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# Improved Stochastic Process Models for Linear Structure Behavior

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*Abstract*— **Linear mathematical models frequently provide good approximations to the input-output relations for real systems. However, ensembles of systems that are nominally identical cannot, usually, be adequately represented with a single model because real systems are stochastic. The randomness in real systems must be modeled if the randomness bears on critical behaviors of the system. The behavior of linear systems can be represented in parametric or non-parametric form; the latter framework is used, here. Among the frameworks available for characterization of system behavior, we choose the frequency response function (FRF). We choose to work with the FRF because many system attributes can be interpreted by inspection of the FRF, and it can be used directly for control design. This paper improves a previously developed Karhunen-Loeve expansion (KLE) representation for linear system behavior based on FRF data. The improvement yields a compact representation of the uncertainty inherent in an ensemble of systems and avoids the introduction of unwanted features in the system representation. This non-parametric, compact representation of the distribution of linear systems can then be used to characterize the performance and stability of a given feedback control law, as well as for control law design.**

## INTRODUCTION

The frequency response function (FRF) is often used to approximate the relation of inputs to outputs for a physical system that can be adequately represented as linear [1,2]. Use of the FRF is convenient because it permits the expression of a response as a linear algebraic function of an input in the frequency domain. Moreover, many system properties can be determined by visual inspection of the FRF. However, when it is necessary to represent the input-output relationships of an ensemble of nominally identical systems, use of a single, deterministic FRF may prove inadequate. Systems intended to be identical are often not identical and are better described in a stochastic framework. Similarly, data collected from a single system, tested under different environmental conditions, may appear stochastic.

Our motivation for considering system randomness is this. We frequently design a controller to control the behavior of a stochastic ensemble of systems. Typically, the controller is deterministic. It is designed to be stable and to optimally satisfy a control objective with respect to one realization from the ensemble of systems. It is critical to establish the probability of stability and the probability of achieving a control objective for an arbitrarily chosen member of the stochastic ensemble. Once the goal of establishing these two probabilities is achieved, the control system designer may consider the extended goal of controller optimization with respect to a measure of the joint probability of stability and satisfaction of the control objective [3-6].

There are many frameworks for expressing the randomness in signals and systems. For example, Ghanem and Spanos [7] developed a method for incorporating randomness into structural dynamic models using the finite element framework. Ghanem, et al. [8], model random structures with a functional analysis and polynomial chaos-based probabilistic approach. Soize [9,10] uses random matrix theory for simulating model stochasticity; his approach accommodates both data and model uncertainties.

The effects of system randomness in controller design have been addressed with design margins (e.g., classical gain and phase margins) bounded uncertainty representations (e.g., $H_\infty$, $\mu$-synthesis) [11], and adaptive control [12]. Optimal estimation theory [13] deals with stochastic signals applied to deterministic systems. None of these approaches uses the stochastic characteristics (e.g., probability distributions) of dynamic systems directly in the controller design problem.

In this work, we use a non-parametric representation of a linear system (the open-loop FRF) and consider it a random process for which we either have a probabilistic model or must create one. We select the Karhunen-Loeve expansion (KLE) [14,7,15] to represent the FRF random process. It represents a random process as a mean function plus deviations from the mean. The deviations are products of shape functions, amplitudes, and randomizing factors. In order to use the KLE in a practical framework, it is necessary to express the joint probability distribution of the randomizing factors. This can be accomplished, approximately, with the kernel density estimator (KDE) [16]. In order to generate new realizations from the identified KLE, a means for generating random samples from the probability distribution of the randomizing factors is required; such a means is provided by the Markov chain Monte Carlo (MCMC) method [17].

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The KLE was developed in a previous paper [18], along with the KDE and the MCMC method, and applied to the problem of modeling the FRF as a random process. Direct application of the KLE to modeling real FRFs of stochastic, second order systems revealed shortcomings. In particular, some FRF realizations generated from the modeled stochastic source reflected unrealistic features with respect to the measured ensemble upon which the model was built. For example, some generated FRF realizations displayed what appeared to be more modes than the systems in the measured ensemble. The problem arises from the detailed representation embodied in the KLE. (An alternate approach to modeling the FRF as a random process was also proposed in [18], but it suffers from the requirement that parameter estimation be performed in a modal analysis-like framework; our hope is to avoid that type of parameter estimation.)

Our goal in this study was to overcome the problems that occurred in [18], and we have developed a technique that does so based on a modified application of the MCMC technique for sampling from random processes. Our technique specifically incorporates additional information about the systems that is normally neglected.

Section I summarizes the results developed in [18]. The shortcomings of the direct approach are described in Section II. A modification of the MCMC method for generation of random process realizations is described in Section III. Section IV discusses how the stochastic FRF model can be used to assess the probability of stability and the probability that a controller will achieve a control objective for a random structure. Section V presents an example application of the modified technique.

## I. Review of the Karhunen-Loeve Expansion and some Tools Required to use it

The KLE [7,15] is a framework for the compact representation of multivariate, continuous-valued, continuous- or discrete-parametered (i.e., continuous or discrete time, frequency, or space) nonstationary random processes. Reference [18] developed the particular KL expansion that approximately represents a univariate, continuous-valued, discrete-parametered (frequency), nonstationary random process. The random process of interest is an $n \times 1$ column vector of random variables denoted $\boldsymbol{X(f)} \in \mathbb{R}^n$. (Subsequently the explicit dependence of $\boldsymbol{X}$ on $\boldsymbol{f}$ is omitted.) The random process mean is an $n \times 1$ column vector denoted $\boldsymbol{\mu_X} \in \mathbb{R}^n$, and its autocovariance matrix is a nonnegative definite matrix denoted $\boldsymbol{C_{XX}} \in \mathbb{R}^{n\times n}$. The KLE representation of the random process is:

$$\boldsymbol{X} - \boldsymbol{\mu_X} = \boldsymbol{v}\boldsymbol{w}^{1/2}\boldsymbol{U}, \qquad (1)$$

where $\boldsymbol{w} \in \mathbb{R}^{m\times m}$ $(m \le n)$ is a diagonal matrix of the dominant eigenvalues of $\boldsymbol{C_{XX}}$, and a submatrix of $\boldsymbol{W} \in \mathbb{R}^{n\times n}$, the diagonal matrix containing all the eigenvalues of $\boldsymbol{C_{XX}}$; $\boldsymbol{v} \in \mathbb{R}^{n\times m}$ is the matrix containing the corresponding columns from $\boldsymbol{V} \in \mathbb{R}^{n\times n}$, the orthonormal matrix whose columns are the eigenvectors of $\boldsymbol{C_{XX}}$. $\boldsymbol{v}^T\boldsymbol{v} = \boldsymbol{I}$. $\boldsymbol{U} \in \mathbb{R}^m$ is a vector of zero-mean, unit-variance, uncorrelated random variables, whose joint distribution is to be approximated.

The columns of $\boldsymbol{v}$ assume the role of shape functions characterizing underlying components that appear in realizations of $\boldsymbol{X}$. The elements in the diagonal matrix $\boldsymbol{w}^{1/2}$ assume the role of amplitudes characterizing the relative contributions of the individual shapes in $\boldsymbol{v}$. The elements in the zero-mean, unit-variance, vector of uncorrelated random variables $\boldsymbol{U}$ are randomizing factors.

Reference [18] shows that when the joint probability distribution of the elements in the vector $\boldsymbol{X}$ is known, then it can be used to establish the joint probability distribution of the elements in the vector $\boldsymbol{U}$, and vice versa. When measured data from the random process $\boldsymbol{X}$, denoted $\boldsymbol{x}_j \in \mathbb{R}^n, j \in \{1, \dots, M\}$, are available the parameters of the KLE can be estimated. The estimates are denoted $\overline{\boldsymbol{x}}$, $\widehat{\boldsymbol{w}}$ and $\widehat{\boldsymbol{v}}$. Further, realizations of the random vector $\boldsymbol{U}$ that correspond to the measured realizations of $\boldsymbol{X}$ can be obtained. These are denoted $\boldsymbol{u}_j \in \mathbb{R}^m, j \in \{1, \dots, M\}$.

The ultimate goal of this investigation is to compute the probability of closed-loop stability and the probability of achieving a control objective. Our initial means for accomplishing this will be Monte Carlo analysis, therefore, a method is required for generating realizations of the random process $\boldsymbol{X}$. Development of such a capacity requires characterization of the joint probability density function (PDF) of the random vector $\boldsymbol{U}$, and reference [18] proposes to approximate that using the KDE. The KDE approximation to the joint PDF of $\boldsymbol{U}$ is

$$\hat{f}_{\boldsymbol{u}}(\boldsymbol{\alpha}) = \frac{1}{M}\sum_{j=1}^{M} \frac{1}{(2\pi\varepsilon^2)^{m/2}} exp\left[-\frac{1}{2\varepsilon^2}\left\|\boldsymbol{\alpha} - \boldsymbol{u}_j\right\|^2\right], \qquad (2)$$

where $\varepsilon > 0$ is a user-selected smoothing parameter, and the symbol $\|\cdot\|$ refers to the 2-norm.

The Metropolis-Hastings version of MCMC can be implemented using the KDE of (2) to generate realizations of the random process $\boldsymbol{X}$ with the following steps.

1. Initiate sampling at a point in the space of $\boldsymbol{U}$.
2. Generate another point in the space of $\boldsymbol{U}$ that is a random deviation from the initial point.
3. Compute likelihoods of the points described in 1 and 2.
4. Compare the likelihoods, probabilistically, to form a decision about whether the point in Step 2 is accepted or rejected as a realization of $\boldsymbol{U}$.

Repeat Steps 1-4 as many times as desired to generate the required number of realizations of $\boldsymbol{u}_j^{(gen)} \in \mathbb{R}^m, j \in \{1, \dots, n_{gen}\}$. Use these realizations in place of $\boldsymbol{U}$ in (1) to obtain generated realizations $X_j^{(gen)} \in \mathbb{R}^n, j \in \{1, \dots, n_{gen}\}$.

## II. Shortcomings of the method of Reference [18]

The scheme described in Section I was used in reference [18] with the imaginary part of measured FRF data from a second order system to identify the KLE that approximates the random source of a measured ensemble, and then generate realizations from the random source. (Only the imaginary part is needed because the real part is simply the Hilbert transform of the imaginary part for causal, linear systems.) Because of the inherent nature of KLE, some of the generated realizations of the imaginary part of the FRF had characteristics that made them implausible representations of the measured ensemble. Among other things, some of the generated realizations of the imaginary parts of the FRF had more peaks than the imaginary part of FRF in the measured ensemble.

Figure 1 shows the ensemble upon which the analysis was based, in reference [18]. The ensemble includes the imaginary part of single-input/single-output (SISO) FRFs from twenty nominally identical structures. The FRFs are denoted $\boldsymbol{H}_j(f) \in \mathbb{C}^n, j \in \{1, \ldots, 20\}$. The FRFs are shown at $n = 512$ discrete frequencies. The frequency increment is $\Delta f = 1.2\,Hz$. The system has five modes in the frequency range shown. Modal parameters vary randomly.

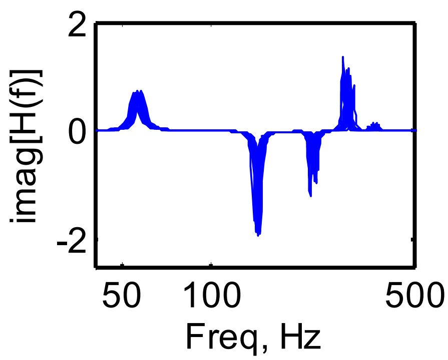


Figure 1. Imaginary parts of the FRFs of 20 structures from a stochastic ensemble.

Application of the analysis from Section I to the data in Figure 1 yields KDE parameter estimates $\bar{\boldsymbol{x}}$, $\widehat{\boldsymbol{W}}$ and $\widehat{\boldsymbol{v}}$, for the model of the imaginary part of the FRF. The number of terms retained in the model was $m = 12$, out of a possible 20, and this yielded a 99% accuracy.

When the estimated parameters were used with the model form of (1), the KDE of (2), and the MCMC method for generation of model realizations, the results shown as green in Figure 2 were generated. (The original ensemble is also shown in blue, to emphasize the contrast between the original ensemble and the synthesized ensemble.)

Two things are wrong with the synthesized (green) curves. First, and easier to remedy, is the fact that the simulations, particularly near modes two, three and four, assume positive values when they should remain strictly negative, and vice versa. This problem is overcome by using a transformation of values in the imaginary parts of the FRFs, such as a logarithmic transformation, prior to KLE modeling, to assure that synthesized data maintain the correct signs.

A second, and more important, problem is illustrated by the FRF imaginary part shown in Figure 3. (This is one of the elements of the generated ensemble shown as green in Figure 2.) It appears from the figure that there are multiple modes where the second, third and fourth modes appeared in Figure 1. The reason is not that the ensemble of random structures has more than five modes; rather, the reason is that (1) permits the superposition of components (shape functions times amplitudes) in combinations that should not occur. This is fundamentally a problem of the simulation of dependence in $\boldsymbol{U}$ in the KLE. It is possible to avoid this problem by introducing a mechanism to include information about the system, such as order; the following section describes how to do so.

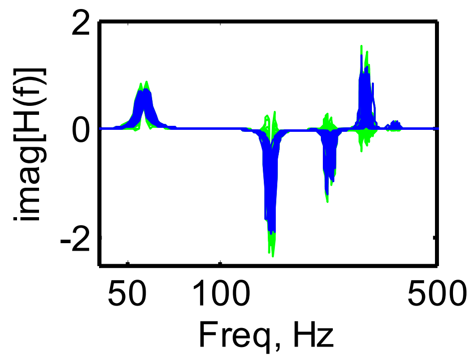


Figure 2. Imaginary parts of the FRFs generated using the KLE (green), and the imaginary parts of the FRFs from Figure 1 (blue).

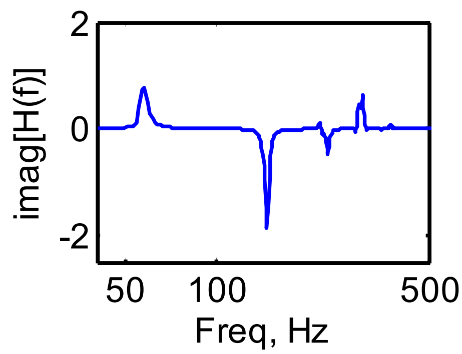


Figure 3. Imaginary part of one of the generated FRFs.

## III. Modification of the MCMC Method

The first problem described in the previous section is treated as specified there; prior to estimation of the KLE parameters, the absolute value of the imaginary part of each of the measured data, $\boldsymbol{H}_j(f) \in \mathbb{C}^n, j \in \{1, \ldots, 20\}$, is computed. Then the common logarithm of the result is computed and the parameters estimated. This yields parameter estimates $\bar{\boldsymbol{x}}$, $\widehat{\boldsymbol{W}}$ and $\widehat{\boldsymbol{v}}$, as in the previous analysis. Use of the parameters in (1) yields a model for the transformed random process. The logarithmic transformation and that the absolute value operation must be inverted to obtain realizations of the imaginary part of the FRF. The first inversion is trivial. The second inversion requires tracking the signs of the imaginary parts of the FRF and using them in the second inversion. These operations yield MCMC-generated samples of the imaginary part of the FRF. Denote the generated FRFs $\boldsymbol{H}_j^{(gen)}(f) \in \mathbb{C}^n, j \in \{1, \ldots, n_{gen}\}$.

The latter problem described in Section II is attacked in the following way. During MCMC generation of FRF imaginary parts the random variates, $\boldsymbol{u}_j \in \mathbb{R}^m, j \in \{1,\dots,20\}$, corresponding to the measured realizations, $\boldsymbol{H}_j(f) \in \mathbb{C}^n, j \in \{1,\dots,20\}$, are used to guide the selection of random variates $\boldsymbol{u}_j^{(gen)} \in \mathbb{R}^m, j \in \{1,\dots,n_{gen}\}$ corresponding to the $\boldsymbol{H}_j^{(gen)}(f) \in \mathbb{C}^n, j \in \{1,\dots,n_{gen}\}$. The fact that some of the $\boldsymbol{H}_j^{(gen)}(f)$ appear implausible indicates that some combinations of random variates in $\boldsymbol{u}_j^{(gen)}$ yield unacceptable realizations. At present, we do not hope to decipher the source of the implausibility, directly, from observation of the $\boldsymbol{u}_j^{(gen)}$, but it is entirely feasible to judge implausibility of a realization through observation of each $\boldsymbol{H}_j^{(gen)}(f)$. We require an algorithm to automatically judge whether or not a particular realization $\boldsymbol{H}_j^{(gen)}(f)$ is a plausible representation of the original, measured system. The entire ensemble of generated realizations, $\boldsymbol{H}_j^{(gen)}(f) \in \mathbb{C}^n, j \in \{1,\dots,n_{gen}\}$, is observed, and for each realization, $\boldsymbol{H}_j^{(gen,NP)}(f)$, judged implausible (The NP in the superscript indicates "not plausible," or implausible.), its corresponding vector of underlying random variates, $\boldsymbol{u}_j^{(gen,NP)}$, is saved; the collection of saved vectors forms the ensemble $\boldsymbol{u}_j^{(gen,NP)}, j\epsilon\{1,\dots,n_{NP}\}$. The source of the vectors, so defined, has a joint PDF, and that PDF can be approximated with a KDE with the form of (2). The vectors $\boldsymbol{u}_j^{(gen,NP)}, j\epsilon\{1,\dots,n_{NP}\}$ occupy a space denoted $\boldsymbol{U}^{(NP)}$. We denote the space of vectors generated and not rejected during this initial round of MCMC simulation as $\boldsymbol{U}^{(AC)}$. The collections $\boldsymbol{U}^{(NP)}$ and $\boldsymbol{U}^{(AC)}$ are mutually exclusive.

To alleviate the second problem described in Section II, we perform MCMC sampling of the KLE, as before, but when a generated realization, $\boldsymbol{u}^{(gen)}$, of the random vector $\boldsymbol{U}$ is accepted by MCMC, its likelihood $L_{\boldsymbol{U}}[\boldsymbol{u}^{(gen)}]$, computed from known realizations of $\boldsymbol{U}$ is compared to its likelihood, $L_{\boldsymbol{U}^{(NP)}}[\boldsymbol{u}^{(gen)}]$, in the space of the known realizations of $\boldsymbol{U}^{(NP)}$. We accept $\boldsymbol{u}^{(gen)}$ with probability

$$p_{acc} = \frac{L_U[\boldsymbol{u}^{(gen)}]}{L_U[\boldsymbol{u}^{(gen)}] + L_{\boldsymbol{U}^{(NP)}}[\boldsymbol{u}^{(gen)}]}. \tag{3}$$

(Either of both of the likelihoods can be weighted with a positive factor to emphasize one over the other in definition of the acceptance probability.) The probability always lies in the interval $(0,1)$. When $L_{\boldsymbol{U}}[\boldsymbol{u}^{(gen)}]$ is great compared to $L_{\boldsymbol{U}^{(NP)}}[\boldsymbol{u}^{(gen)}]$, the acceptance probability is high because $\boldsymbol{u}^{(gen)}$ tends to be relatively far from the space of $\boldsymbol{U}^{(NP)}$. When $L_{\boldsymbol{U}}[\boldsymbol{u}^{(gen)}]$ is small compared to $L_{\boldsymbol{U}^{(NP)}}[\boldsymbol{u}^{(gen)}]$, the acceptance probability is low because $\boldsymbol{u}^{(gen)}$ tends to lie within the space of $\boldsymbol{U}^{(NP)}$. Denote the ensemble of vector random variates accepted during this step as $\boldsymbol{u}_j^{(gen,PR)}, j\epsilon\{1,\dots,n_{PR}\}$. (The PR superscript denotes the realizations as "probable.") These vectors form the space of $\boldsymbol{U}^{(PR)}$. These are used in (1) to form the imaginary parts of the ensemble $\boldsymbol{H}_j^{(gen,PR)}(f) \in \mathbb{C}^n, j \in \{1,\dots,n_{PR}\}$.

## IV. Control of Stochastic Systems

The framework developed in this paper interprets a linear stochastic system as one whose FRF ensemble consists of random functions. (This is not to be confused with a deterministic system excited by stochastic inputs and disturbances.) Control of a linear system can be based on the system FRF, and, for example, deterministic controller design can be based on one system, $\boldsymbol{H}_0$, in the stochastic ensemble, or an estimated median behavior of systems in a measured ensemble. The controller would be designed to yield stable closed-loop behavior and to optimally achieve an objective in terms of a response measure. An important feature of the deterministic controller, relative to the stochastic system, is the probability that it yields stable control of members of the system ensemble, and the probability that it achieves the performance objective for members of the system ensemble.

The development of the previous section permits approximation of the probability of stability of closed-loop response and the probability of meeting the performance objective for a deterministic controller applied to a stochastic plant. Once a controller is designed, its stability when used in closed-loop operation with a system with FRF, $\boldsymbol{H}$, can be analyzed. Consider an ensemble of systems with the FRFs $\boldsymbol{H}_j^{(gen,PR)}(f) \in \mathbb{R}^n, j \in \{1,\dots,n_{PR}\}$, as developed in the previous section. Assess the stability of the controller when used in closed-loop operation with each of the systems in the ensemble. Let the event, $E$, with outcomes $0$ and $1$, denote, respectively, the occurrences "closed-loop behavior unstable," and "closed-loop behavior stable." By analyzing the controller with each of the systems in $\boldsymbol{H}_j^{(gen,PR)}(f) \in \mathbb{R}^n, j \in \{1,\dots,n_{PR}\}$, we are assessing the value of an outcome of $E$, namely $e_j, j\epsilon\{1,\dots,n_{pr}\}$, and each $e_j$ takes the value $0$ or $1$. The estimate of the probability that the deterministic controller yields stable control of the stochastic ensemble is

$$\hat{p}_s = \frac{\sum_{j=1}^{n_{pr}} e_j}{n_{pr}} \tag{4}$$

Of course, the accuracy of the probability estimate improves with increasing $n_{PR}$. The development of Section III facilitates the generation of large numbers of FRF realizations so that arbitrary accuracy of the estimate in (4) can be achieved.

A goal of controller design is to make a measure of closed-loop system behavior satisfy an objective. It may be desired to optimize a performance measure, $M$, of the closed-loop system using the deterministic controller, i.e., $M_{opt} = M(\boldsymbol{H}_0)$. In addition, for the ensemble of plants it may be desired (or necessary) that a closed-loop measure of behavior satisfy $M \leq M_{req}$. We can estimate the probability that $M \leq M_{req}$ by assessing the measure of response for each system, $M\left[\boldsymbol{H}_j^{(gen,PR)}\right], j\epsilon\{1,\ldots,n_{pr}\}$. As in the previous paragraph, we define an event, $E$, with outcomes 0 and 1, indicating, respectively, $M\left[\boldsymbol{H}_j^{(gen,PR)}\right] \leq M_{req}$, and the complement of that event. The collection of outcomes is $e_j, j\epsilon\{1,\ldots,n_{pr}\}$, and the estimate of probability that the controller satisfies the design objective takes precisely the same form as (4).

## V. EXAMPLE

An example, similar to the one in reference [18], is summarized. An ensemble of twenty nominally identical structures was modeled; the SISO FRF was estimated for each of the $M = 20$ structures. The imaginary parts of the FRFs are shown in Figure 4. The FRFs are shown at $n = 512$ discrete frequencies; the frequency increment is 1.2 Hz. The system has five modes in the frequency range shown. Modal frequencies, modal damping factors, and modal amplitudes vary randomly.

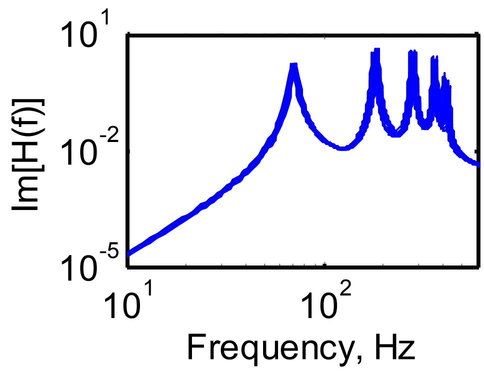


Figure 4. Imaginary parts of FRFs of 20 structures from the ensemble of a stochastic system.

The imaginary parts of the FRFs were transformed through computation of their absolute values (not necessary, in this case, because the signals are all positive), and then computation of their logarithms. The plot of the log-transformed FRF imaginary parts duplicates Figure 4 when the vertical scale is linear.

KLE parameters were estimated, as specified in previous sections, and 74 realizations of the transformed FRF imaginary part were generated via MCMC. A criterion was established to judge the plausibility of the generated signals and applied to each of the generated signals to form the ensemble $\boldsymbol{u}_j^{(gen,NP)}, j\epsilon\{1,\ldots,n_{NP}\}$. The criterion used, in the present case, simply counts the peaks in the generated signals, and rejects a generated signal as implausible when it had more than five peaks. (A peak occurs in the imaginary part of an FRF when the slope of the curve changes from negative to positive.) This is the additional information constraint we are imposing. Twenty-five of the realizations were judged implausible; $n_{NP} = 25$. One of the implausible signals is shown in Figure 5. (It is plotted on linear axes to accentuate the features that lead to the judgment that it is implausible.)

The additional MCMC sampling outlined at the end of the previous section was applied, using both the ensembles of $\boldsymbol{U}$ and $\boldsymbol{U}^{(NP)}$. Some scatterplots of elements from the vectors of of $\boldsymbol{U}$, $\boldsymbol{U}^{(NP)}$, $\boldsymbol{U}^{(PR)}$, and $\boldsymbol{U}^{(AC)}$, are shown in Figures 6a and 6b. A striking feature is that elements of the vectors from the four overlapping spaces appear substantially intermixed. This indicates that subtle features in the scatterplots shown, and features in scatterplots of other elements, not shown, enforce the exclusion of vectors from $\boldsymbol{U}^{(NP)}$.

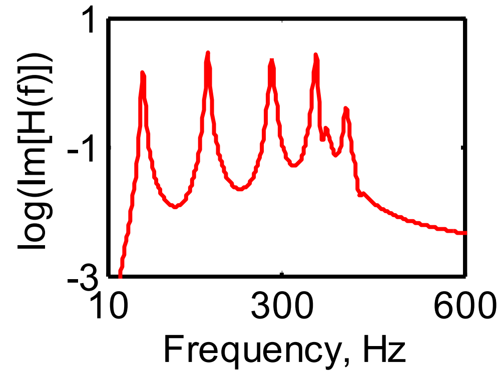


Figure 5. One example of an implausible, generated transformed imaginary part of an FRF.

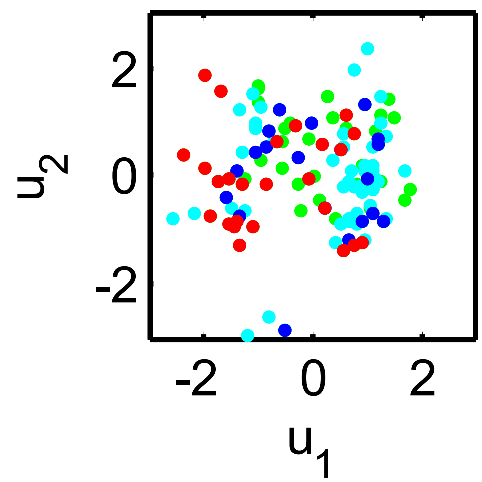


Figure 6a. Scatterplot of second element versus first element in $\boldsymbol{U}$ (blue), $\boldsymbol{U}^{(NP)}$ (red), $\boldsymbol{U}^{(PR)}$ (cyan), and $\boldsymbol{U}^{(AC)}$ (green).

The data $\boldsymbol{u}_j^{(gen,PR)}, j\epsilon\{1,\ldots,n_{PR}\}$ were used in (1) to obtain the imaginary parts of the ensemble $\boldsymbol{H}_j^{(gen,PR)}(f) \in \mathbb{C}^n, j \in \{1,\ldots,n_{PR}\}$. The latter are plotted in Figure 7, along with the data from Figure 3 and realizations that come from the $\boldsymbol{U}^{(AC)}$. Though they are not perfect representations of the original, measured data, the simulations appear to present plausible representations of the random source. Agreement could be improved by specifying a more stringent criterion for forming $\boldsymbol{U}^{(NP)}$.

## CONCLUSIONS

The Karhunen-Loeve expansion was developed to model the source of frequency response functions of a stochastic system. Both the kernel density estimator and a modified Markov chain Monte Carlo method were developed for use in conjunction with the KLE. The current approach augments an approach developed in a previous study. The effect of the augmentation is to eliminate adverse, unrealistic effects present in the earlier implementation. Further, the current approach introduces the idea that additional information can be brought into an MCMC analysis so long as the additional information can be quantifiably expressed.

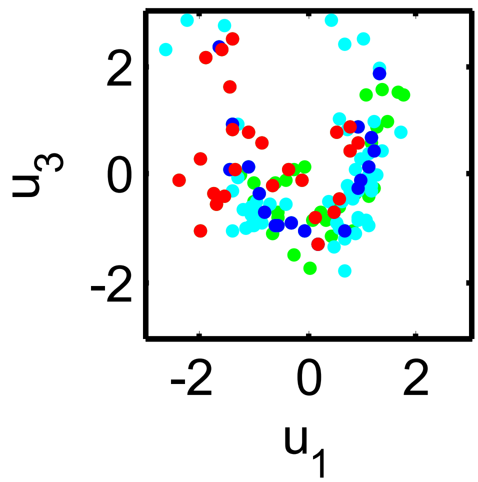


Figure 6b. Scatterplot of third element versus first element in $\boldsymbol{U}$ (blue), $\boldsymbol{U}^{(NP)}$ (red), $\boldsymbol{U}^{(PR)}$ (cyan), and $\boldsymbol{U}^{(AC)}$ (green).

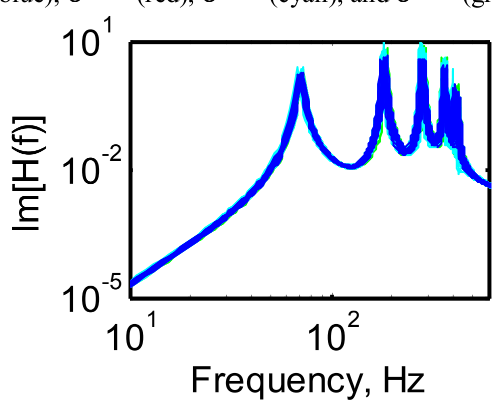


Figure 7. Generated realizations of the imaginary part of the FRF from the first MCMC simulation (green), the second simulation (cyan), and data upon which the model is based (blue).

The current technique yields realistic simulations of the frequency response function of a stochastic, second order system.

Although the current work has developed a technique that is superior to methods previously available, much work remains to develop the KLE as a general purpose tool for stochastic system modeling and simulation for identification and control problems. Among other things, the direct representation of FRFs must be improved even more, perhaps, through finer definition of acceptable FRF features.

Only the imaginary part of the FRF was modeled, here, and while the corresponding real part will normally be required, that is a matter of deterministic computation, for second order, linear systems, because the real and imaginary parts form a Hilbert transform pair.

Finally, methods for modeling and generating FRF matrices that involve multiple input locations and multiple output locations must be developed. Then the stochastic FRFs can be used to design MIMO controllers to meet closed-loop performance and stability objectives which can be expressed stochastically. This will permit a probabilistic characterization of the performance and stability of the ensemble of closed-loop systems.

The present work shows that in KLE there is ample promise that accurate, general-purpose FRF modeling will be realized.

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