2011 American Control Conference
on O'Farrell Street, San Francisco, CA, USA
June 29 - July 01, 2011

Optimal Input Design for Flat Systems using B-Splines

W. Van Loock, G. Pipeleers, J. Swevers
Division PMA, Departement of Mechanical engineering
Katholieke Universiteit Leuven
Email: wannes.vanloock@mech.kuleuven.be

***Abstract*— This paper deals with optimal design of input signals for linear, controllable systems, by means of their flat output. The flat output is parametrized by a polynomial spline and a linear problem is formulated in which both the spline coefficients and the knot locations are found simultaneously. Conservative constraints on the spline coefficients ensure that semi-infinite bounds are never violated and numerical results show that the amount of conservatism is little.**

I. Introduction

This paper considers input design for so-called flat systems. For such systems there exists a minimal set of particular outputs, i.e. the flat outputs, that characterize all the state-space motions and the corresponding input history [1]. Solving a motion control problem through flatness avoids integrating the differential equations and amounts to finding the best flat output motion that obeys the path constraints expressed in terms of the flat outputs and their derivatives [2].

In [2] this motion planning problem is reformulated as a B-spline positivity problem, which can be cast as a semidefinite program using a sum of squares decomposition. However, the authors consider the knot locations to be fixed, resulting in a suboptimal solution. In this paper we improve upon [2] by also optimizing over the spline knots using an indirect approach as in [3]. In addition, we formulate the problem as a linear program for improved numerical efficiency at the cost of introducing conservatism.

II. Optimization Problem

The system considered in this paper is an overhead crane with fixed cable length as shown in Fig. 1. With the small-angle approximation, the state-variable description of this system is

$$L\ddot{\theta}(t) = -\ddot{u}(t) - g\theta(t), \tag{1}$$

where $u(t)$ is the system input, $\theta(t)$ is the angle between the cable and the vertical axis, L is the cable length and g is the gravitational acceleration. This linearized system is differentially flat with $y(t) = L\theta(t) + u(t)$, the coordinate of the load, as the flat output:

$$\theta(t) = -\frac{\ddot{y}(t)}{g} \text{ and } u(t) = y(t) + L\frac{\ddot{y}(t)}{g}\,. \tag{2}$$

Now consider the basic problem of moving the load from y_0, at time $t = 0$ to y_T as fast as possible. At the boundaries we wish that the input is smooth. Therefore we impose $y^{(r)}(0) = y^{(r)}(T) = 0$ for $r = 1, 2, 3$. This also ensures

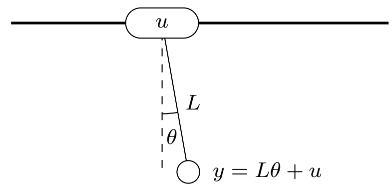

Fig. 1. Illustration of an overhead crane, with input u, states θ and $\dot{\theta}$,and output y, the coordinate of the load.

residual vibrations are canceled at the end of the motion. The acceleration of the trolley is limited, $\underline{a} \le \ddot{u}(t) \le \overline{a}$ to avoid actuator saturation. The considered problem amounts to solving the following integration-free optimization problem

$$\begin{aligned}
&\underset{T,y(t)}{\text{minimize}} && T && \text{(3a)}\\
&\text{subject to} && y(0) = y_0,\, y(T) = y_T && \text{(3b)}\\
& && y^{(r)}(0) = 0,\, r = 1, 2, 3 && \text{(3c)}\\
& && y^{(r)}(T) = 0,\, r = 1, 2, 3 && \text{(3d)}\\
& && \underline{a} \le \ddot{y}(t) + L\frac{y^{(4)}(t)}{g} \le \overline{a} && \text{(3e)}\\
& && \underline{\theta} \le -\frac{\ddot{y}(t)}{g} \le \overline{\theta}\,. && \text{(3f)}
\end{aligned}$$

Equation (3f) constrains the angle so the small-angle approximation remains valid. From the solution, $y^*(t), T^*$ of this problem, we then calculate the optimal input $u^*(t)$ as in (2).

A. Trajectory Parametrization

In order to find a solution of problem (3), the trajectory $y(t)$ is parametrized by a polynomial spline of degree $k = 4$. The main challenge when optimizing splines is determining the location of the knots, $\{\lambda_1, \ldots, \lambda_n\}$. This requires treating the knots as variables, resulting in a highly nonlinear and nonconvex optimization problem [4]. To overcome the nonlinearity, [3] propose to optimize the spline knot locations indirectly by supplying many candidate knot locations ($n =$ 500 à 1000) and using a regularization to favor solutions with few active[1] knots when many equally optimal solutions exist. In this work we apply this indirect approach and use B-splines as a basis to represent polynomial splines, which

[1]A knot λ_i is called active if the kth-order derivative of the spline is discontinuous at λ_i

allows us to deal with constraints (3e,3f) in an elegant manner (see Section II-B).

Assuming the knot sequence $\{0 = \lambda_{-k} = \cdots = \lambda_0 \leq \lambda_1 \leq \cdots \leq \lambda_n \leq \lambda_{n+1} = \cdots = \lambda_{n+k+1} = T\}$, $y(t)$ is written as

$$y(t) = \sum_{i=-k}^{n} c_i B_{i,k+1}(t)\,, \tag{4}$$

where c_i are the B-spline coefficients of $y(t)$ and $B_{i,k+1}(t)$ are the B-spline basis functions [4]. The coefficients $\mathbf{c} = (c_{-k} \cdots c_n)^T$ become the optimization variables for (3).

B. Inequality Constraints

To impose $\underline{C} \leq \sum_{i=-k}^{n} c_i B_{i,k+1}(t) \leq \overline{C}$ for all t, it follows from the convex hull property of B-splines that $\underline{C} \leq c_i \leq \overline{C}$ for $i = -k, \ldots, n$, is a sufficient condition. However, it is not a necessary condition and may introduce conservatism [5].

In our problem, we impose such constraints on the spline coefficients of $\theta(t)$ (3f) and $\ddot{u}(t)$ (3e). By virtue of the Curry-Schoenberg theorem $\theta(t)$ and $\ddot{u}(t)$ also have a B-spline representation, whose basis can easily be determined from $y(t)$ [2]. The spline coefficients of $\theta(t)$ and $\ddot{u}(t)$ depend linearly on the spline coefficients of $y(t)$ via matrices Θ and A, which are found by comparing sampled values.

C. Time Optimality

The objective T in problem (3) would render the optimization problem non-convex. Therefore a bisection is performed to find the optimal time, where each bisection step requires solving the following linear feasibility problem

$$\begin{aligned}
&\underset{\mathbf{c}\in\mathbb{R}^{g+k+1}}{\text{minimize}} && 0 && \text{(5a)}\\
&\text{subject to} && c_{-k} = y_0,\, c_n = y_T && \text{(5b)}\\
& && c_{-k+i} = c_{-k},\, i = 1,2,3 && \text{(5c)}\\
& && c_{n-i} = c_n,\, i = 1,2,3 && \text{(5d)}\\
& && \underline{a} \leq A\mathbf{c} \leq \overline{a} && \text{(5e)}\\
& && \underline{\theta} \leq \Theta\mathbf{c} \leq \overline{\theta}\,, && \text{(5f)}
\end{aligned}$$

where (5c, 5d) constrain the derivatives at the boundaries according to (3c,3d) and (5e, 5f) replace (3e, 3f) by constraining the spline coefficients.

III. Results

An optimal input to move the load from $y_0 = 0\,\text{m}$ to $y_T = 0.6\,\text{m}$, is calculated for $L = 0.45\,\text{m}$, $\overline{a} = 5\,\text{m}\,\text{s}^{-2} = -\underline{a}$, $\overline{\theta} = 5^\circ = -\underline{\theta}$, $k = 4$ and $n = 500$. Fig. 2 shows the results for $\ddot{u}(t)$ (top) and $\theta(t)$ (bottom). Only seven active knots are found.

The motion looks optimal since at each time instant one of the inequality constraints seems to be active. Though not visible in Fig. 2, a closer inspection reveals that some conservatism is introduced by (5e), related to the approach from II-B. However, the relative difference between the optimal time by gridding (3e, 3f), as in [3], and our approach is only $0.005\,\%$. Contrary to [3], constraints are guaranteed

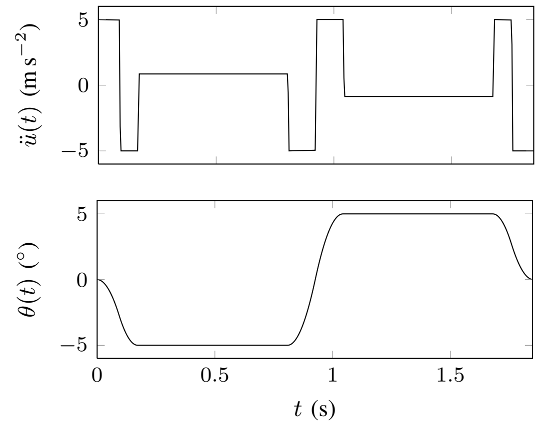

Fig. 2. Time optimal input acceleration (top) and the corresponding swing angle (bottom)

to be satisfied between candidate knots without any post-analysis. Also, the flat system approach allows us to formulate constraints on the states and outputs elegantly, without having to introduce additional variables and integrate the differential equations.

The conservative linear program is solved more efficiently than the exact semidefinite program [2] and allows for optimizing over the knot locations. Moreover, by choosing the candidate knots where active knots are expected, the problem size can be vastly reduced allowing for solutions to be found even more efficiently.

IV. Future Work

The presented method will be validated experimentally on an overhead crane. Since efficient algorithms exist to solve linear programs, online implementations are subject for future research. Also, the framework is being extended to nonlinear flat systems, which will require convex approximations of the feasible set.

Acknowledgement

Goele Pipeleers is a Postdoctoral Fellow of the Research Foundation – Flanders (FWO – Vlaanderen). This work benefits from K.U.Leuven – BOF PFV/10/002 Center-of-Excellence Optimization in Engineering (OPTEC), the Belgian Programme on Interuniversity Attraction Poles, initiated by the Belgian Federal Science Policy Office (DYSCO), research project G.0377.09 of the Research Foundation – Flanders (FWO – Vlaanderen), and K.U.Leuven's Concerted Research Action GOA/10/11 "Global real-time optimal control of autonomous robots and mechatronic systems".

References

[1] M. Fliess, J. Lévine, and P. Rouchon, "Flatness and defect of nonlinear systems: Introductory theory and examples," *International Journal of Control*, vol. 61, pp. 1327–1361, 1995.

[2] C. Louembet, F. Cazaurang, and A. Zolghadri, "Motion planning for flat systems using positive b-splines: An lmi approach," *Automatica*, vol. 46, no. 8, pp. 1305 – 1309, 2010.

[3] B. Demeulenaere, G. Pipeleers, J. De Caigny, J. Swevers, J. De Schutter, and L. Vandenberghe, "Optimal splines for rigid motion systems: A convex programming framework," *Journal of Mechanical Design*, vol. 131, no. 10, p. 101004, 2009.

[4] P. Dierckx, *Curve and surface fitting with splines*. New York, NY, USA: Oxford University Press, Inc., 1993.

[5] C. de Boor and J. W. Daniel, "Splines with nonnegative B-spline coefficients," *Mathematics of Computation*, vol. 28, no. 126, pp. 565–568, 1974.