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A Generalized Scaling Based Control Design for Nonlinear Nontriangular Systems with Input and State Time Delays

P. Krishnamurthy and F. Khorrami

***Abstract*— A general class of nonlinear systems containing delays in both state and input is considered and a dynamic high-gain scaling based control design is proposed. The class of systems considered is of a general structure which is not necessarily of any triangular form. All functions appearing in the system dynamics are allowed to be uncertain as long as some polynomial bounds on ratios of uncertain system terms are available. Both state and input delays are allowed to be time-varying and uncertain. The control design is based on our recent results on generalized scaling utilizing appropriate (not necessarily successive) powers of the scaling parameter. The control implementation does not require knowledge of the time delays or any magnitude bounds thereof; the only information about the time delays that is required is a bound on the rate of variation of time delays.**

I. Introduction

The control of systems with state and input time delays has attracted significant research interest in the literature (see [1]–[11] and the references therein) for a variety of classes of nonlinear systems. Control Lyapunov-Razumikhin functions and control Lyapunov-Krasovskii functionals have been considered to provide constructive tools for control design [1], [2] for delayed systems. A domination redesign approach based on control Lyapunov-Razumikhin functions and backstepping based design yielding delay-independent feedback were proposed in [4]. Adaptive backstepping based on a LaSalle-Razumikhin approach using a Lyapunov-Razumikhin function was proposed in [5]. Robust backstepping of time delayed systems has also been considered in [7]. The case of input delays (or equivalently measurement delays) has been addressed in [6], [9]–[11].

The dynamic high-gain scaling based design technique [12], [14], [16], [25]–[27] has been developed in a recent sequence of papers and has been demonstrated to be a versatile control design approach for various classes of nonlinear systems including both strict-feedback and feedforward classes of systems as well as polynomially bounded nontriangular systems. High-gain scaling is a popular technique for the control of strict-feedback systems and various high-gain based controller and observer designs have been considered in the literature ([18]–[21] and references therein). A combination of a high-gain observer and a backstepping based controller was proposed in [15], [22] with the dynamics of the scaling parameter r being of the form of a scalar Riccati equation. The dual observer/controller dynamic high-gain scaling technique was introduced in [16], [23] and shown to be a flexible design technique capable of handling uncertain terms dependent on all states and uncertain ISS appended dynamics with nonlinear gains from all the system states and the input (previous results allowed the ISS appended dynamics to have a nonzero gain only from the output). The dynamic high-gain scaling technique provides a unified framework for state-feedback and output-feedback control of both strict-feedback [16], [24]–[26] and feedforward [14] systems as well as state-feedback control of nontriangular polynomially-bounded systems [27]. The control of a specific structure of systems with state delay (but no input delays) that admit a linear observer/controller design as a particular special case of the dual high-gain scaling approach was addressed in [31]. The control of feedforward systems with input and state delays was addressed in [29], [30] based on the dynamic dual high-gain scaling technique. The application of the dual dynamic high-gain scaling approach to control of strict-feedback systems with input and state delays was considered in [17].

The authors are with the Control/Robotics Research Laboratory, Dept. of ECE, Polytechnic Institute of NYU, Brooklyn, NY 11201, USA. This work was supported in part by the NSF under grant ECS-0501539. Corresponding author: F. Khorrami, khorrami@smart.poly.edu.

In this paper, we further investigate the robustness of the dynamic high-gain scaling based controllers as applied to the following class of nontriangular nonlinear systems that features state and input time delays:

$$\begin{aligned}\dot{x}_i(t) &= \phi_{(i,i+1)}(t, x(t), u(t))x_{i+1} \\ &\quad + \phi_i(t, x(t), u(t), x(t-\tau_i), u(t-\tau_i)) \\ &\quad \text{for } i = 1, \ldots, n-1 \\ \dot{x}_n(t) &= \phi_n(t, x(t), u(t), x(t-\tau_n), u(t-\tau_n)) + \mu(t, x, u)u(t) \quad (1)\end{aligned}$$

where $x = [x_1, \ldots, x_n]^T \in \mathcal{R}^n$ is the state and $u \in \mathcal{R}$ is the input. $\phi_{(i,i+1)} : \mathcal{R}^{n+2} \to \mathcal{R}, i = 1, \ldots, n-1$, $\phi_i : \mathcal{R}^{2n+3} \to \mathcal{R}, i = 1, \ldots, n$, and $\mu : \mathcal{R}^{n+2} \to \mathcal{R}$ are uncertain continuous functions[1]. $\tau_1, \ldots, \tau_n$ are uncertain non-negative values representing time-varying time delays in state and input signals. It is assumed that sufficient conditions (e.g., local Lipschitz property) on ϕ_i needed for local existence and uniqueness of solutions of (1) are satisfied. A general class of non-triangular systems with system uncertainties and uncertain time delays is considered and a state-feedback controller is proposed based on the generalized scaling technique. As a motivating example of the class of systems that is addressed by the proposed control design methodology, consider the third-order system given by:

$$\begin{aligned}\dot{x}_1(t) &= (1+x_1^2(t))x_2(t)+x_1^5(t)+x_1^3(t-\tau_1) \\ \dot{x}_2(t) &= (1 + x_3^2(t) + \sqrt{|u(t)|})x_3(t) + x_1^2(t)x_2(t)x_3(t) \\ &\quad +x_1^2(t-\tau_2)\sqrt{|x_3(t-\tau_2)|} \\ &\quad +x_1(t-\tau_2)x_2(t-\tau_2)\sin(x_3(t) + x_3(t-\tau_2)) \\ \dot{x}_3(t) &= u(t) + x_1^2(t)x_2^3(t) + x_1^2(t-\tau_3)|u(t-\tau_3)|^{\frac{1}{4}} \\ &\quad +x_1^2(t-\tau_3)x_2^2(t-\tau_3)\cos(u(t-\tau_3)) \quad (2)\end{aligned}$$

where τ_1, τ_2, and τ_3 are uncertain possibly time-varying time delays. The system (2) is in neither strict-feedback nor feedforward triangular structures. The global stabilization and asymptotic regulation (to zero) problem for this system cannot be addressed using any systematic control design methodology currently available in the literature. If τ_1, τ_2, and τ_3 are zero, then the generalized scaling technique [27] provides a globally stabilizing high-gain scaling based control design that regulates $x(t)$ and $u(t)$ exponentially to zero as $t \to \infty$. The goal of this paper is to extend the generalized scaling technique to handle time delayed functions of input and state in the system dynamics and thereby to provide control designs that achieve global asymptotic results for systems such as (2).

II. Statement of Main Result

The problem addressed in this paper is the design of a dynamic state-feedback controller of the following form to globally stabilize the system (1) and regulate the signals $x(t)$ and $u(t)$ to zero as $t \to \infty$:

$$\dot{\varpi}(t) = \Omega_1(x(t), \varpi(t)) \;;\; u(t) = \Omega_2(x(t), \varpi(t)). \quad (3)$$

This stabilization problem will be addressed under the assumptions listed below. Note that the dynamic control law (3) is delay-free in the sense that it does not utilize delayed versions of the state or input. The functions $\phi_i, i = 1, \ldots, n$, $\phi_{(i,i+1)}, i = 1, \ldots, n-1$, and μ are uncertain functions, regarding which no

[1] For simplicity, a single time delay value is considered in (1) for all components entering into the dynamics of each state variable. The proposed controller design can evidently be applied (with appropriate additional terms in the overall system Lyapunov function) to the case when there are multiple time delay values in the system dynamics.

knowledge is assumed to be available beyond what is stated in the assumptions listed below. Furthermore, the time delay values $\tau_1, \ldots, \tau_n$ are allowed to be time-varying and uncertain, with no knowledge assumed beyond what is stated in Assumption A6. Before stating the main result of the paper, some required notations and terminology are summarized below:

$\mathcal{R}$, $\mathcal{R}^+$, $\mathcal{R}^k$, and $\mathcal{R}^{k^*}$ denote the set of real numbers, the set of non-negative real numbers, the set of real k-dimensional column vectors, and the set of real k-dimensional row vectors, respectively. diag$(\eta_1, \ldots, \eta_k)$ denotes the $k \times k$ diagonal matrix with the i^{th} diagonal element being η_i. upperdiag$(\eta_1, \ldots, \eta_{k-1})$ denotes the $k \times k$ matrix with the $(i, i+1)^{th}$ entry being $\eta_i, i = 1, \ldots, k-1$, and zeros elsewhere. If $\eta \in \mathcal{R}^k$, $S(\eta)$ denotes the $k \times k$ diagonal matrix with (i,i) entry being the sign (± 1; sign of zero taken to be $+1$) of the i^{th} element of η. A function $f : \mathcal{R}^l \rightarrow \mathcal{R}$ is a multinomial if it is of the form $f(z_1, \ldots, z_l) = \sum_{k=1}^{N} \chi_k \prod_{i=1}^{l} z_i^{\beta_{(i,k)}}$, $N \geq 1$ with χ_k and $\beta_{(i,k)}, i = 1, \ldots, l, k = 1, \ldots, N$, being nonnegative real numbers. A multinomial f is said to be superlinear if a continuous nonnegative function $\overline{f}$ (called its bounding function) exists such that $|f(z_1, \ldots, z_l)| \leq \overline{f}(z_1, \ldots, z_l)\sqrt{\sum_{i=1}^{l} z_i^2}$. A real number ζ is said to dominate f relative to real numbers $\zeta_1, \ldots, \zeta_l$ if $\zeta \geq \zeta_1 \beta_{(1,k)} + \ldots + \zeta_l \beta_{(l,k)}, k = 1, \ldots, N$. To denote that ζ dominates f relative to $\zeta_1, \ldots, \zeta_l$, we use the notation $\zeta \succ f|_{(\zeta_1, \ldots, \zeta_l)}$. It can be shown that the multinomial of form defined above is superlinear if and only if $\sum_{i=1}^{l} \beta_{(i,k)} \geq 1$ for each $k \in \{1, \ldots, N\}$ for which $\chi_k > 0$. The notation $|\pi|$ denotes the Euclidean norm of a vector (or scalar) π.

The principal result of this paper is stated in Theorem 1 below, the proof of which is presented in Section III.

Theorem 1: Under Assumptions A1-A6 listed below, continuous functions $K : \mathcal{R}^{n+2} \rightarrow \mathcal{R}^{n+1^*}$, $q : \mathcal{R} \rightarrow \mathcal{R}^+$, $\overline{R} : \mathcal{R}^{n+1} \rightarrow \mathcal{R}^+$, and $\alpha : \mathcal{R}^{n+2} \rightarrow \mathcal{R}^+$, a nonnegative constant q_{n+1}, and positive constants $q_1, \ldots, q_n$, and b_q can be chosen such that all solution trajectories of the closed-loop system formed by the dynamic controller [2]

$$\xi_i = \frac{x_i}{r^{q_i}}, i = 1, \ldots, n+1 \, ; \, \xi = [\xi_1, \ldots, \xi_n, \xi_{n+1}]^T \quad (4)$$

$$u = \tilde{\mu}(x) x_{n+1} \ ; \ \dot{x}_{n+1} = v \quad (5)$$

$$v = r^{q_{n+1}} \left[K(x, x_{n+1}, r)\xi - b_q \frac{\dot{r}}{r} \xi_{n+1} \right] \quad (6)$$

$$\dot{r} = rq(\overline{R}(\xi) - r)\alpha(x, x_{n+1}, r) \ ; \ r(0) \geq 1 \quad (7)$$

and system (1) starting from any initial conditions $(x(0), x_{n+1}(0), r(0)) \in \mathcal{R}^n \times \mathcal{R} \times [1, \infty)$ have the following properties: (a) all closed-loop signals are bounded on the time interval $t \in [0, \infty)$, (b) the signals $x_i(t), i = 1, \ldots, n+1$, and $u(t)$ asymptotically converge to zero as $t \rightarrow \infty$.

Assumption A1: A constant $\sigma > 0$ is known such that for all $t \in \mathcal{R}^+$, $x \in \mathcal{R}^n$, and $u \in \mathcal{R}$, $|\phi_{(i,i+1)}(t, x, u)| \geq \sigma > 0$, $i = 1, \ldots, n-1$ and $|\mu(t, x, u)| \geq \sigma > 0$. The sign of each $\phi_{(i,i+1)}, i = 1, \ldots, n-1$, and of μ is independent of its arguments and known.

Assumption A2: Continuous functions $\overline{\phi}_{(i,i+1)} : \mathcal{R}^{n+1} \rightarrow \mathcal{R}^+, i = 1, \ldots, n-1$ and $\overline{\mu} : \mathcal{R}^{n+1} \rightarrow \mathcal{R}^+$, are known such that $|\phi_{(i,i+1)}(t, x, u)| \leq \overline{\phi}_{(i,i+1)}(x, u)$ and $|\mu(t, x, u)| \leq \overline{\mu}(x, u)$ for all $t \in \mathcal{R}^+$, $x \in \mathcal{R}^n$, and $u \in \mathcal{R}$.

Assumption A3: Continuous (possibly uncertain) functions $\phi_{(i,j)} : \mathcal{R}^{n+2} \rightarrow \mathcal{R}^+, i = 1, \ldots, n, j = 1, \ldots, i$, $\Xi_{(i,j)} : \mathcal{R}^{n+1} \rightarrow \mathcal{R}^+, i = 1, \ldots, n, j = 1, \ldots, i$, and $\phi_{fi} : \mathcal{R}^{n+2} \rightarrow \mathcal{R}^+, i = 1, \ldots, n$, exist such that the inequalities in (8) hold for all $t \in \mathcal{R}^+$, $x \in \mathcal{R}^n$, and $u \in \mathcal{R}$. Continuous functions $\overline{\phi}_{(n,j)} : \mathcal{R}^{n+1} \rightarrow \mathcal{R}^+, j = 1, \ldots, n$, are known such that $\phi_{(n,j)}(t, x, u) \leq \overline{\phi}_{(n,j)}(x, u), j = 1, \ldots, n$ for all $t \in \mathcal{R}^+$, $x \in \mathcal{R}^n$, and $u \in \mathcal{R}$.

[2]In cases where all signals appearing in an equation are indexed at the same time (e.g., in equations (4)-(7)), the explicit time argument is omitted for notational convenience and clarity.

Assumption A4: Continuous functions $\tilde{\mu} : \mathcal{R}^n \rightarrow \mathcal{R}^+$ and $\gamma_u : \mathcal{R}^n \rightarrow \mathcal{R}^+$, superlinear multinomials $f_i, i = 1, \ldots, n$, and (not necessarily superlinear) multinomials $f_{(i,j)}, i = 1, \ldots, n, j = 1, \ldots, i$, $\tilde{f}_{(i,j)}, i = 1, \ldots, n, j = 1, \ldots, i$, and $\tilde{f}_i, i = 2, \ldots, n$, are known such that the following inequalities (where $\phi_{(n,n+1)}(t, x, u) \stackrel{\triangle}{=} \tilde{\mu}(x)\mu(t, x, u)$) hold for all $t \in \mathcal{R}^+$, $x \in \mathcal{R}^n$, and $u \in \mathcal{R}$, with $\phi_{(0,1)} \stackrel{\triangle}{=} \phi_{(1,2)}$:

$$\frac{\phi_{(i,j)}(t, x, u)}{\sqrt{|\phi_{(i,i+1)}(t, x, u)||\phi_{(j-1,j)}(t, x, u)|}} \leq f_{(i,j)}\left(|x_1|, \ldots, |x_n|, \gamma_u(x)|u|\right), \quad \text{for } i = 1, \ldots, n, j = 1, \ldots, i \quad (9)$$

$$\frac{|\Xi^2_{(i,j)}(x, u)|}{|\phi_{(1,2)}(t, x, u)|} \leq \tilde{f}_{(i,j)}\left(|x_1|, \ldots, |x_n|, \gamma_u(x)|u|\right) \quad \text{for } i = 1, \ldots, n, j = 1, \ldots, i \quad (10)$$

$$\frac{|\phi_{(i-1,i)}(t, x, u)|}{|\phi_{(i,i+1)}(t, x, u)|} \leq \tilde{f}_i\left(|x_1|, \ldots, |x_n|, \gamma_u(x)|u|\right) \quad \text{for } i = 2, \ldots, n \quad (11)$$

$$\phi_{fi}(t, x, u) \leq f_i(|x_1|, \ldots, |x_n|, \gamma_u(x)|u|), i = 1, \ldots, n \quad (12)$$

and the inequality $\tilde{\mu}(x)\gamma_u(x) \leq \mu^*$ holds for all $x \in \mathcal{R}^n$ with μ^* being a known positive constant.

Assumption A5: Positive constants $c_{(i,j)}, i = 1, \ldots, n, j = 1, \ldots, i$, $\tilde{c}_{(i,j)}, i = 1, \ldots, n, j = 1, \ldots, i$, $c_i, i = 1, \ldots, n+1$, $\tilde{c}_i, i = 2, \ldots, n$, and $q_i, i = 1, \ldots, n$, and a (not necessarily positive) constant q_{n+1} exist such that the set of linear inequalities in (13)-(17) are satisfied. If any of $f_{(i,j)}, i = 1, \ldots, n, j = 1, \ldots, i$, $\tilde{f}_{(i,j)}, i = 1, \ldots, n, j = 1, \ldots, i$, or $\tilde{f}_i, i = 2, \ldots, n$, are non-zero constants, the right hand sides of the corresponding inequalities in (13)-(16) reduce to zero. If any of $f_{(i,j)}, i = 1, \ldots, n, j = 1, \ldots, i$, or $\tilde{f}_{(i,j)}, i = 1, \ldots, n, j = 1, \ldots, i$, are zero, the corresponding inequalities in (13)-(15) can be dropped. None of the $\tilde{f}_i$ can be zero since $\phi_{(i,i+1)}, i = 1, \ldots, n$, are lower bounded in magnitude by σ. If any of $f_i, i = 1, \ldots, n+1$, are zero, the corresponding inequalities in (17) can be dropped. Note that none of the f_i can be a non-zero constant since $f_i, i = 1, \ldots, n+1$, are superlinear multinomials.

Assumption A6: The time-varying time delays $\tau_1, \ldots, \tau_n$ are uniformly bounded in time and satisfy, for all time, the inequalities $|\dot{\tau}_i| \leq \overline{\tau}_i < 1$, $i = 1, \ldots, n$, where $\overline{\tau}_i$ are known constants.

Remark 1: Assumption A1 ensures the controllability of the system (1). Assumption A2 imposes the requirement that the uncertain functions $\phi_{(i,i+1)}$ and μ must have known time-independent upper bounds as functions of x and u. Assumption A3 imposes bounds on the uncertain terms $\phi_{(i,j)}$ in the system dynamics; the structure of the bounds is very general and essentially only requires that the upper bounds on the uncertain functions should admit factorizations into nonlinear and linear functions. Assumption A4 addresses the relative sizes (in a nonlinear function sense) of the terms $\phi_{(i,j)}$, $\Xi_{(i,j)}$, and $\phi_{(i,i+1)}$ and plays a crucial role in ensuring solvability of a pair of coupled Lyapunov inequalities as will be seen during the stability analysis. The inequalities in Assumption A4 are formulated in terms of superlinear multinomial functions of the entire state and input and are quite general and do not particularly impose a tight restriction on the system terms; in particular, inequalities as in Assumption A4 are definitely satisfied if the functions $\phi_{(i,j)}$, $\Xi_{(i,j)}$, and $\phi_{(i,i+1)}$ are themselves upper bounded by superlinear multinomials (a very general class). Assumption A5 prescribes a set of linear inequalities, a solution of which will be seen to play a role as scaling powers of a high-gain scaling parameter. Assumption A6 is a fairly standard assumption in control of systems with time-varying delays and essentially requires that the values of the time delays should not change faster than "real-time". Among the Assumptions A1-A6, it is only Assumption A5 that imposes a relatively strong restriction

$$|\phi_i(t,x(t),u(t),x(t-\tau_i),u(t-\tau_i))| \leq \sum_{j=1}^{i}\phi_{(i,j)}(t,x(t),u(t))|x_j(t)| + \sum_{j=1}^{i}\Xi_{(i,j)}(x(t-\tau_i),u(t-\tau_i))|x_j(t-\tau_i)| + |\phi_{(1,2)}(t,x(t),u(t))|\phi_{fi}(t,x(t),u(t))\ , \ i=1,\ldots,n \quad (8)$$

$$\frac{q_{i+1}+q_i-q_j-q_{j-1}}{2} - c_{(i,j)} \succ f_{(i,j)}|_{(q_1,\ldots,q_{n+1})}, i=2,\ldots,n\ ,\ j=2,\ldots,i \quad (13)$$

$$\frac{q_{i+1}+q_i+q_2-3q_1}{2} - c_{(i,1)} \succ f_{(i,1)}|_{(q_1,\ldots,q_{n+1})}, i=1,\ldots,n \quad (14)$$

$$q_2-q_1-2q_j+2q_i-\tilde{c}_{(i,j)} \succ \tilde{f}_{(i,j)}|_{(q_1,\ldots,q_{n+1})}, i=1,\ldots,n\ ,\ j=1,\ldots,i \quad (15)$$

$$q_{i+1}+q_{i-1}-2q_i-\tilde{c}_i \succ \tilde{f}_i|_{(q_1,\ldots,q_{n+1})}, i=2,\ldots,n \quad (16)$$

$$q_i+q_2-q_1-c_i \succ f_i|_{(q_1,\ldots,q_{n+1})}\ , \ i=1,\ldots,n. \quad (17)$$

on the class of systems that can be handled by the proposed control design approach. The inequalities (13)-(17) determine if scaling powers $q_1,\ldots,q_{n+1}$ can be found to achieve a high-gain scaling based global control design via the generalized scaling technique [27]. However, the verification of Assumption A5 given a specific structure of the bounds from Assumptions A3 and A4 is straightforward since (13)-(17) is simply a system of linear inequalities in the unknowns $q_1,\ldots,q_{n+1}$. The requirement in Assumption A5 that $q_1,\ldots,q_n$ be positive can be captured by appending the n inequalities $q_i>0, i=1,\ldots,n$ to this system of linear inequalities. In this context, note that the factorization of upper bounds in (8) in Assumption A3 is non-unique. For instance, a function such as $x_1^2x_2x_3$ can be bounded as any one of $(|x_1||x_2||x_3|)|x_1|$, $(x_1^2|x_2|)|x_3|$, or $(x_1^2|x_3|)|x_2|$. This non-uniqueness in the factorization of the upper bounds in Assumption A3 provides a highly useful design freedom to aid in satisfying Assumption A5 since different factorizations effectively result in different structures of right hand sides in the system of linear inequalities in Assumption A5.

Remark 2: With the definition of the $\succ$ property as defined above, it can be seen that if $f:\mathcal{R}^l\to\mathcal{R}$ is a multinomial and $\zeta \succ f|_{(\zeta_1,\ldots,\zeta_l)}$, then for all $\eta\geq 1$ and all $[z_1,\ldots,z_l]^T\in\mathcal{R}^l$, the inequality $\left|f(\eta^{\zeta_1}z_1,\ldots,\eta^{\zeta_l}z_l)\right| \leq \eta^{\zeta}f(|z_1|,\ldots,|z_l|)$ holds. This property will be useful during the stability analysis.

Remark 3: It can be shown (analogous to the reasoning used in Section III in [27] that Assumptions A4 and A5 imply the following statement: A positive constant ρ, continuous functions $R:\mathcal{R}^{n+1}\to\mathcal{R}^+$ and $R_f:\mathcal{R}^{n+1}\to\mathcal{R}^+$, positive constants $q_i, i=1,\ldots,n$, and a (not necessarily positive) constant q_{n+1} are known such that the following inequalities hold for all $t\in\mathcal{R}^+$, $u\in\mathcal{R}$, $\xi\in\mathcal{R}^{n+1}$, and all $r\in[R(\xi),\infty)$:

$$\frac{\hat{\phi}_{(i,j)}(t,\xi,u,r)}{\sqrt{\hat{\phi}_{(i,i+1)}(t,\xi,u,r)\hat{\phi}_{(j-1,j)}(t,\xi,u,r)}} \leq \rho\ ,\ i=2,\ldots,n, \quad \text{for } j=2,\ldots,i \quad (18)$$

$$\frac{\hat{\phi}_{(i,1)}(t,\xi,u,r)}{\sqrt{\hat{\phi}_{(i,i+1)}(t,\xi,u,r)\hat{\phi}_{(1,2)}(t,\xi,u,r)}} \leq \rho \quad \text{for } i=1,\ldots,n \quad (19)$$

$$\frac{\hat{\phi}_{(i-1,i)}(t,\xi,u,r)}{\hat{\phi}_{(i,i+1)}(t,\xi,u,r)} \leq \rho \qquad \text{for } i=2,\ldots,n \quad (20)$$

where $\hat{\phi}_{(i,j)}(t,\xi,u,r) \stackrel{\triangle}{=} r^{q_j-q_i}|\phi_{(i,j)}(t,T^{-1}(r)\xi,u)|, i=1,\ldots,n, j=1,\ldots,i+1$, $\phi_{(n,n+1)}(t,x,u)=\tilde{\mu}(x)\mu(t,x,u)$, and $T(r)\stackrel{\triangle}{=}\text{diag}(\frac{1}{r^{q_1}},\frac{1}{r^{q_2}},\ldots,\frac{1}{r^{q_n}})$. Also, for all $t\in\mathcal{R}^+$, $u\in\mathcal{R}$, $\xi\in\mathcal{R}^n$, and all $r\in[R_f(\xi),\infty)$: $\frac{\phi_{fi}(t,T^{-1}(r)\xi,u)}{r^{q_i+q_2-q_1}}\leq\rho\ , \ i=1,\ldots,n$.

III. Proof of Theorem 1

A dynamic state extension is defined as shown in (5) with x_{n+1} being a new state coordinate and v being the new control input into the extended system. A dynamic scaling of the state variables $x_1,\ldots,x_n$ is defined as shown in (4). The control law for v is defined to be of the form shown in (6) with K being a continuous function and b_q a positive constant. Note that the assumption that $\tilde{\mu}(x)\gamma_u(x)\leq\mu^*$ which is part of Assumption A4 implies that $\gamma_u(x)|u|\leq\mu^*|x_{n+1}|$. Hence, in the extended system, the bounds on the uncertain functions in Assumption A4 are bounded by a function of the states x and ξ_{n+1} and do not involve the new input v. The dynamic high-gain scaling parameter r is a time-varying signal whose dynamics will be designed to be of the form shown in (7) with $\overline{R}:\mathcal{R}^{n+1}\to\mathcal{R}^+$, $q:\mathcal{R}\to\mathcal{R}^+$, and $\alpha:\mathcal{R}^{n+2}\to\mathcal{R}^+$ being continuous functions. The functions $\overline{R}$, q, and α will be designed during the stability analysis below; however, at this stage, it is to be noted that the dynamics of r will be designed such that $\dot{r}(t)\geq 0$ at all times t, i.e., that $r(t)$ is monotonically non-decreasing in time. Furthermore, r will be initialized with $r(0)\geq 1$; hence, $r(t)\geq 1$ for all time t. Local existence of solutions starting from any initial condition is guaranteed by the assumptions on the functions $\phi_{(i,i+1)}$, ϕ_i, and μ and the continuity (by construction) of functions appearing in the overall dynamic controller. By construction, the functions appearing in the dynamic controller inherit any continuity requirements imposed on the functions appearing in the system dynamics and in the bounds in the Assumptions A3 and A4. Hence, uniqueness of solutions is guaranteed if these functions are all locally Lipschitz-continuous; if not, while uniqueness of solutions is not guaranteed, the theorem of Kurzweil [32] can still be used to infer boundedness and convergence properties of all solutions through the Lyapunov arguments in this section. Let the maximal interval of existence of solutions be $[0,t_f)$ where $t_f\in(0,\infty]$. The dynamics of the scaled states ξ are given by

$$\dot{\xi} = A(t,x,u,r)\xi + BK(x,x_{n+1},r)\xi + \Phi_t - \frac{\dot{r}}{r}D\xi$$
$$B=[0,\ldots,0,1]^T\ ;\ D=\text{diag}(q_1,\ldots,q_n,q_{n+1}+b_q) \quad (21)$$

and Φ_t is used to denote the signal

$$\Phi_t = \left[\frac{1}{r^{q_1}(t)}\phi_1(t,x(t),u(t),x(t-\tau_1),u(t-\tau_1)),\ldots,\frac{1}{r^{q_n}(t)}\phi_n(t,x(t),u(t),x(t-\tau_n),u(t-\tau_n)),0\right]^T \quad (22)$$

with $A(t,x,u,r)$ being the $(n+1)\times(n+1)$ matrix with $A_{(i,i+1)}(t,x,u,r)=r^{q_{i+1}-q_i}\phi_{(i,i+1)}(t,x,u)\ , i=1,\ldots,n$ and zeros elsewhere, and B being the $(n+1)\times 1$ vector with a 1 as the last element and zeros everywhere else. Here, $\phi_{(n,n+1)}$ is defined as $\phi_{(n,n+1)}(t,x,u)=\tilde{\mu}(x)\mu(t,x,u)$. The constant b_q is picked such that $b_q>-q_{n+1}$. Under the Assumptions A4 and A5 and using Remark 3, it can be shown as in [27] that, given any positive constant c_0, a continuous function $K:\mathcal{R}^{n+2}\to\mathcal{R}^{(n+1)^*}$, a constant symmetric positive-definite $n\times n$ matrix P, and positive constants ν_1, $\underline{\nu}_2$, and $\overline{\nu}_2$ can be found such that for all $t\in\mathcal{R}^+$, $x\in\mathcal{R}^n$, $u\in\mathcal{R}$, and $r\geq R(\xi)$, the inequalities (23) and (24) are satisfied where I_{n+1} denotes an $(n+1)\times(n+1)$ identity matrix and $\overline{\Phi}_t$ is the $(n+1)\times(n+1)$ matrix with $(i,j)^{th}$ element being $r^{q_j-q_i}(t)\phi_{(i,j)}(t,x(t),u(t))$ if $1\leq i\leq n, 1\leq j\leq i$, and zeros elsewhere. Q_1 and Q_2 are

$$P[A(t,x,u,r)+Q_1\overline{\Phi}_tQ_2+c_0I_{n+1}+BK(x,x_{n+1},r)] +[A(t,x,u,r)+Q_1\overline{\Phi}_tQ_2+c_0I_{n+1}+BK(x,x_{n+1},r)]^TP \leq -\nu_1 r^{q_2-q_1}|\phi_{(1,2)}(t,x,u)|I_{n+1} \quad (23)$$

$$\underline{\nu}_2 I_{n+1} \leq PD+DP \leq \overline{\nu}_2 I_{n+1} \quad (24)$$

arbitrary $(n+1)\times(n+1)$ diagonal matrices with each diagonal entry $+1$ or -1. The demonstration of the simultaneous solvability of the coupled Lyapunov inequalities (23) and (24) is based upon the fact that, under the imposed assumptions, the matrix $\overline{A} = A + Q_1\overline{\Phi}_tQ_2 + c_0I_{n+1}$ is dual w-CUDD $(\tilde{\rho})$ with $\tilde{\rho} = \rho + c_0/\sigma$ for all $r \in [\overline{R}(\xi),\infty)$. From [16], a $(n+1)\times(n+1)$ matrix A with $(i,j)^{th}$ element $A_{(i,j)}$ is said to be dual weakly Cascading Upper Diagonal Dominant with parameter ρ or w-CUDD(ρ) with ρ being a given positive constant if the following hold [28]:
1) A is in lower Hessenberg form, i.e., $A_{(i,j)} = 0$ for $j \geq i+2$.
2) The upper diagonal elements of A are non-zero, i.e., $A_{(i,i+1)} \neq 0$, $i = 1,\ldots,n$.
3) The following inequalities are satisfied: $|A_{(i,j)}|/\sqrt{|A_{(i,i+1)}||A_{(j-1,j)}|} \leq \rho$ for $i = 2,\ldots,n, j = 2,\ldots,i$; $|A_{(i,1)}|/\sqrt{|A_{(i,i+1)}||A_{(1,2)}|} \leq \rho$ for $i = 1,\ldots,n$; $|A_{(i-1,i)}|/|A_{(i,i+1)}| \leq \rho$ for $i = 2,\ldots,n$. Noting that D is a diagonal matrix with positive diagonal entries, and applying the results in [16], [28], the solvability of the coupled Lyapunov inequalities (23) and (24) is inferred. By the construction procedure described in [16], [28], the choice of K depends only on the known upper and lower bounds on $\phi_{(i,i+1)}, i = 1,\ldots,n$, and the known upper bounds on $\phi_{(n,j)}, j = 1,\ldots,n$, and does not require knowledge of the uncertain functions $\phi_{(i,j)}, i = 1,\ldots,n, j = 1,\ldots,i$, and $\phi_{(i,i+1)}, i = 1,\ldots,n$, themselves. Hence, K is a known function of (x, x_{n+1}, r). The continuity of K follows [28] from continuity of $\overline{\phi}_{(i,i+1)}, i = 1,\ldots,n$, and $\overline{\phi}_{(n,j)}, j = 1,\ldots,n$. The choice of P, ν_1, $\underline{\nu}_2$, and $\overline{\nu}_2$ depends only on the choice of ρ which is free to be arbitrarily picked and the signs of $\phi_{(i,i+1)}$ which are known and constant by Assumption A1. Furthermore, K, P, ν_1, $\underline{\nu}_2$, and $\overline{\nu}_2$ do not depend on Q_1 and Q_2. Note that while A and $\overline{\Phi}$ depend explicitly on the control input u, K does not depend on u but instead depends on x_{n+1}, the state variable of the dynamic extension. This is a consequence of the fact that the bounds on the ratios of elements of A and $\overline{\Phi}_t$, and consequently of the corresponding CUDD parameters in the matrix $A+Q_1\overline{\Phi}_tQ_2+c_0I_{n+1}$, involve $\gamma_u(x)u$, the design that $u = \tilde{\mu}x_{n+1}$, and the assumption that $\tilde{\mu}(x)\gamma_u(x)$ is upper bounded by a known positive constant. A *controller Lyapunov function* $V_c : \mathcal{R}^{n+1} \to [0,\infty)$ is defined as $V_c(\xi) = \xi^TP\xi$ where P is a constant matrix satisfying the coupled Lyapunov inequalities (23) and (24) with c_0 being any positive constant. V_c satisfies

$$\dot{V}_c \leq \xi^T\{P[A+BK]+[A+BK]^TP\}\xi + 2\xi^TP\Phi_t - \frac{\dot{r}}{r}\xi^T(PD+DP)\xi. \quad (25)$$

The term $2\xi^TP\Phi_t$ can be upper bounded as:

$$2\xi^T(t)P\Phi_t \leq \xi^T(t)PS(P\xi(t))\overline{\Phi}_tS(\xi(t))\xi(t) + \xi^T(t)S(\xi(t))\overline{\Phi}_t^TS(P\xi(t))P\xi(t) + 2\lambda_{max}(P)|\xi(t)||\tilde{\Phi}(t)| + 2\xi^T(t)P\tilde{\Xi}_t \quad (26)$$

$$\tilde{\Phi}(t) \triangleq |\phi_{(1,2)}(t,x(t),u(t))| \times\Big[\frac{1}{r^{q_1}(t)}f_1(x_1(t),\ldots,x_n(t),\gamma_u(x(t))|u(t)|),\ldots, \frac{1}{r^{q_n}(t)}f_n(x_1(t),\ldots,x_n(t),\gamma_u(x(t))|u(t)|),0\Big]^T \quad (27)$$

and $\tilde{\Xi}_t$ is a $(n+1)\times 1$ vector with i^{th} element given by

$$\tilde{\Xi}_i = \sum_{j=1}^{i}\Xi_{(i,j)}(x(t-\tau_i),u(t-\tau_i))r^{-q_i}(t)|x_j(t-\tau_i)|. \quad (28)$$

The $(n+1)\times 1$ vector $\tilde{\Phi}$ can be bounded as

$$|\tilde{\Phi}| \leq \frac{|\phi_{(1,2)}(t,x,u)|}{r^{q_1-q_2}}|\xi|\Big[\sum_{i=1}^{n}\frac{\overline{f}_i^2(|\xi_1|,\ldots,|\xi_n|)}{r^{2c_i}}\Big]^{\frac{1}{2}}. \quad (29)$$

The term $2\xi^TP\tilde{\Xi}_t$ can be upper bounded as

$$2\xi^TP\tilde{\Xi}_t \leq c_0\xi^TP\xi + \frac{\lambda_{max}(P)}{c_0}\sum_{i=1}^{n}\Big(\sum_{j=1}^{i}\Xi_{(i,j)}(x(t-\tau_i),u(t-\tau_i)) \times r^{-q_i}(t-\tau_i)|x_j(t-\tau_i)|\Big)^2. \quad (30)$$

An overall Lyapunov-Krasovskii functional is defined as

$$V = V_c + \frac{\lambda_{max}(P)}{c_0}\sum_{i=1}^{n}\int_{t-\tau_i}^{t}\frac{\Big(\sum_{j=1}^{i}\tilde{\Xi}_{(i,j)}(\pi)\Big)^2}{1-\overline{\tau}_i}d\pi \quad (31)$$

where $\tilde{\Xi}_{(i,j)}(\pi)$ is used to denote $\Xi_{(i,j)}(x(\pi),u(\pi))|x_j(\pi)|r^{-q_i}(\pi)$. We get

$$\dot{V} \leq \xi^T\{P[A+BK]+[A+BK]^TP\}\xi + 2\xi^TP\Phi_t - \frac{\dot{r}}{r}\xi^T(PD+DP)\xi + \frac{\lambda_{max}(P)}{c_0}\sum_{i=1}^{n}\Bigg[\frac{\Big(\sum_{j=1}^{i}\tilde{\Xi}_{(i,j)}(t)\Big)^2}{1-\overline{\tau}_i} - \frac{(1-\dot{\tau}_i)\Big(\sum_{j=1}^{i}\tilde{\Xi}_{(i,j)}(t-\tau_i)\Big)^2}{1-\overline{\tau}_i}\Bigg]. \quad (32)$$

Using Assumption A6, $|\dot{\tau}_i| \leq \overline{\tau}_i < 1$ for $= 1,\ldots,n$. Also, noting that $r(t) \geq 1$ for all time t, (32) reduces to

$$\dot{V} \leq \xi^T\{P[A+BK]+[A+BK]^TP\}\xi + 2\xi^TP\Phi_t - \frac{\dot{r}}{r}\xi^T(PD+DP)\xi + \frac{\lambda_{max}(P)}{c_0}\sum_{i=1}^{n}\Bigg[\frac{\Big(\sum_{j=1}^{i}\tilde{\Xi}_{(i,j)}(t)\Big)^2}{1-\overline{\tau}_i} - \Big(\sum_{j=1}^{i}\tilde{\Xi}_{(i,j)}(t-\tau_i)\Big)^2\Bigg]. \quad (33)$$

Using (33), (26), (29), and (30), we obtain, by utilizing the coupled Lyapunov inequalities (23) and (24):

$$\dot{V} \leq -\nu_1r^{q_2-q_1}|\phi_{(1,2)}(t,x,u)||\xi|^2 - \frac{\dot{r}}{r}\underline{\nu}_2|\xi|^2 + 2\lambda_{max}(P)|\xi(t)||\tilde{\Phi}(t)| + \sum_{i=1}^{n}\frac{\lambda_{max}(P)}{c_0(1-\overline{\tau}_i)}\Big(\sum_{j=1}^{i}\tilde{\Xi}_{(i,j)}(t)\Big)^2. \quad (34)$$

The term $\sum_{i=1}^{n}\frac{\lambda_{max}(P)}{c_0(1-\overline{\tau}_i)}\Big(\sum_{j=1}^{i}\tilde{\Xi}_{(i,j)}(t)\Big)^2$ can be upper bounded as

$$\sum_{i=1}^{n}\frac{\lambda_{max}(P)}{c_0(1-\overline{\tau}_i)}\Big(\sum_{j=1}^{i}\tilde{\Xi}_{(i,j)}(t)\Big)^2 \leq \Big[\sum_{i=1}^{n}\frac{n\lambda_{max}(P_o)}{c_0(1-\overline{\tau}_i)}\sum_{j=1}^{i}\Xi^2_{(i,j)}(x,u)r^{2q_j-2q_i}\Big]|\xi|^2. \quad (35)$$

Defining a function $\overline{R}$ as in (36) where $\overline{f}_i$ are the bounding functions of f_i, $\lambda_{max}(P)$ is the maximum eigenvalue of P, and $R(\xi)$ is defined as in Remark 3, and inserting $Q_1 = S(P\xi)$ and $Q_2 = S(\xi)$ into (23), we infer that the following inequality (37) is satisfied for all $t \in \mathcal{R}^+$, $x \in \mathcal{R}^n$, $u \in \mathcal{R}$, and $r \in [\overline{R}(\xi),\infty)$,

$$\dot{V} \leq -\underline{\nu}_1r^{q_2-q_1}|\phi_{(1,2)}(t,x,u)||\xi|^2 \quad (37)$$

with $\underline{\nu}_1 = \nu_1/4$. Furthermore, it is also seen that a continuous

$$\overline{R}(\xi) = \max\Bigg(R(\xi), \max_{i=1,\ldots,n}\{[4\lambda_{max}(P)\sqrt{n}\overline{f}_i(|\xi_1|,\ldots,|\xi_n|,\mu^*|\xi_{n+1}|)/\nu_1]^{\frac{1}{c_i}}\}, \max_{i=1,\ldots,n}\max_{j=1,\ldots,i}\{[4n^3\lambda_{max}(P)\tilde{f}_{(i,j)}(|\xi_1|,\ldots,|\xi_n|,\mu^*|\xi_{n+1}|)/(\nu_1 c_0(1-\overline{\tau}_i))]^{\frac{1}{\tilde{c}_{(i,j)}}}\},\Bigg) \quad (36)$$

$$\Delta(x,x_{n+1},r) = 2\lambda_{max}(P)\left\{\sqrt{\sum_{i=1}^{n}\left[\frac{\overline{\phi}_{(i,i+1)}(x)}{r^{q_i-q_{i+1}}}\right]^2}+|K(x,x_{n+1},r)|+\hat{\overline{\Phi}}(x,x_{n+1},r)+\overline{\phi}_{(1,2)}(x)\sqrt{\sum_{i=1}^{n}\frac{\overline{f}_i^2(|\xi_1|,\ldots,|\xi_n|,|\mu^*\xi_{n+1}|)}{r^{2(q_1-q_2+c_i)}}}\right\}$$

$$\hat{\overline{\Phi}}(x,x_{n+1},r) \triangleq \Bigg[\sum_{i=2}^{n-1}\sum_{j=2}^{i} f_{(i,j)}^2(|x_1|,\ldots,|x_n|,|\mu^*\xi_{n+1}|)\overline{\phi}_{(i,i+1)}(x)\overline{\phi}_{(j-1,j)}(x)r^{2q_j-2q_i} + \sum_{i=1}^{n-1} f_{(i,1)}^2(|x_1|,\ldots,|x_n|,|\mu^*\xi_{n+1}|)\overline{\phi}_{(i,i+1)}(x)\overline{\phi}_{(1,2)}(x)r^{2q_1-2q_i} + \sum_{j=1}^{n}\overline{\phi}_{(n,j)}^2(x)r^{2q_j-2q_n}\Bigg]^{\frac{1}{2}}. \quad (39)$$

function $\Delta(x, x_{n+1}, r)$ exists such that for all $t \in \mathcal{R}^+$, $x \in \mathcal{R}^n$, $u \in \mathcal{R}$, and $r \in [1,\infty)$, the inequality

$$\dot{V} + \frac{\dot{r}}{r}\underline{\nu}_2|\xi|^2 \leq \Delta(x,x_{n+1},r)|\xi|^2 \quad (38)$$

holds. One such function Δ can be explicitly constructed using (26) and (29) as shown in (39). The dynamics of r are designed as in (7) with $\alpha(x,r) = \frac{1}{\underline{\nu}_2}[\Delta(x,r)+\Delta_0]$ and with q being a continuous nonnegative function such that

$$q(b) = \begin{cases} 1 & , \ b > 0 \\ 0 & , \ b \leq -\epsilon_r \ , \ \epsilon_r > 0. \end{cases} \quad (40)$$

with Δ as determined above and $\Delta_0 \in (0,\infty)$. From (7), $r(t)$ is monotonically non-decreasing so that $r(t) \geq 1$ for all $t \in [0,t_f)$. Consider the following two cases. Case $\mathcal{A}1$: $r < \overline{R}(\xi)$; Case $\mathcal{A}2$: $r \geq \overline{R}(\xi)$. In Case $\mathcal{A}1$, (40) implies that $q(\overline{R}(\xi)-r) = 1$. Hence, $\dot{V} \leq -\Delta_0|\xi|^2$. In Case $\mathcal{A}2$, we get $\dot{V}(t) \leq -\underline{\nu}_1 r^{q_2-q_1}|\phi_{(1,2)}(t,x,u)||\xi|^2$. Hence, in either case, $\dot{V}(t) \leq 0$ over $[0,t_f)$ implying that the signal $V(t)$, and hence the signal $\xi(t)$, is bounded (in $\mathcal{L}^\infty$ norm sense) on $[0,t_f)$. Noting from (7) and (40) that $\dot{r}(t) = 0$ if $r(t) \geq \overline{R}(\xi(t)) + \epsilon_r$, the boundedness of the signal $\xi(t)$ implies that of $r(t)$. Therefore, the signals $x_i(t) = r^{q_i}(t)\xi_i(t), i = 1,\ldots,n$, and hence $u(t)$ are bounded over $[0,t_f)$, implying that $t_f = \infty$. Therefore, all closed-loop signals are bounded on $t \in [0,\infty)$; also, $r(t) \in [1,\overline{r}] \ \ \forall t \geq 0$ with $\overline{r}$ being some positive constant. Furthermore, using Assumption A1, the bounds on $\dot{V}(t)$ obtained in Cases $\mathcal{A}1$ and $\mathcal{A}2$ can be combined into

$$\dot{V}(t) \leq -\frac{1}{\lambda_{max}(P)}\min(\Delta_0, \underline{\nu}_1\sigma, \underline{\nu}_1\sigma\overline{r}^{q_2-q_1})V(t) \quad (41)$$

implying that $V(t)$ goes to zero exponentially as $t \to \infty$. Hence, from the definition of the functional V, it is seen that $|\xi(t)|$ goes to zero exponentially as $t \to \infty$. Since $|x_i(t)| = r^{q_i}(t)|\xi_i(t)| \leq \overline{r}^{q_i}|\xi_i(t)|$, it follows that the signals $x(t)$ and $u(t)$ also go to zero exponentially as $t \to \infty$.

IV. Illustrative Examples

Example 1: Consider the third order system given in (2) and with τ_1, τ_2, and τ_3 being uncertain time-varying time delays with known bounds smaller than 1 on the rates of time variation of the time delays, i.e., $|\dot{\tau}_i| \leq \overline{\tau}_i < 1$ for $i = 1,2,3$, with known nonnegative constants $\overline{\tau}_i, i = 1,2,3$. Consider, for instance, $\overline{\tau}_1 = \overline{\tau}_2 = \overline{\tau}_3 = 1$. The values of the actual time delays τ_1, τ_2, and τ_3 are not required in the control design. For this system, the functions appearing in the general system description (1) are given by: $\phi_{(1,2)} = 1 + x_1^2(t)$, $\phi_{(2,3)} = 1 + x_3^2(t) + \sqrt{|u(t)|}$, $\mu = 1$, $\phi_1 = x_1^5(t) + x_1^3(t-\tau_1)$, $\phi_2 = x_1^2(t)x_2(t)x_3(t) + x_1^2(t-\tau_2)\sqrt{|x_3(t-\tau_2)|} + x_1(t-\tau_2)x_2(t-\tau_2)\sin(x_3(t)+x_3(t-\tau_2))$, and $\phi_3 = x_1^2(t)x_2^3(t) + x_1^2(t-\tau_3)|u(t-\tau_3)|^{\frac{1}{4}} + x_1^2(t-\tau_3)x_2^2(t-\tau_3)\cos(u(t-\tau_3))$. It can be verified that this system satisfies the Assumptions A1-A6 as follows:

1) Assumption A1 is satisfied with $\sigma = 1$.
2) Assumption A2 is trivially satisfied since the functions $\phi_{(1,2)}$, $\phi_{(2,3)}$, and μ are continuous functions and do not depend explicitly on time.
3) Assumption A3 is satisfied by noting that
   - $|\phi_1| \leq \phi_{(1,1)}(t,x(t),u(t))|x_1(t)| + \Xi_{(1,1)}(x(t-\tau_1),u(t-\tau_1))|x_1(t-\tau_1)|$ where $\phi_{(1,1)}(t,x,u) = x_1^4$ and $\Xi_{(1,1)}(x,u) = x_1^2$,
   - $|\phi_2| \leq \phi_{(2,2)}(t,x(t),u(t))|x_2(t)| + \Xi_{(2,1)}(x(t-\tau_2),u(t-\tau_2))|x_1(t-\tau_2)| + \Xi_{(2,2)}(x(t-\tau_2),u(t-\tau_2))|x_2(t-\tau_2)|$ where $\phi_{(2,2)}(t,x,u) = x_1^2x_3$, $\Xi_{(2,1)}(x,u) = |x_1|\sqrt{|x_3|}$, and $\Xi_{(2,2)}(x,u) = |x_1|$,
   - $|\phi_3| \leq \phi_{(3,2)}(t,x(t),u(t))|x_2(t)| + \Xi_{(3,1)}(x(t-\tau_2),u(t-\tau_2))|x_1(t-\tau_2)| + \Xi_{(3,2)}(x(t-\tau_2),u(t-\tau_2))|x_2(t-\tau_2)|$ where $\phi_{(3,2)}(t,x,u) = x_1^2x_2^2$, $\Xi_{(3,1)}(x,u) = |x_1||u|^{\frac{1}{4}}$, and $\Xi_{(3,2)}(x,u) = x_1^2|x_2||\cos(u)|$.
4) With the functions $\phi_{(i,j)}$ and $\Xi_{(i,j)}$ appearing in the bounds identified above in the verification of Assumption A3, the Assumption A4 is easily verified and the relevant multinomials for the inequalities in Assumption A4 are obtained as: $f_{(1,1)} = |x_1|^2$, $f_{(2,2)} = |x_1|$, $f_{(3,2)} = |x_1|^2|x_2|^2$, $\tilde{f}_{(1,1)} = |x_1|^2$, $\tilde{f}_{(2,1)} = |x_3|$, $\tilde{f}_{(3,1)} = \sqrt{|u|}$, $\tilde{f}_{(3,2)} = |x_1|^2|x_2|^2$, $\tilde{f}_2 = 1 + |x_1|^2$, and $\tilde{f}_3 = 1 + |x_3|^2 + \sqrt{|u|}$. Also, $\gamma_u(x)$ and $\tilde{\mu}(x)$ can be defined to be 1.
5) From the definitions of the functions $f_{(i,j)}$, $\tilde{f}_{(i,j)}$, and $\tilde{f}_i$, in the verification of the Assumption A4 above, the system of linear inequalities in Assumption A5 is obtained to be: $q_2 - q_1 - c_{(1,1)} \geq 2q_1$; $0.5(q_3 - q_1) - c_{(2,2)} \geq q_1$; $0.5(q_4+q_3-q_2-q_1)-c_{(3,2)} \geq 2q_1+2q_2$; $q_2-q_1-\tilde{c}_{(1,1)} \geq 2q_1$; $3q_2-3q_1-\tilde{c}_{(2,1)} \geq q_3$; $q_2-3q_1+2q_3-\tilde{c}_{(3,1)} \geq 0.5q_4$; $-q_2-q_1+2q_3-\tilde{c}_{(3,2)} \geq 2q_1+2q_2$; $q_3+q_1-2q_2-\tilde{c}_2 \geq 2q_1$; $q_3 + q_1 - 2q_2 - \tilde{c}_2 \geq 0$; $q_4 + q_2 - 2q_3 - \tilde{c}_3 \geq 2q_3$; $q_4+q_2-2q_3-\tilde{c}_3 \geq 0.5q_4$; $q_4+q_2-2q_3-\tilde{c}_3 \geq 0$. Note that a multinomial with multiple terms (e.g., $\tilde{f}_2$, $\tilde{f}_3$) results in multiple inequalities in the system of linear inequalities given above. To impose the condition that $q_1,\ldots,q_3$ be positive constants as required by Assumption A5, the additional linear inequalities $q_1 > 0$, $q_2 > 0$, and $q_3 > 0$ are introduced into the system of linear inequalities given above. It is numerically verified that solutions do exist for this overall set of linear inequalities; for instance, $q_1 = 1$, $q_2 = 6$, $q_3 = 14$, $q_4 = 51$, $c_{(1,1)} = 3$, $c_{(2,2)} = 5.5$, $c_{(3,2)} = 15$, $\tilde{c}_{(1,1)} = 3$, $\tilde{c}_{(2,1)} = 1$, $\tilde{c}_{(3,1)} = 5.5$, $\tilde{c}_{(3,2)} = 7$, $\tilde{c}_2 = 1$, and $\tilde{c}_3 = 1$.
6) A6 is satisfied as noted above with $\overline{\tau}_1 = \overline{\tau}_2 = 0.5$.

Hence, the system given in (2) satisfies Assumptions A1-A6 and the control design approach proposed in this paper is applicable to system (2) and yields a dynamic control law of the form (4)-(7) that globally stabilizes the system (2) and regulates $x(t)$ and $u(t)$ to zero exponentially as $t \to \infty$.

Example 2: To illustrate the generality of the design procedure, consider the fairly complex fifth order system given by

$$\dot{x}_1(t) = (1 + x_1^2(t))x_2(t)$$
$$\dot{x}_2(t) = (1 + x_1^4(t)x_2^2(t))x_3(t) + 0.2x_1(t)|x_4(t)|^{\frac{1}{5}}$$

$$+x_1^2(t-\tau_2)x_2(t-\tau_2)$$
$$\dot{x}_3(t) = x_4(t) + x_2^2(t-\tau_3)$$
$$\dot{x}_4(t) = (1 + x_1^2(t)x_2^4(t))x_5(t) + x_2^5(t) + x_3^3(t)|u(t)|^{\frac{1}{5}} + x_1^2(t)x_2^2(t)x_4^2(t)$$
$$\dot{x}_5(t) = u(t) + x_2^4(t)x_3^3(t) + x_1^2(t-\tau_5)x_3(t-\tau_5)|u(t-\tau_5)|^{\frac{1}{5}}$$

with τ_2, τ_3, and τ_5 being uncertain time-varying time delays with known bounds on rates of variation of the form $|\dot{\tau}_i| \leq \overline{\tau}_i = 0.5, i = 2, 3, 5$. As in the analysis of Example 1, it can be verified that the system above also satisfies Assumptions A1-A6. Specifically, we obtain $\tilde{f}_2 = 1 + |x_1|^2$, $\tilde{f}_3 = 1 + |x_1|^4|x_2|^2$, $\tilde{f}_4 = 1$, $\tilde{f}_5 = 1 + |x_1|^2|x_2|^4$, $f_{(2,1)} = 0.2|x_4|^{\frac{1}{5}}$, $f_{(4,2)} = |x_2|^4$, $f_{(4,3)} = |x_3|^2|u|^{\frac{1}{5}}$ $f_{(4,4)} = |x_1||x_3|$, $f_{(5,3)} = |x_2|^4|x_3|^2$, $\tilde{f}_{(2,1)} = |x_1||x_2|^2$, $\tilde{f}_{(3,2)} = |x_2|^2$, and $\tilde{f}_{(5,4)} = |x_1|^3|x_3|^2|u|^{\frac{2}{5}}$. Writing out the system of linear inequalities in Assumption A5, a solution is numerically found as $q_1 = 1$, $q_2 = 5$, $q_3 = 12$, $q_4 = 40.6$, $q_5 = 97.1$, $q_6 = 176.7$, implying that a dynamic control law of form (4)-(7) can be designed using the proposed generalized scaling based control design approach.

V. Conclusion

In this paper, we considered the state-feedback control problem for a class of nontriangular nonlinear systems including delays in both input and state. The design was based on our generalized scaling approach and yields a robust feedback that requires knowledge of only a bound on the rate of time variation of the delays. The class of systems considered is of a general structure which is not necessarily of any triangular form and addresses both lower triangular (strict-feedback) type systems and upper triangular (feedforward) type systems. The control design is delay-independent in two senses that are attractive from a theoretical and implementation standpoint: firstly, that the control algorithm does not utilize any delayed versions of measured states or input and secondly, that the design does not require knowledge of the actual time delays or magnitudes thereof. It is noteworthy that while the systems considered do feature input delays, the input is still required to be "matched" in time with the state at the point where the input most materially enters the system, (i.e., in $\dot{x}_n$); the extension of the methodology to remove this input time-matching requirement remains an interesting topic for further research.

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