}^*G^{-*} = \begin{bmatrix} zI & 0\\ 0 & (P_{21})_{22}^*G_{22}^{-*}\end{bmatrix}$$

and

$$(zI - A_{22})^{-1}H_2E_1^T(P_{21})_{22}^*G_{22}^{-*} = (zI - A_{22})^{-1}A_{22}NV^{\frac{1}{2}} + z^{-1}S(z^{-1}I - (I - NC_{22})^TA_{22}^T)^{-1}C_{22}^TV^{-\frac{1}{2}}$$

Proof. This result follows directly from the construction of G and algebraic manipulations of (5), as in Lemma 7. ■

We can now solve for the $\hat{Q} \in \mathcal{S}$ satisfying our optimality condition (12)

Lemma 9. *For the system in (1), suppose Assumptions A1– A4 hold. Let X, Y, S satisfy (3)–(5), respectively. Also, define K, J, N as in (6)–(8), and let $G \in \mathcal{S}$ be defined as in Lemma 5. Then, the unique $\hat{Q} \in \mathcal{S}$ satisfying (12) is given by*

$$\begin{bmatrix}\hat{Q}_{11}\\ \hat{Q}_{21}\end{bmatrix} = -zK(zI - (A - BK))^{-1}E_1H_1 \tag{13}$$

$$\hat{Q}_{22} = -zJ(zI - (A_{22} - B_{22}J))^{-1}NV^{\frac{1}{2}} \tag{14}$$

Proof. Using Lemma 6, the optimality condition can be solved as two separate problems. Condition (i) of the lemma implies that we must find $\hat{Q}E_1 \in \mathcal{RH}_2$ satisfying

$$P_{12}^*P_{11}P_{21}^*G^{-*}E_1 + P_{12}^*P_{12}\hat{Q}E_1 \in \mathcal{H}_2^\perp \tag{15}$$

From Lemma 8, we have $P_{21}^*G^{-*}E_1 = zE_1$. Letting $R_1 = P_{11}E_1$ and $R_2 = P_{12}$, we define $L \in \mathcal{RH}_\infty$ as in Lemma 7, via (3). Since $L^{-*} \in \mathcal{H}_\infty^-$, then $L^{-*}\mathcal{H}_2^\perp \subset \mathcal{H}_2^\perp$, and (15) is equivalent to

$$zL^{-*}P_{12}^*P_{11}E_1 + L\hat{Q}E_1 \in \mathcal{H}_2^\perp$$

Consequently, $\hat{Q}E_1$ can be found by projecting this expression onto $\mathcal{H}_2$. Using Lemma 7, we obtain

$$\hat{Q}E_1 = -L^{-1}\mathrm{P}_{\mathcal{H}_2}(zL^{-*}P_{12}^*P_{11}E_1) = -L^{-1}(zW^{\frac{1}{2}}K(zI - A)^{-1}E_1H_1)$$

from which (13) follows. For condition (ii), we must find $\hat{Q}_{22} \in \mathcal{RH}_2$ satisfying

$$E_2^TP_{12}^*P_{11}P_{21}^*G^{-*}E_2 + E_2^TP_{12}^*P_{12}E_2\hat{Q}_{22} \in \mathcal{H}_2^\perp \tag{16}$$

From Lemma 8, we have $P_{21}^*G^{-*}E_2 = E_2(P_{21})_{22}^*G_{22}^{-*}$. This time, we let $R_2 = C_{12}(zI - A_{22})^{-1}B_{22} + D_2$, and now define $L \in \mathcal{RH}_\infty$ according to Lemma 7, via (4). Then, (16) is equivalent to

$$L^{-*}(P_{12}^*P_{11})_{22}(P_{21})_{22}^*G_{22}^{-*} + L\hat{Q}_{22} \in \mathcal{H}_2^\perp$$

Projecting this expression onto $\mathcal{H}_2$ follows from Lemmas 7 and 8; the details are omitted due to space constraints.

$$\mathrm{P}_{\mathcal{H}_2}(L^{-*}(P_{12}^*P_{11})_{22}(P_{21})_{22}^*G_{22}^{-*}) = W^{\frac{1}{2}}J(I + (zI - A_{22})^{-1}A_{22})NV^{\frac{1}{2}}$$

The result follows by left-multiplying this expression with L^{-1}. ■

Having found the $\hat{Q} \in \mathcal{S}$ which satisfies (12), we can now prove our main results.

Proof of Theorem 1. From Lemmas 3 and 4, $\mathcal{K} \in \text{lower}(\mathcal{RL}_2)$ is optimal for (2) if and only if $Q = \mathcal{K}(I - P_{22}\mathcal{K})^{-1} \in \mathcal{S}$ satisfies (11). Spectral factorizing $P_{21}P_{21}^*$ according to Lemma 5, the optimality condition is equivalent to (12), where $\hat{Q} = QG$. The unique $\hat{Q}$ satisfying this expression was found in Lemma 9. Thus, the optimal $\mathcal{K}$ is given by

$$\mathcal{K} = \hat{Q}G^{-1}(I + P_{22}\hat{Q}G^{-1})^{-1}$$

from which it follows that

$$\mathcal{K} = \left[\begin{array}{cc|cc} (A_K)_{22} & 0 & (A_K)_{21} & 0 \\ A_N N C_{22} & A_N & A_N N C_{21} & -A_N N \\ \hline -K_{12} & 0 & -K_{11} & 0 \\ -K_{22} + JNC_{22} & J & -K_{21} + JNC_{21} & -JN \end{array}\right] \tag{17}$$

In state-space this corresponds to the controllers in the theorem. ∎

6 Estimation

We end our analysis of this problem by discussing the structure of the optimal controller. To begin, we see that the order of the optimal controller in (17) is equal to twice the state dimension of player 2.

Corollary 10. *The optimal controller $\mathcal{K} \in \text{lower}(\mathcal{RL}_\infty)$ for (2) has order equal to twice the state dimension of x_2.*

This is reasonable since, as we will show next, the dynamics associated with the controller correspond to each player's estimate of x_2.

Recalling our definitions from Theorem 1, let η_1 and η_2 be the states of the optimal controller in (17). Combining this with the dynamics in (1), and letting $e_1 = x_2 - \eta_1$ and $e_2 = \eta_2 + (I - NC_{22})e_1$, the closed-loop dynamics mapping $(w_1, w_2, v) \mapsto (x_1, y_2)$ become

$$\left[\begin{array}{ccc|ccc} A_K & 0 & 0 & E_1 H_1 & 0 & 0 \\ 0 & A_J & B_{22}J & 0 & H_2 & -B_{22}JNH_v \\ 0 & 0 & MA_{22} & 0 & MH_2 & -MA_{22}NH_v \\ \hline E_1^T & 0 & 0 & 0 & 0 & 0 \\ C_2 & C_{22} & 0 & 0 & 0 & H_v \end{array}\right] \tag{18}$$

where $M = (I - NC_{22})$, $C_2 = \begin{bmatrix} C_{21} & C_{22} \end{bmatrix}$, and the states are (x_1, η_1, e_1, e_2). As a result, we obtain a very simple interpretation for η_1.

Lemma 11. *Suppose x_1, η_1, e_1, e_2 are the states of the autonomous system in (18), and w_1, w_2, v are independent, zero mean random processes. Then,*

$$\eta_1(t) = \mathrm{E}\big(x_2(t) \mid x_1(0), \ldots, x_1(t)\big) \tag{19}$$

Proof. For notational convenience, let $x(0 : t) = x(0), \ldots, x(t)$.

From the independence of the noises and the block diagonal structure of the dynamics, it is clear that x_1, η_1 evolve independently of $e_1 = x_2 - \eta_1$. Consequently,

$$\mathrm{E}\big(x_2(t) - \eta_1(t) \mid x_1(0:t), \eta_1(0:t)\big) = \mathrm{E}\big(x_2(t) - \eta_1(t)\big) = 0$$

Thus,

$$\mathrm{E}\big(x_2(t) | x_1(0:t), \eta_1(0:t)\big) = \mathrm{E}\big(\eta_1(t) | x_1(0:t), \eta_1(0:t)\big) \tag{20}$$

Note that the expected value of $\eta_1(t)$, conditioned on itself, is just equal to $\eta_1(t)$. Moreover, since η_1 is a deterministic function of x_1, conditioning on $x_1(0:t), \eta_1(0:t)$ is equivalent to conditioning on just $x_1(0:t)$. As a result, (20) is equivalent to (19). ∎

To gain intuition into the second controller state η_2, we must similarly find player 2's estimate for x_2. This estimate is chosen as the transfer function $F \in \mathcal{RH}_\infty$ which minimizes

$$\min_{F \in \mathcal{RH}_\infty} \left\| x_2 - F \begin{bmatrix} x_1 \\ y_2 \end{bmatrix} \right\|_2 \tag{21}$$

It is well-known that this estimate comes from the steady-state Kalman filter for the autonomous system in (18), as seen in the following lemmas.

Lemma 12. *For the system in (18), suppose $(A_K)_{22}$ and A_N are stable matrices. Then, the estimator $\hat{e}_1 = G \begin{bmatrix} x_1 \\ y_2 \end{bmatrix}$, $G \in \mathcal{RH}_\infty$, which minimizes*

$$\min_{G \in \mathcal{RH}_\infty} \left\| e_1 - G \begin{bmatrix} x_1 \\ y_2 \end{bmatrix} \right\|_2 \tag{22}$$

is given by

$$\hat{e}_1 = \big(N + (zI - A_N)^{-1} A_N N\big) \left(y_2 - C_2 \begin{bmatrix} x_1 \\ \eta_1 \end{bmatrix} \right) \tag{23}$$

Proof. This result is the standard steady-state Kalman filter and is the dual of the classic LQR problem. The proof is omitted here due to space constraints. ∎

Note that the conditions $(A_K)_{22}$ and A_N being stable are simply employed so that the solution for the underlying Riccati equation in the optimization problem (22) reduces to solving the Riccati equation (5).

Theorem 13. *Suppose the conditions of Lemma 12 hold. Then, the controller state η_2 is given by*

$$\eta_2 = -\hat{x}_{2|12} + \eta_1 + N \left(y_2 - C_2 \begin{bmatrix} x_1 \\ \eta_1 \end{bmatrix} \right)$$

where $\hat{x}_{2|12} = F \begin{bmatrix} x_1 \\ y_2 \end{bmatrix}$, $F \in \mathcal{RH}_\infty$, is the estimator of x_2 which minimizes (21).

Proof. It is clear that the two norm optimizations in (22) and (21) are equivalent, if we let $F = G + \begin{bmatrix} \Phi & 0 \end{bmatrix}$, where $\eta_1 = \Phi x_1$. Thus, $\hat{x}_{2|12} = \hat{e}_1 + \eta_1$.

From (17), we can write the dynamics of η_2 as

$$\eta_2 = -(zI - A_N)^{-1} A_N N \left(y_2 - C_2 \begin{bmatrix} x_1 \\ \eta_1 \end{bmatrix} \right)$$

Combining this with (23), we see that

$$\hat{e}_1 + \eta_2 = \hat{x}_{2|12} - \eta_1 + \eta_2 = N \left(y_2 - C_2 \begin{bmatrix} x_1 \\ \eta_1 \end{bmatrix} \right)$$

from which the result follows. ∎

Letting $\hat{x}_{2|1} = \eta_1$ from Lemma 11 and $\hat{x}_{2|12}$ as defined in Theorem 13, it is now straightforward to show that the optimal policy can be written as shown in (9).

7 Conclusions

In this paper, we considered a two-player decentralized control problem, with communication allowed in only one direction. In addition, the problem imposed a partial output feedback structure, in which one player measures his state directly and the other player is limited to a noisy measurement of his own state. The optimal $\mathcal{H}_2$ norm controller was found via a spectral factorization approach. The optimal controller involves dynamics which were shown to be associated with estimation processes for each player. Moreover, the order of the optimal controller was established and is equal to twice the state dimension of subsystem 2.

This work extended the results for the full state feedback problem in [18] to the partial output feedback case. Though omitted here for space, this work extends naturally to more general networks which have a particular partial output feedback structure. Our future work will continue to extend our methodology to more general networks and output feedback problems.

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