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# Fuzzy TSK Positive Systems: Stability and Control

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*Abstract*— **Positive systems play an important role in many fields including biology, chemistry, and economics, among others. This paper discusses the stability and control of discrete-time positive Takagi-Sugeno-Kang (TSK) fuzzy systems. It provides a sufficient condition for their exponential stability, as well as a sufficient condition for their instability. It also presents a new approach to their controller design. We present two examples to demonstrate our results. In the first example we develop a positive TSK fuzzy Volterra predator-prey model and investigate its stability. The second example demonstrates our controller design methodology.**

***Keywords: Positive systems, stability, TSK systems.***

## I. INTRODUCTION

A positive system is one whose state vectors remain nonnegative along its trajectories for any nonnegative initial conditions [1]. Since many physical systems have state variable that cannot assume negative values, positive systems arise in many practical applications such as economics, biology, chemistry, etc. For example, concentrations of reagents in a chemical reaction are clearly governed by positive dynamics. The stability of positive systems was investigated in [2],[3],[4]. Controllers have been proposed for positive systems (see for example [5]). For more on positive systems and their properties the reader is referred to the excellent text by Farina and Rinaldi [6].

In many applications of positive system, only qualitative knowledge of the system behavior is available. This knowledge is in the form of a set of rules known to experts in the field but with no exact mathematical expression. Such knowledge can be written in the form of a set of IF-THEN fuzzy rules that can be used to compute the output of the system with words. However, positive fuzzy systems have only been discussed in one publication [7]. The authors present interesting stability and stabilization results based on the use of piecewise quadratic Lyapunov functions.

While not clearly defined in [7], the Lypaunov function used by the authors indicates the assumption of an equilibrium at the origin. This constitutes a trivial solution for most applications of positive systems. For example, if the state variables of the system represent the concentrations of reagents in a chemical reaction, an equilibrium at the origin corresponds to zero concentration and is of no interest. In addition, it is not possible to simply translate the coordinates of the fuzzy system to obtain an equivalent system with a nonzero equilibrium because of the complex switching associated with the fuzzy system. This limits the applicability of the results of [7] to the atypical situation of a positive system that must be maintained at the origin.

This paper provides simple stability tests for discrete TSK positive dynamic systems. These stability results are similar to those obtained by the authors for TSK systems in [8] and [9]. The stability results include sufficient conditions for the exponential stability and sufficient conditions for the instability of positive TSK systems. The paper also presents a new control system design methodology for positive TSK systems. Both the stability and stabilization results of the paper can be tested using linear matrix inequalities (LMI).

The paper uses the following notation. If $F$ is a matrix with positive entries only we write $F > 0$, while $F–G >0$ is written as $F > G$ . Similar notation is used for matrices with nonnegative entries with the sign (≥) and for negative entries with the sign (<).

The paper is organized as follows. In Section II, we review discrete positive linear systems. In Section III, we introduce discrete positive fuzzy TSK systems. Section IV presents stability conditions for positive TSK systems and Section V presents a method for their control design. Section VI includes a stability analysis example and a controller design example. Conclusions are given in Section VII.

## II. POSITIVE LINEAR SYSTEMS

In this section, we briefly review discrete positive linear systems. We begin with the definition of a positive system.

**Definition 1:** *Positive Linear Systems*
A linear system is positive if and only if for every nonnegative initial state and for every nonnegative input its state and output are nonnegative.

Definition 1 requires that all trajectories of a positive system starting from nonnegative states remain in the nonnegative quadrant and yield nonnegative outputs. Note that positivity depends on the basis set of the state space so that an appropriate basis set must be used for the system to satisfy Definition 1.

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Here, we only consider single-input-single-output systems. The state space representation of these systems is given by

$$x(k+1) = Ax(k) + bu(k)$$
$$y(k) = c^T x(k) \quad (1)$$

where $x(k)$ is an $n$ by 1 state vector, $u(k)$ is a scalar input, and $y(k)$ is a scalar output. The triple $(A, b, \boldsymbol{c}^T)$ of appropriate dimensions which characterizes the positive system is constrained so that the system trajectories will remain in the nonnegative quadrant. The following theorem gives necessary and sufficient conditions for a discrete linear system to be positive.

**Theorem 1 [6]:** A discrete linear system $(\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{c}^T)$ is positive if and only if $\boldsymbol{A} \geq \boldsymbol{0}, \boldsymbol{B} \geq \boldsymbol{0}, \boldsymbol{c}^T \geq \boldsymbol{0}^T$.

Using Theorem 1, it suffices to check for any negative element in its triple $(A, b, c^T)$ to verify the positivity of a linear system.

## III. FUZZY POSITIVE SYSTEMS

This section introduces fuzzy positive systems. We start with a description of the fuzzy systems used in this paper. Then we extend the concept of a positive system to TSK fuzzy systems.

**Definition 2:** *Discrete Dynamic TSK Systems*

A discrete TSK fuzzy system is a TSK fuzzy system with a rule base of the form

$R^i$: $IF\ \boldsymbol{x}(k) is\ \boldsymbol{A^i}\ THEN\ \boldsymbol{x}^i(k+1) = F^i \boldsymbol{x}(k) + G^i \boldsymbol{u}(k)$,

$\boldsymbol{x}(k) = [x_1(k) \dots x_n(k)]^T$

$\boldsymbol{u}(k) = [u_1(k) \dots u_m(k)]^T$

$\boldsymbol{A^i} = [A_1^i \dots A_n^i]^T$

are normal, consistent, and complete fuzzy sets,

$F^i \in \mathbb{R}^{n \times n}, G^i \in \mathbb{R}^n, i = 1, .., M \quad (2)$

where $n$ is the order of the consequent system and $M$ is the number of rules. The consequent system may also include an output defined as

$$y(k) = \boldsymbol{c}^T \boldsymbol{x}(k) \quad (3)$$

The state of the TSK fuzzy system is updated by

$$\boldsymbol{x}(k+1) = \frac{\sum_{i=1}^{M} \boldsymbol{x}^i(k+1) \prod_{j=1}^{n} \mu_{A_j^i}(x_j)}{\sum_{i=1}^{M} \prod_{j=1}^{n} \mu_{A_j^i}(x_j)} \quad (4)$$

To extend the concept of the positive systems to fuzzy systems, we first define the equilibrium state of a discrete-time dynamic system.

**Definition 3:** *Equilibrium state*

$\boldsymbol{x}^*$ is an equilibrium state of a discrete-time dynamic system if $\boldsymbol{x}(k) = \boldsymbol{x}_e$ implies

$$\boldsymbol{x}(\boldsymbol{k}+\boldsymbol{m}) = \boldsymbol{x}_e, \forall \boldsymbol{m} > \boldsymbol{0} \quad (5)$$

Clearly, a fuzzy linear system is not a linear system even if its consequents are because of the dependence of the membership functions on the state in (4). Thus the fuzzy system often has multiple equilibrium points which may or may not include the origin. For most positive systems, the origin is a trivial state and the system must be maintained at a nonzero equilibrium point. The following is a sufficient condition for such an equilibrium point to exist.

***Lemma 1:*** *Equilibrium of Discrete TSK Fuzzy Systems*

If the system of **Definition 2** with rules of the form

$R^i$: $IF\ \boldsymbol{x}(k) is\ \boldsymbol{A^i}\ THEN\ \boldsymbol{x}^i(k+1) = F^i \boldsymbol{x}(k)$,

$i = 1, \dots, M \quad (6)$

satisfies the condition

$$F^l \boldsymbol{x}_e = \boldsymbol{x}_e, \boldsymbol{x}_e = \boldsymbol{e}^l \quad (7)$$

for some rule $R^l$ with unity membership for the fuzzy sets $\boldsymbol{A}^l$ at $\boldsymbol{e}^l$, then $\boldsymbol{x}_e$ is an equilibrium point of the system.

**Proof:** The proof follows directly from the expression for the output of a fuzzy TSK system with only one rule firing.

■

Since the fuzzy antecedent sets of the rule base are consistent, the membership functions for rules other than the one governing the equilibrium must be zero at the equilibrium, i.e. $\boldsymbol{\mu}_{A_j^i}(\boldsymbol{x}_j) = \boldsymbol{0}, \boldsymbol{i} \neq \boldsymbol{l}, \boldsymbol{j} = \boldsymbol{1}, \dots, \boldsymbol{n}$, at $\boldsymbol{x}_e$. Using the basic definition of a dynamic TSK system, we now define a fuzzy positive system.

**Definition 4:** *Positive Fuzzy TSK Systems*

A positive TSK system is a system of Definition 2 where for every nonnegative initial state and for every nonnegative input its state and output are nonnegative.

Definition 4 is a simple extension of Definition 1 where the linear system replaced with a TSK system. But differences of linear systems and TSK systems cause modification in extension of Theorem 1 to TSK systems.

Positivity of a TSK system is highly dependent on the rules for which the support of all membership functions in its antecedent include positive values. We refer to such rules as **positive rules** in the sequel.

**Theorem 2:** A TSK system with linear consequents is positive if

$$F^j \geq 0, \boldsymbol{G}^j \geq \boldsymbol{0}, \boldsymbol{c}^T \geq 0^T, j = i_1, \dots, i_N \quad (8)$$

where $\boldsymbol{N}$ is the number of positive rules.

**Proof:** For the initial condition in the positive subspace only $N$ rules will be fired. The value of the state vector for next time step is a linear positive combination of consequent of these $N$ rules (see ( 4 )). From Theorem 1 and ( 8 ), $\boldsymbol{F}^j\big(x(k),u(k)\big)\geq 0$ and therefore $\boldsymbol{x}(k+1)\geq 0$. Then, using ( 3 ), the output is also positive.

■

Theorem 2 provides sufficient conditions for a TSK system to be positive. The conditions are not necessary and there are positive TSK systems that do not satisfy them. These conditions are necessary for systems with consistent rule-base.

## IV. Stability Analysis

To examine the stability of a positive system, it is sufficient to consider its behavior in the nonnegative orthant. We assume that the fuzzy system has a complete and consistent rule base on the nonnegative orthant with nonnegative support for all antecedent fuzzy sets. We derive sufficient conditions for the convergence to the equilibrium for system trajectories with nonnegative initial conditions.

The assumption of linear consequents clearly implies that the origin is an equilibrium for our TSK systems. In addition to the origin, positive systems can have other equilibrium points that are more relevant to practical applications. The trivial equilibrium at the origin is typically of no interest. We assume that the system has at least one equilibrium point $\boldsymbol{x}_e\neq\boldsymbol{0}$.

For our stability analysis, we assume that the rules in our TSK systems have antecedent membership functions with bounded support, i.e.

$$\mu_{A_j^i}(x_j)=\begin{cases} nonzero, & x_j\in\left[\underline{e_j^i},\overline{e_j^i}\right]\\ 0, & x_j\notin\left[\underline{e_j^i},\overline{e_j^i}\right]\end{cases}$$
$$,\ i=1,\ldots,M,\ j=1,\ldots,n \qquad (9)$$

An example of antecedent membership functions with bounded support are the triangular membership functions shown in Figure 1.

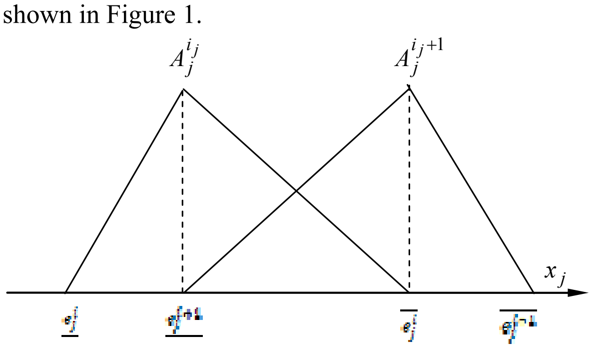

Figure 1- Triangular membership functions.

**Theorem 3:** For a positive TSK system with nonnegative initial condition and zero input, an equilibrium point $\boldsymbol{x_e}$ is exponentially stable if $\exists\boldsymbol{b}\in\mathbb{R}^{\boldsymbol{n}},\boldsymbol{b}>\boldsymbol{0}$ such that

$$\boldsymbol{b}^T\left|F^i\,\boldsymbol{e}^i-\boldsymbol{x}_e\right|\leq\alpha\boldsymbol{b}^T\left|\boldsymbol{e}^i-\boldsymbol{x}_e\right|,$$
$$\boldsymbol{e}^i\in\left\{\overline{\boldsymbol{e}}^i,\underline{\boldsymbol{e}}^i\right\},\overline{\boldsymbol{e}}^i=\left[\overline{e}_1^i\quad\ldots\quad\overline{e}_n^i\right]^T,\underline{\boldsymbol{e}}^i=\left[\underline{e}_1^i\quad\ldots\quad\underline{e}_n^i\right]^T,$$
$$\alpha\in[0,1],i=1,\ldots,M \qquad (10)$$

**Proof:** To prove stability, we show that a weighted metric of the distance from the state to an equilibrium decreases exponentially under condition ( 10 ). The weighted metric at time $k$+1 can be written using ( 4 ) as

$$\boldsymbol{b}^T|\boldsymbol{x}(k+1)-\boldsymbol{x}_e|=\boldsymbol{b}^T\left|\frac{\sum_{i=1}^M F^i\boldsymbol{x}(k)w^i}{\sum_{i=1}^M w^i}-\boldsymbol{x}_e\right|,$$
$$w^i=\prod_{j=1}^n\mu_{A_j^i}(x_j) \qquad (11)$$

Since the consequent matrices are positive, $\left|\frac{\sum_{i=1}^M F^i\boldsymbol{x}(k)w^i}{\sum_{i=1}^M w^i}-\boldsymbol{x}_e\right|$ has its maximum value at the boundaries $\boldsymbol{e}^i\in\left\{\underline{\boldsymbol{e}}^i,\overline{\boldsymbol{e}}^i\right\}$ using ( 10 ) we have

$$\boldsymbol{b}^T|\boldsymbol{x}(k+1)-\boldsymbol{x}_e|\leq\alpha\boldsymbol{b}^T\left|\boldsymbol{e}^i-\boldsymbol{x}_e\right| \qquad (12)$$

It is obvious that $\underline{\boldsymbol{e}}^i\leq\boldsymbol{x}(k)\leq\overline{\boldsymbol{e}}^i$ and therefore

$$\boldsymbol{b}^T|\boldsymbol{x}(k+1)-\boldsymbol{x}_e|\leq\alpha\boldsymbol{b}^T|\boldsymbol{x}(k)-\boldsymbol{x}_e| \qquad (13)$$

This shows that a weighted norm of the state vector is exponentially decreasing and the system is stable.

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Theorem 3 provides sufficient stability conditions for positive TSK systems. Since the conditions are only sufficient, we provide the following sufficient instability result theorem for use if the stability test fails.

**Theorem 4:** For a positive TSK system with nonnegative initial condition and zero input, the origin is unstable if $\exists\boldsymbol{b}\in\mathbb{R}^{\boldsymbol{n}},\boldsymbol{b}>\boldsymbol{0}$ such that

$$\boldsymbol{b}^T F^i\overline{\boldsymbol{e}}^{\boldsymbol{i}}>\boldsymbol{b}^T\underline{\boldsymbol{e}}^i,$$
$$\overline{\boldsymbol{e}}^i=\left[\overline{e}_1^i\quad\ldots\quad\overline{e}_n^i\right]^T,\underline{\boldsymbol{e}}^i=\left[\underline{e}_1^i\quad\ldots\quad\underline{e}_n^i\right]^T,i=1,\ldots,n_r \qquad (14)$$

**Proof:** The proof is similar to the proof of Theorem 3**.** We show that the weighted norm of the state vector is monotonically increasing, instead of exponentially decreasing, using ( 14 ).

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## V. Control Design

We use the stability results of Section IV to obtain feedback controllers for positive TSK systems with the feedback configuration of Figure 2. The controller is a positive TSK controller. To design a general TSK controller that maps the plant states to the plant inputs, we must determine all its membership functions and rules. This may require the evaluation of a large number of control parameters and present a significant burden for the designer. We consider a TSK controller with a similar rule base to the TSK plant and with different consequents. This simplifies the controller design because the only tuning parameters we evaluate are the parameters in the consequent of the TSK controller. The latter are considerably fewer than the number of parameters of a general TSK fuzzy controller.

The plant in Figure 2 is a positive TSK system. The controller is TSK with membership functions the same as the plant. The rule base of the controller is

$R^i: IF\ \boldsymbol{x}(k) is\ \boldsymbol{B}^i\ THEN\ u(k) = K^i \boldsymbol{x}(k)$

$K^i \in \mathbb{R}^{1\times n}, i = 1, .., n_r$

( 15 )

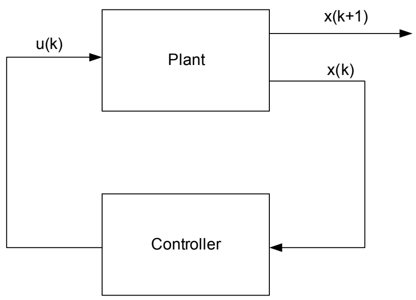

Figure 2 – Block diagram of fuzzy control system.

where $K^i$ are positive matrices.

Note that as it discussed, we choose $\boldsymbol{B}^i$ the same as the membership functions in the antecedents of the plant ($\boldsymbol{A}^i$). The state update of this controller can be calculated using ( 4 ). Replacing the control signal $u(k)$ in the antecedents of the positive TSK plant gives an autonomous closed-loop TSK system with nonlinear consequents

$$\boldsymbol{x}^j(k+1) = F^j \boldsymbol{x}(k) + \boldsymbol{G}^j . \frac{\sum_{i=1}^{M} [K^i \boldsymbol{x}(k)] w^i}{\sum_{i=1}^{M} w^i}$$

( 16 )

where $w^i$ is the firing strength of controller ( 15 ).

To understand why this consequent is nonlinear, note that $w^i$ is a function of $\boldsymbol{x}(k)$. Since the consequent is monotone, the stability conditions of Section IV are applicable to this system provided that the consequents are nonnegative. To obtain proper situation, we choose controller parameters $K^i$ with all nonnegative.

**Theorem 5:** Consider the positive TSK system of Figure 2 with fuzzy controller of ( 15 ). The closed-loop system is exponentially stable with convergence rate $\boldsymbol{\alpha}$ if $\exists \boldsymbol{b} \in \mathbb{R}^{\boldsymbol{n}}, \boldsymbol{b} > \boldsymbol{0}$, and the controller parameters $\boldsymbol{K^i} \in \mathbb{R}^{\boldsymbol{1\times n}}, \boldsymbol{K^i} \geq \boldsymbol{0}, \boldsymbol{i} = \boldsymbol{1}, \dots, \boldsymbol{M}$ such that

$$\boldsymbol{b}^T \left| F^i\ \boldsymbol{e}^i + K^{i+l} \boldsymbol{e}^i - \boldsymbol{x}_e \right| \leq \alpha \boldsymbol{b}^T \left| \boldsymbol{e}^i - \boldsymbol{x}_e \right|,$$

$$[\boldsymbol{e}^i \quad l] \in \{[\overline{\boldsymbol{e}}^i \quad 1], [\underline{\boldsymbol{e}}^i \quad -1]\}, \overline{\boldsymbol{e}}^i = [\overline{e}_1^i \quad \dots \quad \overline{e}_n^i]^T,$$

$$\underline{\boldsymbol{e}}^i = [\underline{e}_1^i \quad \dots \quad \underline{e}_n^i]^T, \alpha \in [0,1], i = 1, \dots, M$$

( 17 )

**Proof:** Applying Theorem 3 to the closed loop system with consequents of ( 16 ) yields ( 17 ).

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## VI. EXAMPLES

*Example 1: Stability Analysis*

In this example, we consider the well-known Lotka-Volterra competition model that governs the population of a single predator and a single prey [10]. The model is given by the second order system

$$\dot{x}_1 = x_1(3 - 2x_1 - x_2)$$

$$\dot{x}_2 = x_2(3 - x_1 - 2x_2)$$

( 18 )

where $x_1$ is the prey population and $x_2$ is the predator population.

The system has two equilibrium points, at $x_1 = x_2 = 1$ population unit, and $x_1 = x_2 = 0$. The second equilibrium point corresponds to the extinction of the predator and prey while the first represents a balance between the two populations. Using Theorem 4, we can easily show that the second equilibrium is unstable since the instability condition ( 14 ) holds for any $\boldsymbol{b} > \boldsymbol{0}$.

The stability of the first equilibrium state $x = [1,1]^T$ is more important in population studies and is investigated next. We first model the predator-prey dynamics as a discrete-time positive TSK fuzzy system. Since we are interested in the local stability of the origin, we assume $x_1 \epsilon [0.01,1.99]$ and $x_2 \epsilon [0.01,1.99]$. The discrete-time fuzzy TSK system can be described by the following rules:

$$R^1: IF\ x_1\ is\ A_1^1\ and\ x_2\ is\ A_1^2\ THEN\ \boldsymbol{x^1}(k+1) = \begin{bmatrix} 1.0002 & 0.0299 \\ 0.0299 & 1.0002 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix}$$

$$R^2: IF\ x_1\ is\ A_1^1\ and\ x_2\ is\ A_2^2\ THEN\ \boldsymbol{x^2}(k+1) = \begin{bmatrix} 0.9999 & 0 \\ 5.833 & 0.9611 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix}$$

$$R^3: IF\ x_1\ is\ A_2^1\ and\ x_2\ is\ A_1^2\ THEN\ \boldsymbol{x^3}(k+1) = \begin{bmatrix} 0.9611 & 5.833 \\ 0 & 0.9999 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix}$$

$$R^4: IF\ x_1\ is\ A_2^1\ and\ x_2\ is\ A_2^2\ THEN\ \boldsymbol{x}^4(k+1) = \begin{bmatrix} 0.961 & 0.0097 \\ 0.0097 & 0.961 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix}$$

The membership functions are shown in Figure 3. Based on Theorem 2 we conclude that the system is positive.

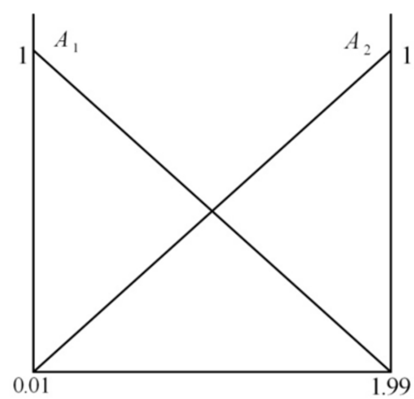

Figure 3 – Membership functions of the Lotka-Volterra fuzzy model.

The stability conditions of Theorem 3 are met with $\boldsymbol{b} = [9,6]^T$. We conclude that $x = [1,1]^T$ is an exponentially stable equilibrium of the system. Figure 4 shows the simulation results for this system for several initial conditions. All the trajectories in the simulation converge to the equilibrium.

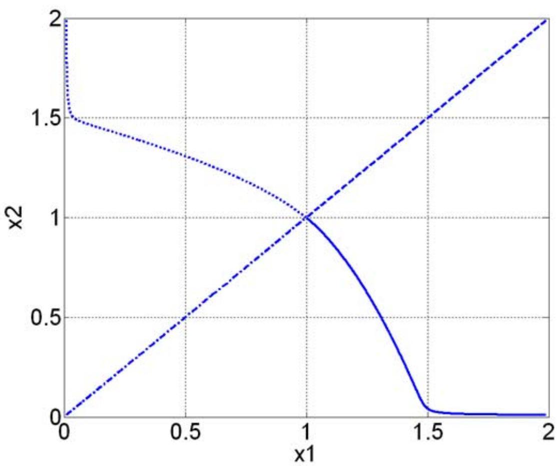

Figure 4 – Simulation results of the Lotka-Volterra fuzzy model.

*Example 2: Control design*

Design a fuzzy controller to asymptotically stabilize the discrete fuzzy system with the rules of the form

$R^1$: $IF\ x_1\ is\ A_1^1\ and\ x_2\ is\ A_1^2\ THEN\ \boldsymbol{x}^{\mathbf{1}}(k+1)$
$$= \begin{bmatrix} 0.44 & 0.33 \\ 0.33 & 0.77 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} 0.27 \\ 0.93 \end{bmatrix} u(k)$$

$R^2$: $IF\ x_1\ is\ A_2^1\ and\ x_2\ is\ A_1^2\ THEN\ \boldsymbol{x}^{\mathbf{2}}(k+1)$
$$= \begin{bmatrix} 0.42 & 0.61 \\ 0.60 & 0.34 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} 0.41 \\ 0.62 \end{bmatrix} u(k)$$

$R^3$: $IF\ x_1\ is\ A_3^1\ and\ x_2\ is\ A_1^2\ THEN\ \boldsymbol{x}^{\mathbf{3}}(k+1)$
$$= \begin{bmatrix} 0.76 & 0.95 \\ 0.95 & 0.21 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} 0.32 \\ 0.11 \end{bmatrix} u(k)$$

$R^4$: $IF\ x_1\ is\ A_1^1\ and\ x_2\ is\ A_2^2\ THEN\ \boldsymbol{x}^4(k+1)$
$$= \begin{bmatrix} 0.92 & 0 \\ 0.96 & 0.82 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} 0.61 \\ 0.92 \end{bmatrix} u(k)$$

$R^5$: $IF\ x_1\ is\ A_2^1\ and\ x_2\ is\ A_2^2\ THEN\ \boldsymbol{x}^5(k+1)$
$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix} u(k)$$

$R^6$: $IF\ x_1\ is\ A_3^1\ and\ x_2\ is\ A_2^2\ THEN\ \boldsymbol{x}^6(k+1)$
$$= \begin{bmatrix} 0.9 & 0.75 \\ 0.70 & 0.41 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} 0.36 \\ 0.29 \end{bmatrix} u(k)$$

$R^7$: $IF\ x_1\ is\ A_1^1\ and\ x_2\ is\ A_3^2\ THEN\ \boldsymbol{x}^7(k+1)$
$$= \begin{bmatrix} 0.17 & 0.10 \\ 0.09 & 0.51 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} 0.45 \\ 0.88 \end{bmatrix} u(k)$$

$R^8$: $IF\ x_1\ is\ A_2^1\ and\ x_2\ is\ A_3^2\ THEN\ \boldsymbol{x}^8(k+1)$
$$= \begin{bmatrix} 0.42 & 0.46 \\ 0.71 & 0.68 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} 0.77 \\ 0.43 \end{bmatrix} u(k)$$

$R^9$: $IF\ x_1\ is\ A_3^1\ and\ x_2\ is\ A_3^2\ THEN\ \boldsymbol{x}^9(k+1)$
$$= \begin{bmatrix} 0.62 & 0.80 \\ 0.95 & 0.44 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} 0.21 \\ 0.51 \end{bmatrix} u(k)$$

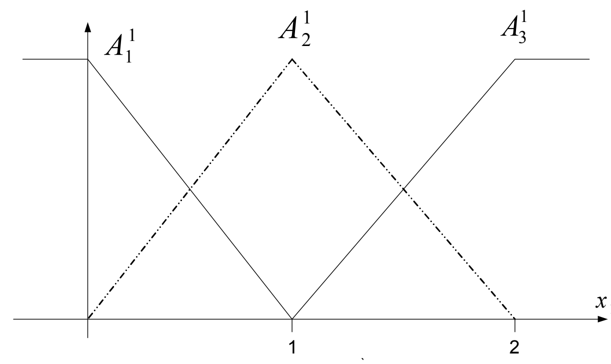

Figure 5 – Membership functions of the first input.

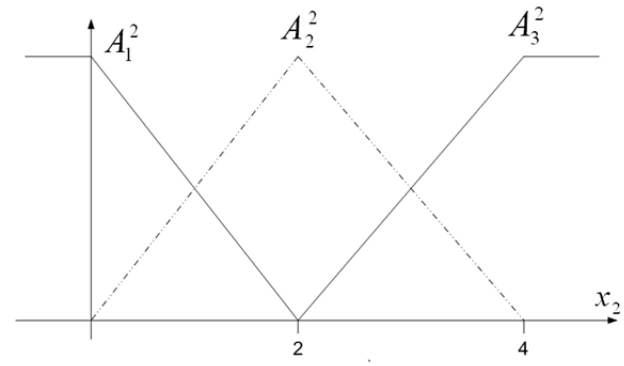

Figure 6 – Membership functions of the second input.

where the antecedent membership functions are shown in Figure 5 for the first input and in Figure 6 for the second input.

We first observe that the system satisfies the conditions of Theorem 2 and is therefore a positive fuzzy TSK system. By *Lemma* 1, the system has an equilibrium at $\boldsymbol{x}_e = [1 \quad 2]^T$. We design a controller to stabilize the system in the vicinity of its equilibrium using the conditions of Theorem 5. The controller has the following rules

$R^1$: $IF\ x_1\ is\ A_1^1\ and\ x_2\ is\ A_1^2\ THEN\ u(k)$
$$= [0.8 \quad 0.3] \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix}$$

$R^2$: $IF\ x_1\ is\ A_2^1\ and\ x_2\ is\ A_1^2\ THEN\ u(k)$
$$= [0.3 \quad 1] \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix}$$

$R^3$: $IF\ x_1\ is\ A_3^1\ and\ x_2\ is\ A_1^2\ THEN\ u(k)$
$$= [0.2 \quad 1] \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix}$$

$R^4$: $IF\ x_1\ is\ A_1^1\ and\ x_2\ is\ A_2^2\ THEN\ u(k)$
$$= [1 \quad 0.7] \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix}$$

$R^5$: $IF\ x_1\ is\ A_2^1\ and\ x_2\ is\ A_2^2\ THEN\ u(k)$
$$= [0 \quad 0] \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix}$$

$R^6$: $IF\ x_1\ is\ A_3^1\ and\ x_2\ is\ A_2^2\ THEN\ u(k)$
$$= [0.3 \quad 1] \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix}$$

$R^7: IF\ x_1\ is\ A_1^1\ and\ x_2\ is\ A_3^2\ THEN\ u(k) = [1 \quad 0.8] \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix}$

$R^8: IF\ x_1\ is\ A_2^1\ and\ x_2\ is\ A_3^2\ THEN\ u(k) = [0.8 \quad 1] \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix}$

$R^9: IF\ x_1\ is\ A_3^1\ and\ x_2\ is\ A_3^2\ THEN\ u(k) = [0.3 \quad 1] \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix}$

For the initial condition $x_0 = [0.1, 0.1]^T$ and zero input, the trajectories of the open-loop system and of the system with fuzzy control are shown in Figure 7. Figure 7 shows that trajectory of the open-loop system diverges while the trajectory of the system with fuzzy control converges to the equilibrium point.

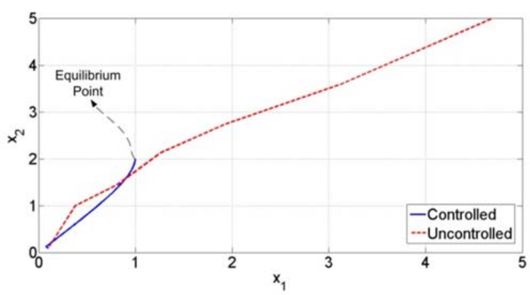


Figure 7 – Simulation result of the system of Example 2.

## VII. Conclusion

Positive systems are used to model many physical systems in biology, economics, etc. This paper discusses positive TSK fuzzy systems and their stability. It provides sufficient conditions for exponential stability for this class of TSK systems and a new control design methodology. The paper presents a positive TSK model of a Volterra predator-prey system. The stability of the model is investigated using our sufficient conditions. A second example demonstrates through how our design methodology can drive a positive system to its equilibrium point with a prescribed rate of convergence selected by the designer.

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