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Cooperative Collision-Free Control of Lagrangian Multi-agent Formations

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Abstract— **In this paper we address the problem of cooperation and collision avoidance for multi-agent Lagrangian systems with input disturbances. Two different disturbance observers with different stability results are used for Lagrangian systems, and collisions are shown to never occur in both cases. Then, using Lyapunov techniques, states of the systems are shown to converge to an ultimately bounded region around the master agent. Theoretical results are illustrated through simulations.**

I. Introduction

Control of multi-agent formations has been a popular area for numerous researchers. A survey of works that deal with autonomous consensus seeking methods where agents utilize local data can be found in [1]. In [2], reaching consensus on the heading of multi-agent systems using nearest neighbor rules and switching communication graphs has been discussed. Consensus under communication delays has been considered in [3], [4]. Potential-based methods have been proposed in [5], and decentralized overlapping control of multi-agent formations have been discussed in [6].

In order to establish safe trajectories in multi-agent formations, many ideas have been proposed. In the case of the noncooperative scenario for a two-agent system, collision avoidance has been studied in [7], [8]. In the context of collision avoidance of multi-agent formations with kinematic models, the problem was addressed using multiobjective and decentralized optimization methods in [9], and attractive/repulsive potentials in [5]. One of the most essential ideas related to the safety of multi-agent formations is the concept of avoidance control that was originally formulated in [10], and later further developed in [11], [12], [13]. The importance of the aforementioned works lies in the fact that the analysis for collision avoidance is naturally integrated into Lyapunov analysis for stability results. Stability results that integrate avoidance control ideas from [10] with Lyapunov analysis for multi-agent formations can be found in [14]. For a chronological survey with more details on avoidance control, we refer the readers to [15]. One of the main ideas for disturbance rejection in dynamical systems is employment of disturbance observers. The main idea of a disturbance observer (DO) is to attenuate the disturbance signal by measuring the states to recreate the dynamic system and constructing additional dynamics for ensuring convergence of the disturbance error. DOs have particularly been utilized for control of robot manipulators as they provide good results in compensating for friction ([16]) and other disturbances ([17]) without the use of additional sensors. In this paper, we utilize DO designs from two papers; the simple DO is designed in [18] and the advanced DO is proposed in [19].

In this paper, we tackle the problem of cooperated coordination and collision avoidance of Lagrangian multi-agent formations with disturbances. Two different disturbance observers are utilized for estimating disturbances of the system. To overcome the drawbacks of the additional dynamics of the DOs, control laws that exploit the bounds on the initial disturbance errors and the derivatives of the disturbances have been proposed, which guarantee collision-free trajectories for the agents. Using Lyapunov techniques, ultimate bounds on the states of the agents are derived.

The remainder of the paper is organized as follows: in Section II, we discuss the dynamics of the multi-agent system, avoidance functions, two versions of DOs and control laws for the system. In Section III, we prove that collisions do not occur in the multi-agent formation, and derive the ultimately bounded region to which all the agents converge in finite time. We provide numerical simulations in Section IV and give final comments in Section V.

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II. Problem Statement

A. Dynamics

Consider N agents whose dynamics are described by

$$M_i(x_i)\ddot{x}_i + C_i(x_i,\dot{x}_i)\dot{x}_i = \tau_i + d_i, \qquad i = 1, ..., N \qquad (1)$$

where $x_i \in \mathbb{R}^n$ are the generalized coordinates, $M_i(x_i)$ is a symmetric positive definite inertia matrix, $C_i(x_i,\dot{x}_i)$ is the Coriolis Matrix, $\tau_i \in \mathbb{R}^n$ is the input force/torque, and d_i is a disturbance to the system.

Agents are assumed to satisfy the following [20]:

P1 $\exists$ constants $0 < \underline{\sigma}_i, \overline{\sigma}_i$ such that $\underline{\sigma}_i \leq \|M_i(x_i)\| \leq \overline{\sigma}_i$.

P2 $\exists$ constants $0 < k_{C_i}$ such that $\|\dot{M}_i(x_i)\| \leq 2\|C_i(x_i,\dot{x}_i)\| \leq 2k_{C_i}\|\dot{x}_i\|$.

P3 Matrices $\dot{M}_i(x_i) - 2C_i(x_i,\dot{x}_i)$ are skew symmetric.

In addition to **P1-P3**, we have the following property for the system utilizing the advanced DO, which is to be introduced in Section II-C.2:

P4 $\exists$ constants $0 < \omega_{d_i}$ such that $\|\dot{d}_i\| \leq \omega_{d_i} < \infty$.

B. Avoidance Functions

In order to guarantee collision avoidance, for each pair of agents we define the following avoidance functions [14]:

$$V_{ij}^a(x_i,x_j) = \left(\min\left\{0, \frac{\|x_i-x_j\|^2-R^2}{\|x_i-x_j\|^2-r^2}\right\}\right)^2 \quad (2)$$

with $i,j \in 1,...,N$, $i \neq j$ and $R > r > 0$. R denotes the radius of the region in which agents can detect other agents, and r denotes the avoidance region, which is the smallest safe distance between the agents.

C. Disturbance Observers

1) Simple Disturbance Observer:

a) Design: For the simple disturbance observer (DO) version, we are going to assume that the time derivative of the disturbance is zero; i.e., $\dot{d}_i = 0$ for all $i \in \{1,...,N\}$. This assumption implies that the disturbance for each agent varies slowly relative to the observer's dynamics. Instead of deriving the equations, we use the results given in [18] and state how we utilize the simple DO. Consider the disturbance d and the error between disturbance and estimation $\hat{d}$ for a Lagrangian system of the form given in (1):

$$d = M(x)\ddot{x} + C(x,\dot{x})\dot{x} - \tau,\ e(t) = d - \hat{d}. \quad (3)$$

Introducing the auxiliary variable δ and the design variables $L(x,\dot{x})$, $p(x,\dot{x})$, the following equations constitute the simple DO dynamics [18]:

$$\delta = \hat{d} - p(x,\dot{x}), \quad (4)$$

$$L(x,\dot{x})M(x)\ddot{x} = \begin{bmatrix} \frac{\partial p(x,\dot{x})}{\partial x} & \frac{\partial p(x,\dot{x})}{\partial \dot{x}} \end{bmatrix} \begin{bmatrix} \dot{x} \\ \ddot{x} \end{bmatrix}, \quad (5)$$

$$\dot{\delta} = -L(x,\dot{x})\left(\delta + p(x,\dot{x})\right) + L(x,\dot{x})\left(C(x,\dot{x})\dot{x} - \tau\right). \quad (6)$$

b) Stability Analysis: Differentiating $e(t)$ with respect to time and utilizing $\dot{d} = 0$ assumption, (4), (5) and (6) yields $\dot{e} = -L(x,\dot{x})e$. It can be seen that $\hat{d}$ asymptotically converges to the actual disturbance d if $L(x,\dot{x})$ is chosen properly. However, $L(x,\dot{x})$ cannot simply be chosen as a constant positive definite matrix since it also has to satisfy the nonlinear equation in (5), so $p(x,\dot{x})$ must now be taken into consideration. One way of choosing $p(x,\dot{x})$ by utilizing the inertial matrix of a two-link robotic manipulator has been proposed in [18], yet we propose a different $p(x,\dot{x})$ function. By using $V(e,x) = e^T M e$ as a Lyapunov function, with $x \in \mathbb{R}^N$, we can state the following lemma:

Lemma 1: Let $p(x,\dot{x}) = s[\dot{x}_1^T,...,\dot{x}_N^T]^T$, such that s satisfies $2sI - \dot{M}(x) \succ 0 \ \ \forall t \geq 0$ where I is the N by N identity matrix and $\dot{M}$ is the derivative of the inertia matrix. Then, the observer described in (4), (5) and (6) is globally asymptotically stable.

c) Bound on Simple Disturbance Observer Error: Using the disturbance observer dynamics described in the previous section, it can be shown that $L_i(x_i,\dot{x}_i) = s_i M_i^{-1}(x_i)$ where $M^{-1}(x_i)$ is the inverse of the inertia matrix of an agent. Utilizing the Lyapunov function $V(e_i) = \frac{1}{2}e_i^T e_i$ for $i = 1,...,N$, the following inequality can be shown to hold:

$$\|e_i(t)\|^2 \leq \|e_i(0)\|^2 e^{-\frac{s_i}{\overline{\sigma}_i}t} \Rightarrow \|e_i(t)\| \leq \|e_i(0)\| \quad (7)$$

with $\overline{\sigma}_i$ defined as in Property **P1**. For all practical purposes, we assume that at $t = 0$, discrepancy between actual and estimated disturbance is bounded; i.e., $\|d_i(0) - \hat{d}_i(0)\| = \|e_i(0)\| \leq \eta_i$ where $\eta_i > 0\ \forall i \in \{1,...,N\}$. It can be seen that the observer error is bounded from above by the initial error for each agent. Define $\|D - \hat{D}\| = \|[e_1(t),...,e_N(t)]^T\|$, and $\eta := \max_{i\in\{1,...,N\}} \eta_i$. Using the bound, we have:

$$\|D - \hat{D}\| \leq \|[\eta,...,\eta]^T\| \leq \sqrt{N}\eta. \quad (8)$$

2) Advanced Disturbance Observer:

For the advanced observer design, we consider the Lagrangian model and design the observer for a single agent, so subscripts will be dropped again. The main difference between the simple and advance DO is that we do not have $\dot{d} = 0$ assumption for the advanced DO. Again, we won't derive the dynamics; instead, we will use the results given in [19].

a) Design: Consider the Lagrangian agent given in (1). We have the exact same equations as in (3) and (4). The only difference here is L is a constant matrix with positive eigenvalues . Similar to the simple DO design, introducing the auxiliary variable δ given in (4), with design parameters L, $p(x,\dot{x})$ and $r(x)$, advanced DO dynamics are described by the following equations [19]:

$$p(x,\dot{x}) = LM(x)\dot{x} + r(x), \quad (9)$$

$$\dot{\delta} = -L\delta - \left(L\frac{\partial(M(x)\dot{x})}{\partial x} + \frac{\partial r(x)}{\partial x}\right)\dot{x} + L\left(C(x,\dot{x})\dot{x} - p(x,\dot{x}) - \tau\right). \quad (10)$$

b) Stability Analysis: Let us define the observer estimation error $e_d = d - \hat{d}$. Taking derivative and using (4), (9) and (10), we get $\dot{e}_d = -Le_d + \dot{d}$. Note that $\dot{d}$ is explicitly taken into account in observer dynamics. Since $\dot{e}_d$ is a stable linear system, it can be integrated to give the following solution [19]:

$$e_d(t) = e^{-Lt}e_d(0) + \int_0^t e^{-L(t-\tau)}\dot{d}(\tau)d\tau. \quad (11)$$

It follows that $\lim_{t\to\infty} \|e^{-Lt}e_d(0)\| = 0$. The ultimate bound on $e_d(t)$ can be shown to be [19]:

$$\lim_{t\to\infty} \|e_d(t)\| \leq c \sup_{t\in(0,\infty]} \|\dot{d}(t)\|, \quad (12)$$

where $c := \int_0^\infty \|e^{-Lt}\|dt$. Hence, the observer error is ultimately bounded.

c) Bound on Advanced Observer Error: Using the bound found in (12), property **P4**, and defining $\omega_d := \max_{i\in\{1,...,N\}} \omega_{d_i}$, $\rho := \max_{i\in\{1,...,N\}} c_i$ where $c_i := \int_0^\infty \|e^{-L_i t}\|dt$, $\|D - \hat{D}\|$ can be bounded as:

$$\|D - \hat{D}\| \leq \|[\rho\omega_d,...,\rho\omega_d]^T\| \leq \sqrt{N}\rho\omega_d. \quad (13)$$

D. Control Laws

Every agent i has a group of neighbors $\mathcal{N}_i$ defined as

$$\mathcal{N}_i = \{j \in \{1, ..., N\} : i \sim j\}, \quad i \in \{1, ..., N\}$$

where $i \sim j$ indicates that the agent i communicates with agent j. It is assumed that the communication graph is undirected. The control laws for the agents (master agent is assumed to have index $i = 1$) are given by

$$\tau_i = -b\dot{x}_i - \alpha k x_i - k \sum_{\forall j \in \mathcal{N}_i} (x_i - x_j) - \sum_{\forall j \in \mathcal{N}_i} \frac{\partial V_{ij}^a}{\partial x_i}^T - \hat{d}_i - \begin{cases} ((1-\beta)c_i\omega_{d_i} + \beta\eta_i) \frac{\dot{x}_i}{\|\dot{x}_i\|} & \dot{x}_i \neq 0 \\ 0 & \dot{x}_i = 0 \end{cases} \quad (14)$$

$\forall i \in \{1, ..., N\}$, where $b > 0$, $k > 0$ are control gains, $\hat{d}_i$ are the estimations of the disturbance observers for each agent. α and β_i are binary variables that satisfy the following:

$$\alpha = \begin{cases} 1 & i = 1 \\ 0 & i \neq 1 \end{cases}, \quad \beta = \begin{cases} 1 & \text{Simple DO} \\ 0 & \text{Advanced DO} \end{cases}.$$

Finally, ω_{d_i} are as defined in Property **P4**, c_i are as defined in Section II-C.2.c and η_i are as defined in Section II-C.1.c.

E. Augmented System

Using the control laws for the agents defined in (14), and the equations of motion for each agent given in (1), we can write the augmented closed loop system as follows [20]:

$$M\ddot{X} + C\dot{X} = -b\dot{X} - kS^TSX - \hat{D} - P(\dot{X}) - L(X) + D \quad (15)$$

where $X^T = \left[x_1^T, ..., x_N^T\right] \in \mathbb{R}^{nN}$ is the position vector, $D^T = \left[d_1^T, ..., d_N^T\right]^T$ is the disturbance vector, similarly $\hat{D}^T$ is the disturbance estimation vector and $L(X) = \left[\sum_{j \in \mathcal{N}_1} \frac{\partial V_{1j}^a}{\partial x_1}, ..., \sum_{j \in \mathcal{N}_N} \frac{\partial V_{Nj}^a}{\partial x_N}\right]^T$ is the collision avoidance part. $P(\dot{X}) = \left[p_1^T, ..., p_N^T\right]^T$ is a vector that contains the following term for each agent, i.e., for all $i \in \{1, ..., N\}$:

$$p_i = \begin{cases} ((1-\beta)c_i\omega_{d_i} + \beta\eta_i) \frac{\dot{x}_i}{\|\dot{x}_i\|} & \dot{x}_i \neq 0 \\ 0 & \dot{x}_i = 0 \end{cases}. \quad (16)$$

Inertia and Coriolis matrices are block diagonal matrices and are given by $M = \text{diag}\{M_1(x_1), ..., M_N(x_N)\}$ and $C = \text{diag}\{C_1(x_1, \dot{x}_1), ..., C_N(x_N, \dot{x}_N)\}$, respectively. S is the connection matrix and is defined as follows [20]: Let E contain x_1 as an element as well as all the error vectors of the form $x_i - x_j$, if i, j are neighbors. Then, S is defined such that $E = SX$, with the first element in E being x_1. Since the communication graph is connected, with the use of the term $-kx_1$ in (14), S can be shown to have full column rank and thus S^TS is a symmetric positive definite matrix.

III. Main Results

A. Collision Avoidance

For the overall system (15), let's define the avoidance region for each pair of agents as

$$\Omega_{ij} = \left\{X : \quad X \in \mathbb{R}^{nN}, \quad \|x_i - x_j\| \leq r\right\}$$

and the detection region for each pair of agents as

$$\mathcal{D}_{ij} = \left\{X : \quad X \in \mathbb{R}^{nN}, \quad \|x_i - x_j\| \leq R\right\}.$$

The overall avoidance and detection regions for the augmented system are then given by the unions of pairwise avoidance regions and detection regions:

$$\Omega = \bigcup_{i,j \in \{1,...,N\}, j>i} \Omega_{ij}, \quad \mathcal{D} = \bigcup_{i,j \in \{1,...,N\}, j>i} \mathcal{D}_{ij}. \quad (17)$$

Let's recall the definition for the avoidance of the set [10]:

Definition 1: The system $\dot{x} = f(x, u(x))$ avoids $\Omega \subset \mathbb{R}^{nN}$, if and only if for each solution $x(t, x_0)$, $t \geq 0$, that does not start in Ω, $x(t, x_0)$ stays out of Ω for all $t \geq 0$.

We now give the results for collision avoidance for two systems that utilize different observers. We only show the proof for the system with simple DO, since the proof is very similar for advance DO case.

Lemma 2: Consider N agents with Lagrangian dynamics given in (1), and the control laws (14)), starting from an initial configuration $x(t_0) \coloneqq x_0 \notin \Omega$, where Ω is defined in (17). Also, assume that the system utilizes the simple DO defined in Section II-C.1.

Then, the set Ω is avoidable in the sense of Definition 1.

Proof: Let's define the following Lyapunov function candidate [20]:

$$V_{col} = \frac{1}{2}\dot{X}^T M \dot{X} + \frac{1}{2}kX^TS^TSX + \frac{1}{2}\sum_{i=1}^{N}\sum_{i \neq j} V_{ij}^a. \quad (18)$$

Since we assume the simple DO is utilized, β in (14) becomes 1. Using the property **P3**, the bound on $\|e_i(t)\|$ from Section II-C.1.c and following derivations in [20], the derivative of V_{col} along the trajectories of the augmented system can then be shown to satisfy the following:

$$\begin{aligned} \frac{dV_{col}}{dt} &\leq -b\|\dot{X}\|^2 + \dot{X}^T\left(D - \hat{D}\right) - \dot{X}^T P(X) \\ &\leq -\sum_{i=1}^{N} b\|\dot{x}_i\|^2 + \sum_{i=1}^{N} \|\dot{x}_i\|(\|e_i(t)\| - \eta_i) \\ &\leq -\sum_{i=1}^{N} b\|\dot{x}_i\|^2 \leq -b\|\dot{X}\|^2 \leq 0. \end{aligned} \quad (19)$$

Since $\dot{V}_{col} \leq 0$, the function V_{col} is non-increasing in the sensing region $\mathcal{D}$. Also, the values of V_{col} are finite for finite values of its argument X that are outside the avoidance region Ω, that is when $X \in \Omega^c$ where $\Omega^c = \mathbb{R}^{nN} \backslash \Omega$ denotes the complement of Ω. Due to continuity of solutions of the system in (15), the assumption that initial condition satisfies $x_0 \in \Omega^c$, and that the following conditions hold:

$$\lim_{\|x_i - x_j\| \to r+} V_{ij}^a = +\infty, \ \forall i, j \in \{1, ..., N\}, \ i \neq j, \quad (20)$$

we conclude that $X(t, X_0, \dot{X}_0)$ will never enter Ω.
$\|x_i - x_j\| \to r+$ in (20) denotes convergence to r from above, i.e., $\|x_i - x_j\| = r + \delta$ while $\delta \to 0$ and $\delta \geq 0$. ■

Lemma 3: Consider N agents with Lagrangian dynamics given in (1), and the control laws (14)), starting from an initial configuration $x(t_0) \coloneqq x_0 \notin \Omega$, where Ω is defined in (17). Also, assume that the system utilizes the advanced DO defined in Section II-C.2.

Then, the set Ω is avoidable in the sense of Definition 1.

B. Bound on Collision Avoidance Terms

The collision avoidance terms can be regarded as disturbances with the following bound shown in [20]:

$$\sup_{t\geq 0}\|L(X(t))\| \leq \sqrt{N}\max_{i\in\{1,\dots,N\}}\sum_{j\neq i}\|\frac{\partial V^a_{ij}}{\partial x_i}\| \leq N^{\frac{3}{2}}g \quad (21)$$

with g defined as in [20], where $\underline{r} \in (r, R)$:

$$\frac{4R(R^2-r^2)(R^2-\underline{r}^2)}{(\underline{r}^2-r^2)^3} := g. \quad (22)$$

C. Stability and Coordination

We want to show that agents will converge to a bounded region around the master agent's position; i.e., the final agent positions will be ultimately bounded.

Theorem 1: Consider the augmented system (15), with the simple DO given in (4), (5) and (6). For any initial condition $\xi(0) = \left[X^T(0), \quad \dot{X}^T(0)\right]^T$, such that $X(0) \notin \Omega$, there exist gains b and k large enough such that the system trajectory avoids the set Ω and satisfies the ultimate bound

$$\|\xi(t)\| \leq \sqrt{\frac{c_2}{c_1}}\frac{2\sqrt{N}}{c_3}\sqrt{(Ng)^2+(2\eta+Ng)^2} \quad (23)$$

$\forall t \geq T$, where $T \geq 0$ is a finite time, c_1, c_2 and c_3 are positive constants to be defined, N is the number of agents, η is as defined in Section II-C.1.c , and g is as defined in (22). Moreover, the positions of all the slave agents ultimately converge to a bounded set around the master agent's position.

Proof: Consider the following Lyapunov function [20]:

$$V_{st} = \frac{1}{2}\dot{X}^T M\dot{X} + \frac{1}{2}kX^TS^TSX + \epsilon\varphi^T(X)M\dot{X} \quad (24)$$

where $\epsilon > 0$ and $\varphi(X) := \frac{X}{1+\|X\|}$. By the virtue of property **P1** and the fact that the matrix S has full column rank, it can be shown that for sufficiently small ϵ there exist two constants c_1 and c_2 such that the following lower and upper bounds hold for V_{st} [3], [21]:

$$c_1\|\xi\|^2 \leq V_{st} \leq c_2\|\xi\|^2. \quad (25)$$

Following the derivations of [20], using the properties **P1-P3**, the result from Lemma 2 and manipulation of terms, we get the following inequality:

$$\begin{aligned}\dot{V}_{st} \leq\ & -b\|\dot{X}\|^2 + \|X\|\left(\frac{\epsilon}{1+\|X\|}\|P(X)\|\right) \\ & +\|X\|\left(\frac{\epsilon}{1+\|X\|}\|E_D\|\right) + \|\dot{X}\|\|L(X)\| \\ & +\|X\|\left(\frac{\epsilon}{1+\|X\|}\|L(X)\|\right) + \epsilon(2\overline{\sigma}+k_C)\|\dot{X}\|^2 \\ & +\epsilon b\frac{\|X\|\|\dot{X}\|}{1+\|X\|} - \epsilon\frac{k\|S\|^2\|X\|^2}{1+\|X\|} \end{aligned} \quad (26)$$

where $E_D \triangleq D - \hat{D}$ and $\overline{\sigma}$ is as defined in Property **P1**. We can write the inequality (26) in the following way:

$$\begin{aligned}\dot{V}_{st} \leq\ & -\begin{bmatrix}\|\dot{X}\| \\ \|X\|\end{bmatrix}^T A \begin{bmatrix}\|\dot{X}\| \\ \|X\|\end{bmatrix} \\ & + \begin{bmatrix}\|\dot{X}\| \\ \|X\|\end{bmatrix}^T B \begin{bmatrix}\|E_D\| \\ \|P(X)\|+\|L(X)\|+\|E_D\|\end{bmatrix}\end{aligned} \quad (27)$$

where

$$A \triangleq \begin{bmatrix} b-\epsilon(2\overline{\sigma}+k_C) & \frac{\epsilon b}{2(1+\|X\|)} \\ \frac{\epsilon b}{2(1+\|X\|)} & \frac{\epsilon k\|S\|}{1+\|X\|}\end{bmatrix}, \quad B \triangleq \begin{bmatrix}1 & 0 \\ 0 & \frac{\epsilon}{1+\|X\|}\end{bmatrix}.$$

From (8), (21) and the fact that the system utilizes the simple DO (i.e., $\beta = 1$), we have the following bounds:

$$\|L(X)\| \leq N^{\frac{3}{2}}g, \ \|E_D\| \leq \sqrt{N}\eta, \ \|P(X)\| \leq \sqrt{N}\eta. \quad (28)$$

Then, we can rewrite the inequality (27) as

$$\begin{aligned}\dot{V}_{st} \leq\ & -\begin{bmatrix}\|\dot{X}\| \\ \|X\|\end{bmatrix}^T A \begin{bmatrix}\|\dot{X}\| \\ \|X\|\end{bmatrix} \\ & + \begin{bmatrix}\|\dot{X}\| \\ \|X\|\end{bmatrix}^T B \begin{bmatrix}N^{\frac{3}{2}}g \\ N^{\frac{3}{2}}g + 2\sqrt{N}\eta\end{bmatrix}.\end{aligned} \quad (29)$$

Pick any $c_3 > 0$ and design the control gains such that $\min\{b, k\} > \frac{c_3}{\|S\|^2}$ holds. Then, for $\epsilon < \min\{\frac{b-c_3}{2\overline{\sigma}+k_C}, \frac{2c_3}{b}\}$, we have the following [20]:

$$\begin{aligned}\|\xi\| \geq\ & \frac{2\sqrt{N}}{c_3}\sqrt{(Ng)^2+(2\eta+Ng)^2} := \nu, \\ \Rightarrow\ & \dot{V}_{st} \leq -\frac{c_3}{2}\begin{bmatrix}\|\dot{X}\| \\ \|X\|\end{bmatrix}^T B \begin{bmatrix}\|\dot{X}\| \\ \|X\|\end{bmatrix}\end{aligned} \quad (30)$$

Using (30), the upperbound on V_{st} in (25), and the comparison principle [22], there exists a function $\beta \in \mathcal{KL}$ such that

$$V_{st} \leq \max\left\{\beta(c_2\|\xi(0)\|^2, t), \nu^2\right\}. \quad (31)$$

Using the lower bound in (25), we find the following bound on the state:

$$\|\xi\| \leq \max\left\{\sqrt{\frac{1}{c_1}\beta(c_2\|\xi(0)\|^2, t)}, \sqrt{\frac{c_2}{c_1}}\nu\right\}, \quad (32)$$

for all $t \geq 0$. For a fixed $r > 0$, $\beta(r, s) \to 0$ as $s \to 0$; hence, there exists a finite time $T > 0$ such that

$$\sqrt{\frac{1}{c_1}\beta(c_2\|\xi(0)\|^2, t)} < \sqrt{\frac{c_2}{c_1}}\nu, \quad \forall t \geq T. \quad (33)$$

Following the arguments from [20], since the error between any two neighbors is captured by the vector $E = SX$, using the ultimate bound, we have that

$$\|E(t)\| \leq \|S\|\sqrt{\frac{c_2}{c_1}}\frac{2\sqrt{N}}{c_3}\sqrt{(Ng)^2+(2\eta+Ng)^2}, \forall t \geq T.$$

Also, Since the coordination graph is connected, there is a path from master agent ($i = 1$) to any other agent j.

All the position difference terms are contained in E, so each one is bounded. Using triangle inequality, we can conclude that $\|x_1 - x_j\|$ is also bounded, and since the result holds for any j, the conclusion is true for any agent. ■

Theorem 2: Consider the augmented system (15), with the advanced DO given in (4), (9) and (10). For any initial condition $\xi(0) = \left[X^T(0), \quad \dot{X}^T(0)\right]^T$, such that $X(0) \notin \Omega$, there exist control gains b and k large enough such that the trajectory of the augmented system avoids the set Ω and it satisfies the ultimate bound

$$\|\xi(t)\| \leq \sqrt{\frac{c_2}{c_1}}\frac{2\sqrt{N}}{c_3}\sqrt{(Ng)^2+(2\rho\omega_d+Ng)^2} \quad (34)$$

$\forall t \geq T$, where ρ and ω_d are as defined in (13) , $T \geq 0$ is a finite time, c_1, c_2 and c_3 are positive constants to be defined, N is the number of agents and g is as defined earlier in (22). Moreover, the positions of all the slave agents ultimately converge to a bounded set around the master's position.

Proof: Consider the same Lyapunov function from (24). Using the same arguments from Theorem 1, the inequality in (25) holds. Following the derivations of [20], using the properties **P1-P3**, the result from Lemma 3 and manipulation of terms, we again get the inequalities (26) and (27). From (21), (13) and the fact that the system utilizes the advanced DO, we have the following bounds:

$$\|L(X)\| \leq N^{\frac{3}{2}}g, \quad \|E_D\| \leq \sqrt{N}\rho\omega_d, \quad \|P(X)\| \leq \sqrt{N}\rho\omega_d. \tag{35}$$

Hence, we can rewrite the inequality (27) for the system with the advanced DO:

$$\begin{aligned} \dot{V}_{st} \quad \leq \quad & -\begin{bmatrix} \|\dot{X}\| \\ \|X\| \end{bmatrix}^T A \begin{bmatrix} \|\dot{X}\| \\ \|X\| \end{bmatrix} \\ & + \begin{bmatrix} \|\dot{X}\| \\ \|X\| \end{bmatrix}^T B \begin{bmatrix} N^{\frac{3}{2}}g \\ N^{\frac{3}{2}}g + 2\sqrt{N}\rho\omega_d \end{bmatrix} \end{aligned} \tag{36}$$

where A and B are as defined in (27). Pick any $c_3 > 0$ and design gains exactly as it has been done in Theorem 1. Then, for $\epsilon < \min\{\frac{b-c_3}{2\overline{\sigma}+k_C}, \frac{2c_3}{b}\}$, we have the following:

$$\begin{aligned} \|\xi\| \quad \geq \quad & \frac{2\sqrt{N}}{c_3}\sqrt{(Ng)^2 + (2\rho\omega_d + Ng)^2}, \\ \Rightarrow \quad & \dot{V}_{st} \leq -\frac{c_3}{2}\begin{bmatrix} \|\dot{X}\| \\ \|X\| \end{bmatrix}^T B \begin{bmatrix} \|\dot{X}\| \\ \|X\| \end{bmatrix} \end{aligned} \tag{37}$$

Similar to the arguments made in Theorem 1, it can be shown that the ultimate bound on the state vector is given by

$$\|\xi\| \leq \sqrt{\frac{c_2}{c_1}}\frac{2\sqrt{N}}{c_3}\sqrt{(Ng)^2 + (2\rho\omega_d + Ng)^2}. \tag{38}$$

Following the arguments from [20], since the error between any two neighbors is captured by the vector $E = SX$, using the ultimate bound, we have that

$$\|E(t)\| \leq \|S\|\sqrt{\frac{c_2}{c_1}}\frac{2\sqrt{N}}{c_3}\sqrt{(Ng)^2 + (2\rho\omega_d + Ng)^2},$$

$\forall t \geq T$. Also, since the coordination graph is connected, there is a path from master agent $(i = 1)$ to any other agent j. All the position difference terms are contained in E hence each one is bounded. Using triangle inequality, we can then conclude that $\|x_1 - x_j\|$ is also bounded, and since the result holds for any j, the conclusion is true for any agent. ■

IV. Simulation Results

In this section, we present two illustrative examples. In both examples, the same multi-agent system has been considered. In the first one, the system utilizes the simple DO design, whereas in the second one, the system utilizes the advanced DO design.

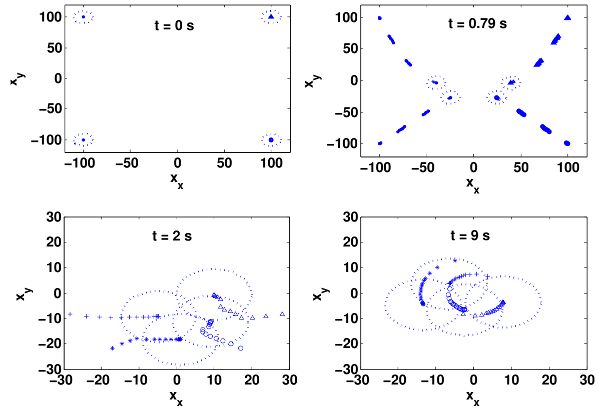

Fig. 1. Snapshots of agents' trajectories (Simple DO)

Consider a multi-robot formation of 4 agents utilizing the simple DO given in Section II-C.1, with the dynamics given by the following:

$$\ddot{x}_i = \tau_i + d_i, \quad \forall i \in \{1, ..., 4\}$$

where $x_i = \begin{bmatrix} x_{x_i} & x_{y_i} \end{bmatrix}^T$, $d_1 = \begin{bmatrix} 5 & 5 \end{bmatrix}^T + d_{uni}$ and $d_i = d_{uni}$ for $i \in \{1, 2, 3, 4\}$, with d_{uni} being a pulse signal that is uniformly distributed in the range $[-3, 3]$. The communication graph is taken as 1~2~3~4, and the control laws given in (14) are used with $b_i = k_i = 10$ for all $i \in \{1, ..., 4\}$. η_i's are selected according to the disturbance signals for each agent. The sensing and avoidance radii are chosen to be $R = 15$ and $r = 10$, respectively. Finally,

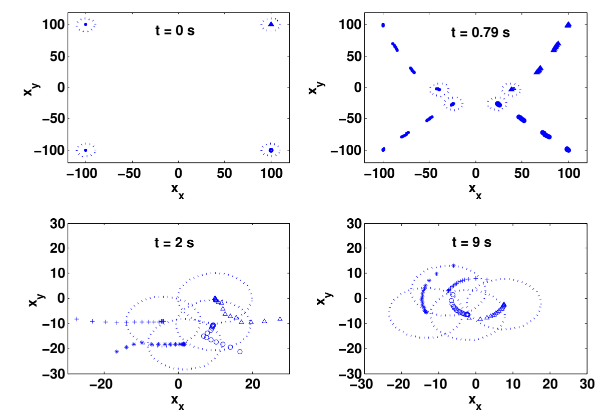

Fig. 2. Snapshots of agents' trajectories (Advanced DO)

the design parameters for the simple DO for each agent are selected to be $s_i = 20$ for all $i \in \{1, ..., 4\}$. The agents are located to lie initially at the corners of a square of length 100 units, as shown in Figure 1. The large circles indicate the avoidance region of each agent. Snapshots of the evolution of the trajectories of the agents are depicted in Figure 1, whereas the distances among all the agents are shown in Figure 3(a). The solid line in Figure 3(a) indicates 10 units of distance which represents the avoidance region for agents. The distances among agents do not cross the solid line, which implies that collisions do not occur. Finally, Figure 4(a) shows the norms of the disturbance errors $\|e_i(t)\|$. Although the disturbances are assumed to be constant in the design of the simple DO, we see from Figure 4(a) that the observers do well even for disturbances that are piecewise constant.

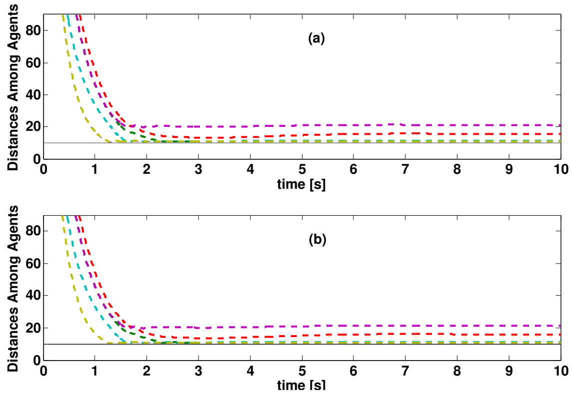

Fig. 3. Distances among agents for the system with (a) Simple DO, (b) Advanced DO

Now, consider the same Lagrangian system, but this time it utilizes the advanced DO, described in Section II-C.2. The difference of this version of the problem is the nature of the disturbances and the design parameters for the advanced DO. The disturbance signals are selected to be periodic, i.e., $d_i = \begin{bmatrix} 3\sin(t) & 2\sin(2t) \end{bmatrix}^T$ for all $i \in \{1, ..., 4\}$. L_i's are selected to be diag$\{7, 7\}$, and $r_i(x_i)$'s, which are depicted in (9), are chosen to be $r_i(x_i) = x_i$. Other design parameters, ω_{d_i}'s and c_i's given in (14) are selected according to the disturbance signals and L_i matrices. Snapshots of the trajectories of the agents can be seen in Figure 2, whereas the distances among all the agents are shown in Figure 3(b). Finally, Figure 4(b) shows the norms of the disturbance errors $\|e_i(t)\|$. Notice that the disturbance errors do not converge to 0; they stay bounded after $t \approx 2s$. This is an expected behavior since the stability result for the advanced DO guarantees an ultimate bound for the disturbance error, rather than asymptotic convergence to 0.

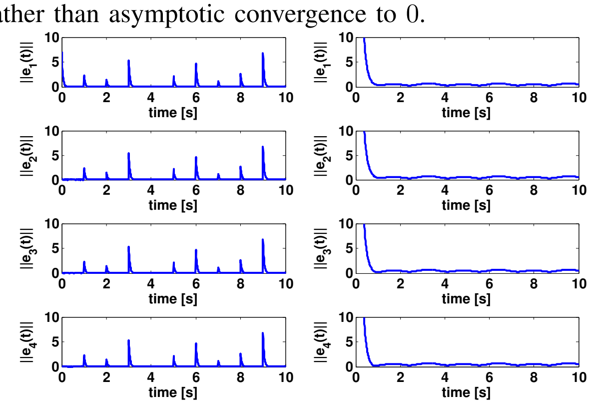

Fig. 4. Norms of Disturbance Errors (a) Simple DO (b) Advanced DO

V. Conclusion

In this paper we presented a control scheme that guarantees collision free trajectories for Lagrangian multi-agent systems while ensuring convergence to an ultimately bounded region via cooperation of agents. The scheme utilizes disturbance observers for attenuating disturbances of various structures. As an extension of the present research, communication delays will be considered for more realistic scenarios, as well as time-delay Lagrangian systems, where states are measured with time delays and control laws are applied with time delays.

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