2011 American Control Conference
on O'Farrell Street, San Francisco, CA, USA
June 29 - July 01, 2011

Derivation of an Optimal Boundary Layer Width for the Smooth Variable Structure Filter

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Abstract—**In this paper, an augmented form of the smooth variable structure filter (SVSF) is proposed. The SVSF is a state estimation strategy based on variable structure and sliding mode concepts. It uses a smoothing boundary to remove chattering (excessive switching along an estimated state trajectory). In its current form, the SVSF defines the boundary layer by an upper-bound on the uncertainties present in the estimation process (i.e., modeling errors, magnitude of noise, etc.). This is a conservative approach as one would be limiting the gain by assuming a larger smoothing boundary subspace than what is necessary. A more well-defined boundary layer will yield more accurate estimates. This paper derives a solution for an optimal boundary layer width by minimizing the trace of the a posteriori covariance matrix. The results of the derivation are simulated on a linear mechanical system for the purposes of control, and compared with the Kalman filter.**

I. Introduction

STATE and parameter estimation theory is particularly important for the successful control of mechanical and electrical systems. In most control scenarios, a number of states may be required, however direct observations or measurements may not always be available. Estimation tools such as filters can be used to extract information on the states from the system. The term filter is used because one needs to remove unwanted signals such as noise from the measurements, in order to obtain an accurate estimate of the states. Clearly, for the successful control of a system, accurate knowledge of the states is critical.

The most popular and well-studied estimation method is the Kalman filter (KF). Introduced in the early 1960's, it yields a statistically optimal solution for linear estimation problems in the presence of Gaussian noise [1]. The KF is formulated in a predictor-corrector manner, such that one first predicts the state estimates using knowledge of the system model. These estimates are termed a priori, meaning 'prior to' knowledge of the observations. A correction term is then added based on the innovation (also called residuals or measurement errors), thus forming the updated or a posteriori (meaning 'subsequent to' the observations) state estimates.

Manuscript received September 27, 2010.

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The extended Kalman filter (EKF) was introduced for nonlinear systems and measurements [2]. Essentially the EKF works in a similar fashion to the KF, but requires linearization (i.e., first-order Taylor series) [3]. This comes at a cost of optimality. The act of truncating the higher-orders through the process of linearization removes information on how the system behaves or the mapping of the measurements, thus introducing uncertainties in the estimation process. For mild nonlinearities, the EKF has been shown to work very well. However, the EKF is shown to fail in cases where the model or measurements have significant nonlinearities [4].

Sliding mode control and estimation techniques have been around for quite a few decades, and are mainly popular due to their relative ease of implementation and robustness to modeling uncertainties [5,6]. In a typical sliding mode control scenario, one utilizes a discontinuous switching gain to maintain the states along some desired trajectory [7]. This plane is quite often referred to as a sliding surface, in which the purpose is to keep the state values along this surface by minimizing the state errors. Ideally, if the state value is off or away from the surface, a switching gain would be used to push the state towards the sliding surface. The gain is calculated based on the proximity of the states from a switching hyperplane. Once on the surface, the states slide along the surface referred to as a sliding mode [7]. The discontinuous switching brings an inherent amount of stability to the control or estimation strategy, while also introducing excessive chattering (from the switching). These sliding mode concepts are based on variable structure control, in which one alters the dynamics of a system by the introduction of high-frequency switching [5].

A smoothing boundary layer may be introduced along the sliding surface in order to saturate and smooth out the chattering within the boundary region. In the KF, one may derive the optimal gain (to be applied on the *a priori* estimate) by taking the partial derivative of the *a posteriori* covariance (trace) with respect to the gain [2,8]. The trace of the covariance is taken because it represents the state error vector. Similarly, one can solve for the 'optimal' boundary layer widths for the SVSF by performing the same action, but with respect to the smoothing boundary layer term ψ.

II. Estimation Strategies

Consider a linear dynamic system defined by using the following two equations:

$$x_{k+1} = Ax_k + Bu_k + w_k \quad (1)$$
$$z_{k+1} = Hx_{k+1} + v_{k+1} \quad (2)$$

The Appendix contains a list of nomenclature. It is the goal of any estimation strategy to obtain the best possible state estimate $\hat{x}$ by minimizing the effects of modeling uncertainties (typically in A or H), as well as the system and measurement noises (w and v, respectively).

A. Kalman Filter

The following five equations form the core of the KF algorithm, and are used in an iterative fashion. Equation (3) defines the *a priori* estimate based on the system definition, and (4) is the corresponding state error covariance matrix. The Kalman gain is defined by (5), and is used to update the state estimate as shown in (6). The *a posteriori* state error covariance matrix is then calculated by (7).

$$\hat{x}_{k+1|k} = \hat{A}\hat{x}_{k|k} + \hat{B}u_k \quad (3)$$
$$P_{k+1|k} = HP_{k|k}H^T + Q_k \quad (4)$$
$$K_{k+1} = P_{k+1|k}H^T\left[HP_{k+1|k}H^T + R_{k+1}\right]^{-1} \quad (5)$$
$$\hat{x}_{k+1|k+1} = \hat{x}_{k+1|k} + K_{k+1}\left[z_{k+1} - H\hat{x}_{k+1|k}\right] \quad (6)$$
$$P_{k+1|k+1} = [I - K_{k+1}H]P_{k+1|k} \quad (7)$$

A number of different methods have extended the classical KF to nonlinear systems, with the most popular one being the EKF [4,9]. The EKF is conceptually similar to the KF; however, the nonlinear system is linearized according to its Jacobian. This linearization process introduces uncertainties that can sometimes cause instability [9].

B. Smooth Variable Structure Filter

The variable structure filter (VSF) was first proposed in 2003, and was introduced as a new type of predictor-corrector estimator based on the sliding mode concept [10]. It is a type of sliding mode estimator, where gain switching is used to ensure that the estimates converge to within a boundary of the true state (i.e., existence subspace). The smooth variable structure filter (SVSF) was later derived from the VSF, and uses a much simpler and less complex gain calculation [11]. In its present form, the SVSF is stable and robust to modeling uncertainties and noise, given an upper bound on the level of un-modeled dynamics or knowledge of the magnitude of noise.

The basic estimation concept of the SVSF is shown in the following figure. An initial estimate of the states is made based on probability distributions or designer knowledge. Through the use of the SVSF gain, the estimated state will be forced to within a region around the state trajectory referred to as the existence subspace. Once the estimate enters the existence subspace, it is forced into switching along the system state trajectory. A saturation term may be used in this region to reduce the magnitude of chattering and smooths-out the result. The SVSF is robust to uncertainties, making this strategy an attractive method for control problems when not all of the dynamics are well defined.

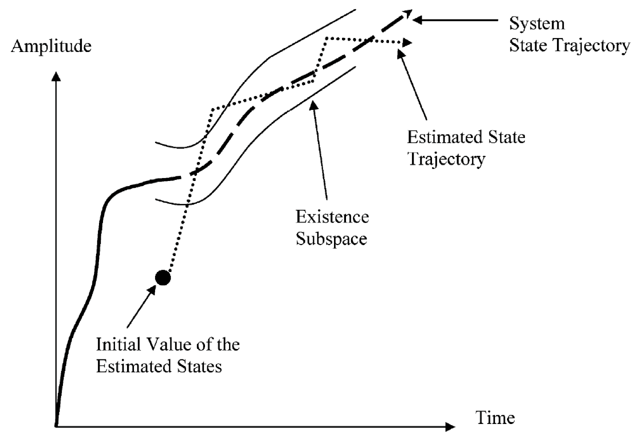

Fig. 1. The smooth variable structure filter estimation concept is shown in the above figure [11].

The SVSF method is model based and applies to differentiable linear or nonlinear dynamic equations. The estimation process is iterative and may be summarized by the following set of equations (for a linear control or estimation problem). Like the KF, the system model is used to calculate *a priori* state and measurement estimates. A corrective term, referred to as the SVSF gain, is calculated as a function of the error in the predicted output and a smoothing boundary layer. This gain is then used to update the state estimates. Note that the estimation process is stable due to the gain calculation of (11), which keeps the estimates bounded [11].

The switching found within the existence subspace is smoothed out by using the saturation term of (11), which is defined by the *a priori* output error and some predetermined smoothing boundary layer width. In its current form, the boundary layer width is defined by using the upper bound knowledge of the uncertainties and noise levels. However, an equation for the boundary layer width can be derived in the following section, to make it less conservative.

$$\hat{x}_{k+1|k} = \hat{A}\hat{x}_{k|k} + \hat{B}u_k \quad (8)$$
$$\hat{z}_{k+1|k} = \hat{H}\hat{x}_{k+1|k} \quad (9)$$
$$e_{z_{k+1|k}} = z_{k+1} - \hat{z}_{k+1|k} \quad (10)$$
$$K_{k+1} = \left(\left|e_{z_{k+1|k}}\right| + \gamma\left|e_{z_{k|k}}\right|\right) \circ sat\left(\frac{e_{z_{k+1|k}}}{\psi}\right) \quad (11)$$
$$\hat{x}_{k+1|k+1} = \hat{x}_{k+1|k} + K_{k+1} \quad (12)$$
$$e_{z_{k+1|k+1}} = z_{k+1} - \hat{H}\hat{x}_{k+1|k+1} \quad (13)$$

Two critical variables in the SVSF estimation process are the *a priori* and *a posteriori* output error estimates, defined by (10) and (13), respectively.

III. Derivation of the Boundary Layer

It is typically assumed that a boundary layer exists for each state trajectory. For the case when there are fewer measurements than states, one needs to implement the reduced order form of the SVSF [11]. This allows the creation of a full measurement matrix, typically in the form of an identity. Consequently, each measurement error can be mapped to its corresponding state boundary layer. For the case when there are more measurements than states, the system output (or measurements) can be mapped to the states. Therefore, it is assumed that the measurements are completely observable.

From [12], the revised SVSF gain and the *a posteriori* covariance (with an identity measurement matrix) are defined respectively as follows:

$$K_{k+1} = diag\left[\left(\left|e_{z_{k+1|k}}\right| + \gamma\left|e_{z_{k|k}}\right|\right) \circ sat\left(\bar{\psi}^{-1} e_{z_{k+1|k}}\right)\right] diag\left(e_{z_{k+1|k}}\right)^{-1} \quad (15)$$

$$P_{k+1|k+1} = (I - K_{k+1})P_{k+1|k}(I - K_{k+1})^T + K_{k+1}R_{k+1}K_{k+1}^T \quad (16)$$

A solution of the 'optimal' value may be found by solving for ψ from the following:

$$\frac{\partial\left(trace[P_{k+1|k+1}]\right)}{\partial\psi} = 0 \quad (17)$$

Essentially, one needs to substitute (15) into (16), and (17). Prior to doing this, consider the following definitions to simplify the process of determining a solution, let:

$$A = \left|e_{z_{k+1|k}}\right| + \gamma\left|e_{z_{k|k}}\right| \quad (18)$$

$$a \circ b = diag(b)a = \bar{b}a \quad (19)$$

Also, consider the following rules for solving partial derivatives with respect to a matrix [13]:

For D symmetric: $$\frac{\partial(trace[BCB^T])}{\partial B} = 2BC \quad (20)$$

For D not symmetric: $$\frac{\partial(trace[BC])}{\partial B} = C^T \quad (21)$$

According to the chain rule [14], one may be able to rewrite (17) as follows:

$$\frac{\partial\left(trace[P_{k+1|k+1}]\right)}{\partial\psi} = \frac{\partial\left(trace[P_{k+1|k+1}]\right)}{\partial K_{k+1}}\frac{\partial K_{k+1}}{\partial\psi} = 0 \quad (22)$$

Each partial derivative in (22) will be solved next. The first term is solved by starting with an expansion of (16):

$$P_{k+1|k+1} = P_{k+1|k} - K_{k+1}P_{k+1|k} - P_{k+1|k}K_{k+1}^T + K_{k+1}P_{k+1|k}K_{k+1}^T + K_{k+1}R_{k+1}K_{k+1}^T \quad (23)$$

Next, each term in (23) will be solved, as per (20) and (21):

$$\frac{\partial\left(trace[P_{k+1|k+1}]\right)}{\partial K_{k+1}} = 0 \quad (24)$$

$$\frac{\partial\left(trace[-K_{k+1}P_{k+1|k}]\right)}{\partial K_{k+1}} = -P_{k+1|k}^T \quad (25)$$

$$\frac{\partial\left(trace[-P_{k+1|k}K_{k+1}^T]\right)}{\partial K_{k+1}} = \frac{\partial\left(trace[-K_{k+1}P_{k+1|k}^T]\right)}{\partial K_{k+1}} = -P_{k+1|k} \quad (26)$$

$$\frac{\partial\left(trace[K_{k+1}P_{k+1|k}K_{k+1}^T]\right)}{\partial K_{k+1}} = 2K_{k+1}P_{k+1|k} \quad (27)$$

$$\frac{\partial(trace[K_{k+1}R_{k+1}K_{k+1}^T])}{\partial K_{k+1}} = 2K_{k+1}R_{k+1} \quad (28)$$

Combining (24) through (28) yields a solution for the first partial derivative in (22), as follows:

$$\frac{\partial\left(trace[P_{k+1|k+1}]\right)}{\partial K_{k+1}} = -P_{k+1|k}^T - P_{k+1|k} + 2K_{k+1}P_{k+1|k} + 2K_{k+1}R_{k+1} \quad (29)$$

Due to the fact that the state error covariance matrix is symmetric, (29) may be simplified further:

$$\frac{\partial\left(trace[P_{k+1|k+1}]\right)}{\partial K_{k+1}} = -2P_{k+1|k} + 2K_{k+1}\left(P_{k+1|k} + R_{k+1}\right) \quad (30)$$

It is important to note that the Kalman gain may be obtained by solving for K_{k+1} in the above equation (by setting it to zero) [8]. The second partial derivative in (22) will now be solved. Note that the region of interest for the value of the boundary layer width is inside the saturation term of (15). Furthermore, using definitions (18) and (19), the gain may be rewritten as:

$$K_{k+1} = diag\left(\bar{\psi}^{-1}\bar{e}_{z_{k+1|k}}A\right) diag\left(e_{z_{k+1|k}}\right)^{-1} \quad (31)$$

Simplifying (31) yields:

$$K_{k+1} = \bar{\psi}^{-1}\bar{A} \quad (32)$$

Also, note the following two properties for matrix derivatives [14], assuming that matrix D is invertible and t is some parameter of interest:

$$\frac{\partial DE}{\partial t} = \frac{\partial D}{\partial t}E + D\frac{\partial E}{\partial t} \quad (33)$$

$$\frac{\partial D^{-1}}{\partial t} = -D^{-1}\frac{\partial D}{\partial t}D^{-1} \quad (34)$$

From (32) and (33), one has:

$$\frac{\partial K_{k+1}}{\partial\psi} = \frac{\partial\bar{\psi}^{-1}}{\partial\psi}\bar{A} + \bar{\psi}^{-1}\frac{\partial\bar{A}}{\partial\psi} \quad (35)$$

Where (35) simplifies to the following:

$$\frac{\partial K_{k+1}}{\partial \psi} = \frac{\partial \bar{\psi}^{-1}}{\partial \psi}\bar{A} \quad (36)$$

Utilizing (34), (36) can be written as:

$$\frac{\partial K_{k+1}}{\partial \psi} = -\bar{\psi}^{-1}\frac{\partial \bar{\psi}}{\partial \psi}\bar{\psi}^{-1}\bar{A} \quad (37)$$

Finally, the second term in (22) may be found by simplifying (37), such that:

$$\frac{\partial K_{k+1}}{\partial \psi} = -\bar{\psi}^{-2}\bar{A} \quad (38)$$

Substituting (30) and (38) into (22) yields:

$$\left[-2P_{k+1|k} + 2K_{k+1}\left(P_{k+1|k} + R_{k+1}\right)\right]\left(-\bar{\psi}^{-2}\bar{A}\right) = 0 \quad (39)$$

Recall that the measurement innovation covariance (with an identity for the measurement matrix) is defined by [2]:

$$S_{k+1} = P_{k+1|k} + R_{k+1} \quad (40)$$

Such that (39) may be written as:

$$\left[-2P_{k+1|k} + 2K_{k+1}S_{k+1}\right]\left(-\bar{\psi}^{-2}\bar{A}\right) = 0 \quad (41)$$

Simplifying (41) further yields:

$$\left[-P_{k+1|k} + K_{k+1}S_{k+1}\right]\bar{A} = 0 \quad (42)$$

Substituting (32) into (42) and expanding further gives:

$$-P_{k+1|k} + \bar{\psi}^{-1}\bar{A}\left(P_{k+1|k} + R_{k+1}\right) = 0 \quad (43)$$

Now what remains is to simplify (43) and solve for the boundary layer width ψ. This is accomplished by isolating the ψ term:

$$\bar{\psi}^{-1} = P_{k+1|k}\overline{\left[\bar{A}\left(P_{k+1|k} + R_{k+1}\right)\right]}^{-1} \quad (44)$$

Finally, solving (44) yields an equation for a variable boundary layer (to be calculated at each time step) in a matrix form:

$$\psi_{k+1} = diag\left[\overline{P_{k+1|k}\overline{\left[\bar{A}\left(P_{k+1|k} + R_{k+1}\right)\right]}^{-1}}^{-1}\right] \quad (45)$$

Furthermore, one can define a simplified version of (45) as follows:

$$\psi_{k+1} = \left(\overline{P_{k+1|k} + R_{k+1}}\right)\overline{P_{k+1|k}}^{-1}\left(\left|e_{z_{k+1|k}}\right| + \gamma\left|e_{z_{k|k}}\right|\right) \quad (46)$$

The proposed boundary layer equation (46) is found to be a function of the *a priori* state error covariance, measurement covariance, *a priori* and previous *a posteriori* measurement errors, and the convergence rate or SVSF 'memory'. It appears that the width of the boundary layer is therefore directly related to the level of modeling uncertainties (by virtue of the errors) and measurement noise. There is no need to define the boundary layer widths as before, as they now may be solved directly at each time step. A revised SVSF estimation process may be summarized as follows. Note that [12] describes the covariance derivations in detail.

$$\hat{x}_{k+1|k} = \hat{A}\hat{x}_{k|k} + \hat{B}u_k \quad (47)$$

$$P_{k+1|k} = \hat{A}P_{k|k}\hat{A}^T + Q_{k+1} \quad (48)$$

$$e_{z_{k+1|k}} = z_{k+1} - \hat{H}\hat{x}_{k+1|k} \quad (49)$$

$$\psi_{k+1} = \left(\overline{P_{k+1|k} + R_{k+1}}\right)\overline{P_{k+1|k}}^{-1}\left(\left|e_{z_{k+1|k}}\right| + \gamma\left|e_{z_{k|k}}\right|\right) \quad (50)$$

$$K_{k+1} = \left(\left|e_{z_{k+1|k}}\right| + \gamma\left|e_{z_{k|k}}\right|\right) \circ sat\left(\frac{e_{z_{k+1|k}}}{\psi_{k+1}}\right) \quad (51)$$

$$\hat{x}_{k+1|k+1} = \hat{x}_{k+1|k} + K_{k+1} \quad (52)$$

$$P_{k+1|k+1} = (I - K_{k+1})P_{k+1|k}(I - K_{k+1})^T + K_{k+1}R_{k+1}K_{k+1}^T \quad (53)$$

$$e_{z_{k+1|k+1}} = z_{k+1} - \hat{H}\hat{x}_{k+1|k+1} \quad (54)$$

IV. Simulation Example

In this section, an electrohydrostatic actuator (EHA) is simulated. The system is based on an actual prototype built for control experimentation [11,15]. The purpose of this simulation is to demonstrate that the new form of the SVSF is numerically stable, and provides an alternative to the Kalman filter (KF) for systems when modeling uncertainties are present. However, recall that for known linear systems, the KF will yield the optimal solution (i.e., best estimate).

The EHA is a third order system with state variables related to its position, velocity, and acceleration. It is assumed that all three states have measurements associated with them (i.e., full measurement matrix). The input to the system is a random normal distribution with magnitude 1. A step change is inserted into the input of the system half-way through the duration. The sample time of the system is 0.001 seconds. The entire EHA system description may be found in [15], however for the purpose of this paper, the discrete state-space model of the system is simply defined as follows:

$$x_{k+1} = \begin{bmatrix} 1 & 0.001 & 0 \\ 0 & 1 & 0.001 \\ -557.02 & -28.616 & 0.9418 \end{bmatrix} x_k + \begin{bmatrix} 0 \\ 0 \\ 557.02 \end{bmatrix} u_k \quad (55)$$

The initial state values are set to zero. The system and measurement noises are considered to be Gaussian with maximum amplitude corresponding to 5 to 10% error ($W_{Max} = [0.01 \quad 0.1 \quad 1]^T$ and $V_{Max} = [0.1 \quad 1 \quad 10]^T$). The initial state error covariance, system noise covariance, and measurement noise covariance are defined respectively as follows:

$$P_{0|0} = 10Q \quad (56)$$
$$Q = 10diag(W_{Max}) \quad (57)$$
$$R = 10diag(V_{Max}) \quad (58)$$

For the SVSF estimation process, the boundary layer widths were initialized as $\psi_0 = [1 \quad 10 \quad 100]^T$. The boundary widths were initialized based on some *a priori* knowledge of the noise; however, their values are not very sensitive to the success of the estimation process. The main results of applying the KF and the new SVSF on the EHA problem are shown in the following figure. This figure shows the true position of the EHA with the corresponding estimates. The estimation results of both filters are relatively the same. It is important to remind the reader that the KF is an optimal strategy given white noise and in the absence of uncertainties, so at the very best the SVSF can only match the estimation accuracy.

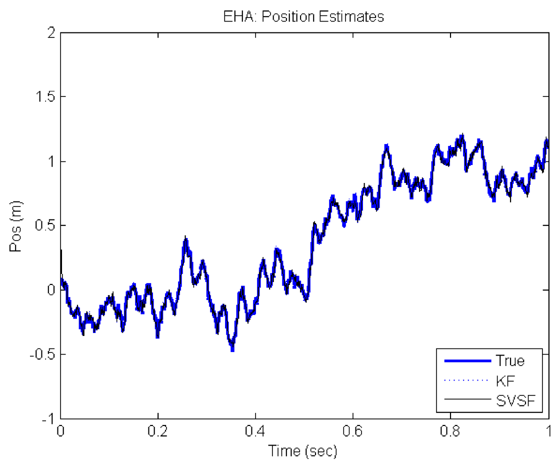

Fig. 2. EHA simulation results with the Kalman filter and the proposed smooth variable structure filter. The results are nearly the same; hence the lines are difficult to distinguish.

The velocity and acceleration estimates were relatively the same as those shown in the above figure (and were thus omitted for space constraints). The RMSE results of running the simulation are as follows:

TABLE I
RMSE SIMULATION RESULTS

Filter	Position (m)	Velocity (m/s)	Acceleration (m/s^2)
KF	0.0233	0.2370	2.4415
SVSF	0.0246	0.2381	2.4422

As shown in the above table, the KF provides the optimal result. However, the new form of the SVSF yields a near-optimal estimate of the states. Although the previous form of the SVSF yielded relatively good results, it was still not optimal [12]. An advantage of using the new form of the SVSF over the KF is its robustness to modeling errors and uncertainties.

Consider another case where there are modeling errors and the filter model is defined incorrectly as follows:

$$x_{k+1} = \begin{bmatrix} 1 & 0.001 & 0 \\ 0 & 1 & 0.001 \\ -240 & -28 & 0.9418 \end{bmatrix} x_k + \begin{bmatrix} 0 \\ 0 \\ 557.02 \end{bmatrix} u_k \quad (59)$$

The following figure shows the results of applying the KF and the new SVSF on the EHA problem with an incorrect system model introduced at 0.5 seconds, as defined by (59).

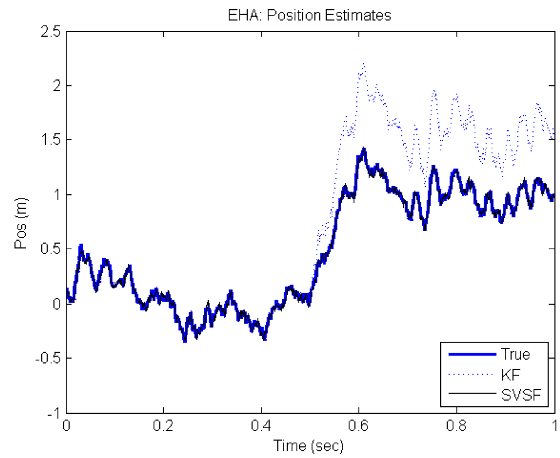

Fig. 3. EHA simulation results with modeling uncertainties introduced at 0.5 seconds. Notice how the KF fails to yield a reasonable estimate.

The RMSE results of running the simulation with modeling uncertainties are as follows:

TABLE II
RMSE SIMULATION RESULTS

Filter	Position (m)	Velocity (m/s)	Acceleration (m/s^2)
KF	0.4036	0.5018	2.7031
SVSF	0.0234	0.2276	2.7125

For this case, as shown in the previous table, the SVSF provides the best result in terms of estimation error. The position error for the KF increased by roughly 2,000%, and the velocity error increased by about 200%. The modeling errors were introduced in the acceleration term of the system model described by (59). Both the KF and SVSF yielded the same acceleration estimate. However, the SVSF remained stable and near-optimal for the first two states, unlike the KF. The following figure shows the boundary layer width over-time for the acceleration estimate. Notice how the boundary layer increases at the inception of the modeling uncertainties at 0.5 seconds. This compensates for the increased errors and uncertainties.

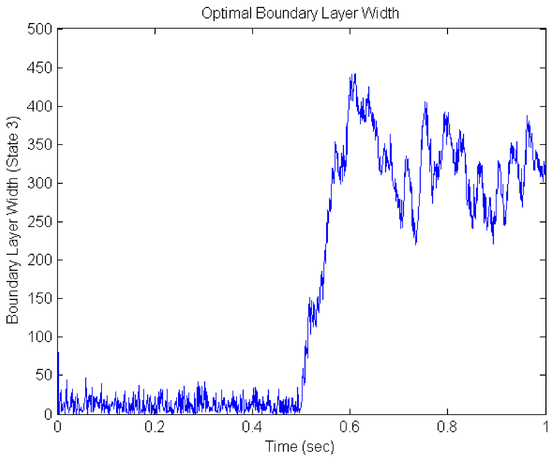

Fig. 4. The acceleration boundary layer width is shown above. Notice how the width increases at the introduction of modeling uncertainties.

V. Conclusions

This paper introduced a revised form of the smooth variable structure filter with an optimal variable boundary layer. The proposed estimation strategy was applied to an electrohydrostatic actuator for comparison with the popular Kalman filter. For known linear systems, the estimation errors are comparable (1% difference). However, in the presence of modeling uncertainties or errors, the SVSF was shown to yield the best result. The revised form of the SVSF yields a near-optimal estimation and is demonstrated to be robust to modeling errors.

Appendix

The following is a table of important nomenclature used throughout this paper:

TABLE III
List of Nomenclature

Parameter	Definition
x	State vector or values
z	Measurement (system output) vector or values
u	Input to the system
w	System noise vector
v	Measurement noise vector
A	Linear system transition matrix
B	Input gain matrix
H	Linear measurement (output) matrix
K	Filter gain matrix (i.e., KF or SVSF)
P	State error covariance matrix
Q	System noise covariance matrix
R	Measurement noise covariance matrix
e	Measurement (output) error vector
$diag(a)$ or $\bar{a}$	Defines a diagonal matrix of some vector *a*
$sat(a)$	Defines a saturation of the term *a*
γ	SVSF 'convergence' or memory parameter
ψ	SVSF boundary layer width
$\|a\|$	Absolute value of some parameter *a*
T	Transpose of some vector or matrix
^	Estimated vector or values
$k+1\|k$	A priori time step (i.e., before applied gain)
$k+1\|k+1$	A posteriori time step (i.e., after update)

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