2011 American Control Conference
on O'Farrell Street, San Francisco, CA, USA
June 29 - July 01, 2011

Piecewise smooth solutions to the nonlinear output regulation PDE

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***Abstract*— The solution to the nonlinear output regulation problem requires one to solve a first order PDE, known as the Francis–Byrnes–Isidori (FBI) equations. In this paper we propose a method to compute approximate solutions to the FBI equations when the zero dynamics of the plant are hyperbolic and the exosystem is two-dimensional. Our method relies on the periodic nature of two-dimensional analytic center manifolds.**

I. INTRODUCTION

Consider the smooth nonlinear control system

$$\begin{aligned} \dot{x} &= f(x) + g(x)u + p(x)w \\ \dot{w} &= s(w) \\ y &= h(x) + q(w) \end{aligned} \tag{1}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $w \in \mathbb{R}^q$ is an exogenous signal and $y \in \mathbb{R}^p$. We say that the feedback $u = \alpha(x, w)$ solves the *output regulation problem (ORP)* for (1) if $\dot{x} = f(x) + g(x)\alpha(x, 0)$ has $x = 0$ as an exponentially stable equilibrium and if $\lim_{t\to\infty} y(t) = 0$ for $(x(0), w(0))$ sufficiently small. In [4], it is shown that under suitable conditions, the ORP is solvable if and only if there exists a pair (π, κ), defined locally about $w = 0$, satisfying the FBI equations

$$\begin{aligned} \frac{\partial \pi}{\partial w}(w)\, s(w) =& f(\pi(w)) + g(\pi(w))\kappa(w) + p(\pi(w))w \\ & h(\pi(w)) + q(w) = 0. \end{aligned} \tag{2}$$

Given a solution (π, κ) to (2), a feedback that solves the ORP is $\alpha(x, w) = \kappa(w) + K(x - \pi(w))$, where K is any matrix rendering the linear system $\dot{x} = \frac{\partial f}{\partial x}(0)x + g(0)u$ asymptotically stable. As shown in [4], solving (2) can be reduced to the solvability of the center manifold PDE for a dynamical system of the form

$$\begin{aligned} \dot{z} &= f_0(z, \varphi(w)) \\ \dot{w} &= s(w) \end{aligned} \tag{3}$$

where $\dot{z} = f_0(z, 0)$ represent the zero dynamics of the plant, $\varphi(w) = -(q(w), L_s q(w), \ldots, L_s^{r-1} q(w))$ and $1 \le r < n$ is the relative degree of the triple $\{f, g, h\}$ at $x = 0$. It is well-known that center manifolds suffer from subtleties in regards to uniqueness and differentiability [2]. A case that seems to have gone unnoticed in the nonlinear control community is the case of two-dimensional real analytic (C^ω) center manifolds [1]. It is shown in [1] that if the local center manifold dynamics of the C^ω system

$$\begin{aligned} \dot{z} &= Bz + \bar{Z}(w_1, w_2, z) \\ \dot{w}_1 &= -w_2 + P(w_1, w_2, z) \\ \dot{w}_2 &= w_1 + Q(w_1, w_2, z) \end{aligned} \tag{4}$$

are Lyapunov stable and not attractive then (4) has a uniquely determined local center manifold which is C^ω and generated by a family of periodic solutions. The matrix B in (4) is assumed to have no eigenvalues on the imaginary axis and $z \in \mathbb{R}^n$, $w_1, w_2 \in \mathbb{R}$. Hence, in the C^ω case with a two-dimensional exosystem and hyperbolic zero dynamics, Aulbach's theorem can be applied directly to the ORP since it is assumed that the exosystem is neutrally stable [4] thereby ensuring Lyapunov stability and non-attractivity. Using Aulbach's result and the patchy technique in [5], we propose a method to obtain piecewise smooth approximate solutions to the center manifold PDE for a system of the form (4). The main idea of our method is to use the periodicity of the solution and build a power series approximation along the solutions of the exosystem. Other methods for solving the FBI equations are based on direct Taylor polynomial approximations [3] and finite-element methods [6]. The novelty in our approach, albeit restricted to two-dimensional exosystems, is that it takes advantage of the geometric structure of the solution and produces an approximate solution that is straightforward to evaluate.

II. PATCHY METHOD FOR THE CENTER MANIFOLD PDE

Applying the change of coordinates $(w_1, w_2, z) = (r\cos\theta, r\sin\theta, z)$ to (4) and eliminating the time variable, one obtains a system of the form

$$\frac{dr}{d\theta} = R(\theta, r) \tag{5a}$$

$$\frac{dz}{d\theta} = Bz + Z(\theta, r, z), \tag{5b}$$

where R and Z are C^ω converging for each $\theta \in [0, 2\pi]$, $|r| \le a$, $\|z\| \le a$, for some $a > 0$ [1]. In polar coordinates (θ, r), the solution to the center manifold PDE of (4) takes the form $\psi(\theta, r) = \sum_{i=1}^{\infty} e_i(\theta) r^i$ and converges on a cylinder $\theta \in [0, 2\pi], |r| < \epsilon$, with 2π-periodic coefficients $e_i(\theta)$. The solution ψ satisfies the PDE

$$B\psi(\theta, r) + Z(\theta, r, \psi(\theta, r)) = \frac{\partial \psi}{\partial \theta} + \frac{\partial \psi}{\partial r} R(\theta, r). \tag{6}$$

For simplicity, let us suppose that the exosystem is given by $\dot{w}_1 = -w_2$ and $\dot{w}_2 = w_1$. Let $\phi_0(w_1, w_2)$ denote the solution to the center manifold PDE for the linear part of (4). The mapping $\psi_0(\theta, r) = \phi_0(r\cos\theta, r\sin\theta)$ is accepted

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as our initial approximation to ψ on the annular region $\theta \in [0, 2\pi], 0 \le r < r_0$, for some $r_0 > 0$. Now define $\Psi_1(\theta, \sigma) = \psi(\theta, r_0 + \sigma)$. Then Ψ_1 has a power series representation

$$\Psi_1(\theta, \sigma) = \Psi_1(\theta, 0) + \sum_{i=1}^{\infty} \frac{\partial^i \Psi_1}{\partial \sigma^i}(\theta, 0)\frac{\sigma^i}{i!}$$

converging for $\theta \in [0, 2\pi]$ and $|\sigma|$ sufficiently small. By construction, Ψ_1 is a radial perturbation of ψ along the solution of the exosystem with initial condition $(w_1(0), w_2(0)) = (r_0, 0)$. We compute a Taylor series approximation to Ψ_1 of the form $\tilde{\Psi}_1(\theta, \sigma) = \Psi_1(\theta, 0) + \Sigma_{i=1}^{N} \frac{\partial^i \Psi_1}{\partial \sigma^i}(\theta, 0)\frac{\sigma^i}{i!}$ for some desired N. Using (6), it is straightforward to show that $\frac{\partial^i \Psi_1}{\partial \sigma^i}(\theta, 0)$ satisfy linear inhomogeneous ODEs:

$$\frac{d\eta_i}{d\theta} = A(\theta)\eta + F_i(\theta, \eta_0(\theta), \ldots, \eta_{i-1}(\theta)), \tag{7}$$

where F_i is 2π-periodic and $\eta_i = \frac{\partial^i \Psi_1}{\partial \sigma^i}(\theta, 0)$, $i = 1, \ldots, N$. To compute $\frac{\partial^i \Psi_1}{\partial \sigma^i}(\theta, 0)$, we solve a BVP using (7) with 2π-periodic boundary conditions. Similarly, to compute $\Psi_1(\theta, 0) = \psi(\theta, r_0)$ we solve a BVP using (5b) and 2π-periodic boundary conditions. Let now $\psi_1(\theta, r) = \tilde{\Psi}_1(\theta, r - r_0)$ and define, for $\theta \in [0, 2\pi]$ and $r_1 > r_0$,

$$\tilde{\psi}(\theta, r) = \begin{cases} \psi_0(\theta, r), & 0 \le r < r_0 \\ \psi_1(\theta, r), & r_0 \le r \le r_1. \end{cases}$$

The mapping $\tilde{\psi}$ is a piecewise smooth approximation to the solution ψ on the cylinder $\theta \in [0, 2\pi]$, $0 \le r \le r_1$. This procedure can be iterated as follows. Let

$$\tilde{\psi}(\theta, r) = \begin{cases} \psi_0(\theta, r), & 0 \le r < r_0 \\ \psi_1(\theta, r), & r_0 \le r < r_1 \\ \quad\vdots & \quad\vdots \\ \psi_k(\theta, r), & r_{k-1} \le r \le r_k \end{cases} \tag{8}$$

where $\psi_j(\theta, r) = \tilde{\Psi}_j(\theta, r - r_{j-1})$ and $\tilde{\Psi}_j$ is a Nth order Taylor approximation of $\Psi_j(\theta, \sigma) = \psi(\theta, r_{j-1} + \sigma)$, $j = 1, \ldots, k$. Now consider $\Psi_{k+1}(\theta, \sigma) = \psi(\theta, r_k + \sigma)$ and its Nth order Taylor approximation $\tilde{\Psi}_{k+1}(\theta, \sigma) = \Psi_{k+1}(\theta, 0) + \Sigma_{i=1}^{N} \frac{\partial^i \Psi_{k+1}}{\partial \sigma^i}(\theta, 0)\frac{\sigma^i}{i!}$, $r_k > r_{k-1}$. The curve $\Psi_{k+1}(\theta, 0) = \psi(\theta, r_k)$ is computed by solving a BVP problem using (5b) with 2π-periodic boundary conditions. As an initial guess for the BVP we take $\psi(\theta, r_k) \approx \psi_k(\theta, r_k)$, i.e., we use the previously computed approximation as an initial guess. Similarly, the coefficients $\frac{\partial^i \Psi_{k+1}}{\partial \sigma^i}(\theta, 0)$ are computed by solving a BVP problem using (7) with 2π-periodic boundary conditions. As an initial guess for the BVP we take the previously computed coefficients, i.e., $\frac{\partial^i \Psi_{k+1}}{\partial \sigma^i}(\theta, 0) \approx \frac{\partial \Psi_k}{\partial \sigma^i}(\theta, 0)$. We then define $\psi_{k+1}(\theta, r) = \tilde{\Psi}_{k+1}(\theta, r - r_k)$ and extend our running approximation (8) to the annulus $\theta \in [0, 2\pi], r_k \le r \le r_{k+1}$ by augmenting ψ_{k+1} to it. In the next section we illustrate our method on a standard control problem.

III. Example

The dynamics of a single pendulum attached to a cart moving in a straight line perpendicular to gravity can be written in the form $\dot{x}_1 = x_2$, $\dot{x}_2 = u$, $\dot{x}_3 = x_4$, $\dot{x}_4 = \frac{g}{\ell}\sin(x_3) - \frac{1}{\ell}\cos(x_3)u$, where x_1 is the position of the cart, x_3 is the angle the pendulum makes with the vertical, u is the control, g is the acceleration due to gravity and ℓ is the length of the rod. For simplicity we set $g = 10$ and $\ell = 1/3$. As output we take $h(x) = x_1$ and reference trajectory $y_{\text{ref}}(t) = A\cos(\beta t)$. Hence we choose exosystem $\dot{w}_1 = -\beta w_2$, $\dot{w}_2 = \beta w_1$ and $q(w) = -w_1$. In the normal coordinates $\xi = (x_1, x_2)$ and $z = (x_3, x_4 + \frac{x_2}{\ell}\cos(x_3))$, the zero dynamics are hyperbolic. Using our method we computed an approximate solution to the associated center manifold PDE for this system and used it in a tracking controller of the form $\alpha(x, w) = \kappa(w) + K(x - \pi(w))$. The matrix K was chosen so that the closed-loop eigenvalues are $-6, -3.5, -3, -2.5$. We used $k = 11$ annuli and order $N = 2$ for the Taylor approximations to Ψ_i. A radius of $r_0 = 0.1$ is used for the initial approximation ψ_0 and each subsequent annulus is of thickness $\sigma = 0.1$. The parameters $\omega = 1.25$ and $A = 1.1$ were selected. Figure 1 shows the output and reference trajectory and Figure 2 shows the tracking error. The initial condition of the cart was initialized to $x_1(0) = -0.25$.

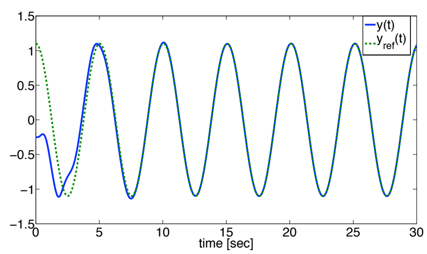

Fig. 1. Output $y(t) = x_1(t)$ and reference $y_{\text{ref}}(t) = w_1(t)$.

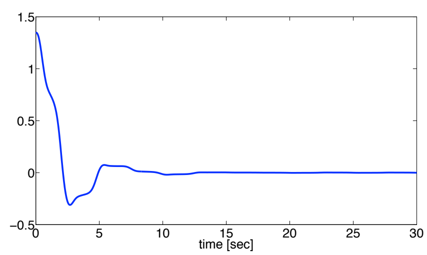

Fig. 2. Tracking error $e(t) = y(t) - y_{\text{ref}}(t)$.

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