2011 American Control Conference
on O'Farrell Street, San Francisco, CA, USA
June 29 - July 01, 2011

Estimation and Control of UAV Swarms for Distributed Monitoring Tasks

Fabio Morbidi, Randy A. Freeman, Kevin M. Lynch

***Abstract*— This paper proposes a distributed estimation and control strategy for cooperative monitoring by swarms of *unmanned aerial vehicles* (UAVs) modeled as constant-speed unicycles. The *geometric moments*, encoding an abstraction of the swarm, are controlled via a nonlinear gradient descent to match those of a discrete set of particles describing the occurrence of some event of interest to be monitored. Because of its limited sensing capabilities, each agent can measure the position of only a subset of the overall particles, from which it locally estimates the desired moments of the swarm running a *proportional-integral* (PI) *average consensus estimator*. The closed-loop stability of the system arising from the combination of the gradient-descent controllers and the consensus estimators is studied and simulation results are provided to illustrate the proposed theory.**

I. Introduction

Recent years have witnessed an acceleration in research efforts aimed at designing environmental monitoring algorithms for mobile sensor networks [1], [2]. In fact, as known, *mobile sensors* offer distinctive advantages over *static* ones, in terms of quality of sensing and estimation, area coverage, adaptability to changing conditions and robustness against failures. In this paper we are interested in monitoring a set of moving particles describing the occurrence of some event of interest in a 2-D environment, with a team of *unmanned aerial vehicles* (UAVs) flying at fixed altitude. With the term *particle*, we refer to any discrete entity belonging to a given ensemble, whose position in the plane has to be tracked over time: examples include animals in a group, people in a crowd, smoke particles in a plume, multiple wildfire spots, droplets in an oil spill. The "shape" of the UAV swarm and of the ensemble of particles is synthetically described in terms of their *geometric moments*. Because of its limited sensing capabilities, each agent, modeled as a *unicycle* with constant positive forward velocity, can only measure the position of a subset of the overall particles. Our goal is to design a distributed estimation and control strategy to match the moments of the swarm with those of the particles: this in turn guarantees the UAVs to properly cover the region of interest.

A. Literature review

Two literature domains are relevant to the present work. In the first one, the goal is to design *distributed algorithms for multiple agents to detect and track the boundary of a region of interest*. In [3], a "snake algorithm" is adopted to identify and track the boundary of harmful algae blooms using a team of agents equipped with chemical sensors. In [4], a random coverage controller is used to detect and surround oil spills, and in [5] an algorithm is described to allow multiple UAVs to cooperatively monitor and track the propagation of large forest fires. Recently, in [6], a method has been proposed to optimally approximate an environmental boundary with a polygon. The mobile agents rely only on sensed local information to position some interpolation points and define the approximating polygon whose vertices are uniformly distributed along the boundary of the target region.

In the second literature domain of interest for this work, the goal is to synthesize *decentralized simultaneous estimation and control strategies for multiple agents*. In [7], a general framework to design collective behaviors for groups of mobile robots has been proposed: each agent communicates with its neighbors and estimates the global performance properties of the robotic network needed to make a local control decision. In [8], a decentralized strategy for modeling of environmental parameters is presented and a gradient control is used to move the agents in order to maximize their sensory information relative to the current uncertainty in the model. A distributed learning and cooperative control strategy has been proposed in [9]. Each agent recursively estimates an unknown field of interest from noisy measurements and moves towards its peaks using the gradient of its estimated field, while maintaining network-wide connectivity. Other distributed approaches to controlled sampling and modeling of deterministic or stochastic scalar fields, include [10]–[12]. In [10], the agents move in a fixed platoon along an estimated gradient, while in [11] they are controlled to track a level set of the field. Recently, in [12], a procedure to adapt local interpolations to represent spatial fields as they are measured by a mobile sensor network, has been presented.

B. Original contributions and organization

This paper proposes a distributed estimation and control strategy for cooperative monitoring by swarms of UAVs. Differently from the first literature domain above, our focus is not on the boundary of the region spanned by the particles (that may be faint or fuzzy in real settings, and thus hard to detect and track), but on controlling the (first- and second-order) *geometric moments* encoding an abstraction of the swarm [13], [14], to match the moments of the ensemble of particles observed by the UAVs. Although full aircraft dynamics are quite complex, the essential components for level cruise flight can be captured by the model of a planar *constant-speed unicycle*. This model is challenging to control because the vehicle cannot stop (nor move directly sideways), and so it is not small-time locally controllable [15]. A new nonlinear gradient-descent angular control is proposed in this paper to steer our team of unicycle-type vehicles.

Differently from [16], [17] and related works in the coverage literature, in this paper we assume that the agents have *not* access to a *distribution density function*, providing an *a priori global* measure of information or probability that

F. Morbidi and K. M. Lynch are with the Dept. of Mechanical Engineering and R. A. Freeman is with the Dept. of Electrical Engineering and Computer Science, Northwestern University, Evanston, IL 60208 USA. Email of the corresponding author: morbidi@dii.unisi.it. This material is based upon work partially supported by the Office of Naval Research.

some event takes place over the region of interest. On the contrary, similarly to [18] where sensor measurements are used to learn the distribution of sensory information in the environment, each agent is equipped with a limited-footprint sensor (e.g., a camera pointing downward), which allows it to detect only a fraction of the overall particles. We will abstract from any actual sensor model, and assume that each agent processes only the particles lying within the Voronoi cell that it generates (and ignores those possibly located outside): from them each UAV locally estimates the desired moments of the swarm running a *proportional-integral* (PI) *average consensus estimator* [8]. As known, if the inputs of the PI estimators as well as the topology of the connected network are slowly varying (this is true in our setting since the dynamics of the environment is assumed to be significantly slower than that of the swarm, c.f. Th. 2), small estimation errors are achieved. The closed-loop stability of the system arising from the combination of the gradient-descent angular controllers and the PI estimators has been studied and simulation experiments have been performed to illustrate the proposed theory.

The rest of the paper is organized as follows. In Sect. II we introduce the gradient-based control, assuming that the desired moments of the swarm are *a priori known* to each agent. In Sect. III and Sect. IV we deal with the distributed estimation problem and the closed-loop stability analysis, respectively. Finally, in Sect. V simulation results are presented and in Sect. VI the main contributions of the paper are summarized and possible avenues for future research are highlighted.

II. CONTROL DESIGN

Consider a swarm of n unmanned aerial vehicles (hereafter, simply *agents* or *vehicles*) flying at fixed altitude, with the following unicycle model,

$$\begin{cases} \dot{p}_{ix}(t) = v_i(t)\cos(\theta_i(t)), \\ \dot{p}_{iy}(t) = v_i(t)\sin(\theta_i(t)), \quad i \in \{1,\dots,n\}, \\ \dot{\theta}_i(t) = \omega_i(t), \end{cases} \tag{1}$$

where $\mathbf{p}_i(t) = [p_{ix}(t),\, p_{iy}(t)]^T \in \mathbb{R}^2$ denotes the position of agent i at time t in the plane of motion, $\theta_i(t) \in [-\pi,\, \pi)$ its heading and $[v_i(t),\, \omega_i(t)]^T \in [v_{\min}, +\infty) \times \mathbb{R}$, $v_{\min} > 0$, its forward and angular velocities. Let $\mathbf{p} = [\mathbf{p}_1^T, \dots, \mathbf{p}_n^T]^T \in (\mathbb{R}^2)^n$. The configuration of the agents is described by using a *swarm moment function* $\mathbf{f} : (\mathbb{R}^2)^n \to \mathbb{R}^\ell$ that we will assume to be of the form:

$$\mathbf{f}(\mathbf{p}) = \frac{1}{n}\sum_{i=1}^{n} \boldsymbol{\phi}(\mathbf{p}_i),$$

where the *moment-generating function* $\boldsymbol{\phi} : \mathbb{R}^2 \to \mathbb{R}^\ell$ is defined as,

$$\boldsymbol{\phi}(\mathbf{p}_i) \triangleq [\,p_{ix},\, p_{iy},\, p_{ix}^2,\, p_{iy}^2,\, p_{ix}p_{iy},\, p_{ix}^3,\, p_{iy}^3, \dots]^T. \tag{2}$$

Note that $\ell = \frac{1}{2}(r+1)(r+2) - 1$ where $r \in \mathbb{Z}_{>0}$ is the maximum order of the moments appearing in (2), and that if ℓ moment constraints are specified on n agents then there is in general a $(2n-\ell)$-dimensional algebraic set of swarm configurations that satisfy them. The primary objective of the agents is to move so that their final arrangement minimizes the error $\mathbf{f}(\mathbf{p}) - \mathbf{f}^\star$, where the *goal vector* $\mathbf{f}^\star \in \mathrm{im}(\mathbf{f})$ defines the desired shape of the formation. For the sake of simplicity, thorough this section we will assume that $\mathbf{f}^\star$ is *constant* and *a priori known* to each agent: we will relax this hypothesis in Sect. III, where each agent independently estimates the goal vector from the environmental data. Our control strategy relies on the gradient of the *potential function* $\Pi : (\mathbb{R}^2)^n \to \mathbb{R}_{\geq 0}$,

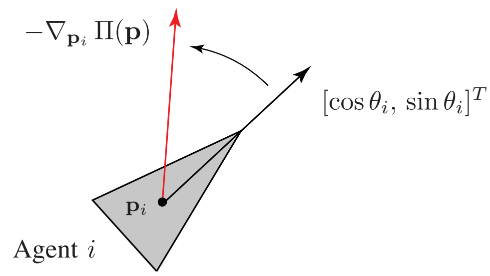

Fig. 1. The angular control of agent i forces its heading direction $[\cos\theta_i,\, \sin\theta_i]^T$ to align with the antigradient of $\Pi(\mathbf{p})$.

$$\Pi(\mathbf{p}) = (\mathbf{f}(\mathbf{p}) - \mathbf{f}^\star)^T\, \boldsymbol{\Gamma}\, (\mathbf{f}(\mathbf{p}) - \mathbf{f}^\star), \tag{3}$$

where $\boldsymbol{\Gamma} \in \mathbb{R}^{\ell\times\ell}$ is an assigned symmetric positive-definite gain matrix. Let $\mathrm{Crit}(\Pi) \triangleq \{\mathbf{p} \in (\mathbb{R}^2)^n : \nabla_{\mathbf{p}}\, \Pi(\mathbf{p}) = \mathbf{0}\}$ denote the set of critical points of (3) and classify such points as *good critical points* where $\mathbf{f}(\mathbf{p}) = \mathbf{f}^\star$ (these are the global minima of Π) and *bad critical points* where $\mathbf{f}(\mathbf{p}) \neq \mathbf{f}^\star$. Given a closed set of swarm configurations $\mathcal{P} \subset (\mathbb{R}^2)^n$ and a goal vector $\mathbf{f}^\star \in \mathbf{f}(\mathcal{P})$, let $\mathcal{G}(\mathbf{f}^\star, \mathcal{P})$ be the convex cone of all symmetric positive-definite matrices $\boldsymbol{\Gamma}$ such that no bad critical points of Π in $\mathcal{P}$ are local minima of Π. To reduce the risk of the swarm "getting stuck" at bad critical points of Π, we would ideally choose a gain matrix $\boldsymbol{\Gamma}$ belonging to $\mathcal{G}(\mathbf{f}^\star, \mathcal{P})$ for a large set $\mathcal{P}$. Actually, for $r = 2$ one can always compute members of $\mathcal{G}(\mathbf{f}^\star, \mathcal{P})$ when $\mathcal{P}$ contains all possible configurations of at least 3 agents. This idea is made precise in the following theorem, readapted from [7, Th. 2]:

Theorem 1: Let $\mathcal{P} = (\mathbb{R}^2)^n$ with $n \geq 3$ and $\mathbf{f}^\star \in \mathbf{f}(\mathcal{P})$. Then there exists a symmetric positive-definite matrix $\boldsymbol{\Gamma}$ such that for every bad critical point $\mathbf{p} \in \mathcal{P}$ of Π, the Hessian matrix $\mathcal{H}(\Pi(\mathbf{p}))$ has at least one strictly negative eigenvalue (hence, $\mathbf{p}$ cannot be a local minimum of Π). In particular, $\boldsymbol{\Gamma} \in \mathcal{G}(\mathbf{f}^\star, \mathcal{P})$. ■

In the rest of this paper, we will then restrict to the case of $r = 2$, (i.e., $\ell = 5$), i.e., only the *first- and second-order moment statistics* will be considered. To provide concise statements in the sequel, we introduce the function $\mathrm{proj}(\cdot)$, which maps the angle $\alpha \in \mathbb{R}$ into the interval $[-\pi,\, \pi)$,

$$\mathrm{proj}(\alpha) \triangleq ((\alpha + \pi) \bmod 2\pi) - \pi, \tag{4}$$

where "mod" stands for the modulo operator which returns the remainder after division. The symbol $\mathbf{A}[i,j]$ will be used to denote the (i,j)-th component of a matrix $\mathbf{A}$.

The geometric intuition behind our control strategy (see equ. (5) below) is simple: in fact, the forward velocity of the agents is set to the same positive constant value and the angular velocities are chosen so that the heading direction of each vehicle is forced to align with the antigradient of the potential function $\Pi(\mathbf{p})$ (see Fig. 1). Let $n \geq 3$ and choose $\boldsymbol{\Gamma}$ and $\mathcal{P}$ as in Th. 1. Let us define the vector function $\mathbf{g}_i : \mathbb{R}_{\geq 0} \to \mathbb{R}^2$, $i \in \{1,\dots,n\}$,

$$\mathbf{g}_i(t) \triangleq -\nabla_{\mathbf{p}_i}\Pi(\mathbf{p}(t)) = -(\mathcal{J}\boldsymbol{\phi}(\mathbf{p}_i(t)))^T\, \boldsymbol{\Gamma}\, (\mathbf{f}(\mathbf{p}(t)) - \mathbf{f}^\star),$$

where $\mathcal{J}\phi(\cdot) \in \mathbb{R}^{\ell\times 2}$ is the Jacobian matrix of $\phi(\cdot)$ and set $\alpha_i(t) \triangleq \text{proj}(\arg(\mathbf{g}_i(t)) - \theta_i(t))$ where $\arg : \mathbb{R}^2 \to [-\pi, \pi)$.

Consider the following control input for agent i,

$$v_i(t) = v, \quad \omega_i(t) = \rho\,\alpha_i(t), \tag{5}$$

where $v \geq v_{\min}$ is a positive constant and ρ is a positive gain.

We observe the following two properties, for almost every initial configuration of the agents:

a) For any $\epsilon > 0$, there exists a sufficiently large gain ρ such that the error on the desired swarm configuration is uniformly ultimately bounded with an ultimate bound ϵ, i.e., for every $\zeta > 0$ there is a positive constant $t_0 = t_0(\zeta)$ such that:

$$\Pi(\mathbf{p}(0)) < \zeta \;\Rightarrow\; \Pi(\mathbf{p}(t)) \leq \epsilon, \; \forall\, t \geq t_0.$$

b) Let $d_{ij}^{\theta}(t) \triangleq \text{proj}(\theta_i(t) - \theta_j(t))$ and $d_{ij}^{\omega}(t) \triangleq \omega_i(t) - \omega_j(t)$, $i, j \in \{1, \dots, n\}$, $i \neq j$, be the functions measuring the disagreement between the heading directions and angular velocities of agents i and j at time t, respectively. Then, for any $\epsilon_\theta, \epsilon_\omega > 0$ there exists a sufficiently large constant $\mu \in \mathbb{R}_{>0}$ satisfying,

$$\boldsymbol{\Gamma}[1,1], \boldsymbol{\Gamma}[2,2] \geq \mu\big|\boldsymbol{\Gamma}[h,l]\big|, \quad \begin{array}{c} h, l \in \{1, \dots, \ell\}, \\ (h,l) \neq \{(1,1), (2,2)\}, \end{array}$$

such that $|d_{ij}^{\theta}(t)|$, $|d_{ij}^{\omega}(t)|$ are uniformly ultimately bounded with ultimate bounds ϵ_θ, ϵ_ω, respectively.

Point *b)* states that $d_{ij}^{\theta}(t)$ becomes bounded as $t \to +\infty$: this behavior is typically referred to as *phase locking* in the literature of coupled oscillators [19].

Remark 1: Note that the angular control in (5) presents discontinuities: in fact, the function proj$(\cdot)$ in (4) is discontinuous at $(2m+1)\pi$, $m \in \mathbb{Z}$, and $\arg(\cdot)$ is not defined at the origin. It is possible to modify the control in order to make it smooth: however, in order to simplify our subsequent analysis we will not pursue this direction herein. ◇

Remark 2: To compute the angular control in (5), agent i needs to know the position of *all* the other agents (i.e., the vector $\mathbf{p}$) at each time instant. This means that control (5) is *not* implementable in a distributed fashion. We will overcome this issue in Sect. III where we will introduce distributed estimators of the swarm moment function $\mathbf{f}(\mathbf{p})$ (as well as of the goal vector, using the environmental data) for each agent. ◇

As an illustration, Fig. 2 shows the trajectory of $n = 4$ agents implementing control (5) with $v = 1$ m/s, $\rho = 0.5$, $\mathbf{f}^\star = [10, 5, 800, 100, 10]^T$ and $\boldsymbol{\Gamma} = \text{diag}(1000, 1000, 0.1, 0.1, 0.1)$. The red dashed ellipse graphically represents the *initial* moments of swarm (i.e., the uniform-density ellipse has the same mass and the same first- and second-order moments as the swarm at the initial time), the red solid ellipse the moments of the swarm at the *final time instant*, and the black ellipse the *desired* moments of the swarm. Note that differently from [7], where double-integrator agents are considered, the red solid and black ellipses are not perfectly superimposed at steady state and the vehicles hover about four points corresponding to a minimum of $\Pi(\mathbf{p})$. In other words, the swarm converges to a configuration that satisfies the desired moment statistics up to an inherent error, consequence of the inability of the agents to stop moving forward. Actually, with a control of the form (5), spinning around the equilibrium turns out to be the "best strategy" for the vehicles to meet the goal.

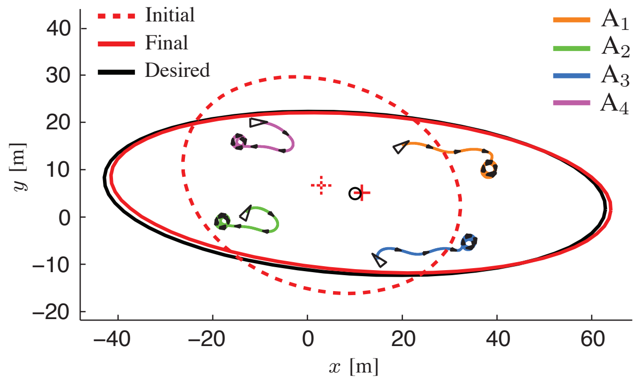

Fig. 2. *Example*: Trajectory of $n = 4$ agents using control (5). The red dashed ellipse graphically represents the *initial* moments of the swarm, the red solid ellipse the moments of the swarm at the *final time instant* ($t = 80$ sec.), and the black ellipse the *desired* moments of the swarm.

III. Distributed Estimation

In this section we present a distributed algorithm to locally estimate the swarm moment function $\mathbf{f}(\mathbf{p})$ and the vector of desired geometric moments using the environmental data.

The following notion of *Voronoi partition* [16, Sect. IIB] is essential for the forthcoming developments.

***Definition 1** (Voronoi partition):* Given a set $\mathcal{Q} \subset \mathbb{R}^2$ and n distinct points $\{\mathbf{p}_1, \dots, \mathbf{p}_n\}$ in $\mathcal{Q}$, the *Voronoi partition* of $\mathcal{Q}$ generated by $\{\mathbf{p}_1, \dots, \mathbf{p}_n\}$ is the collection of sets $\{V_1, \dots, V_n\}$ defined by, for each $i \in \{1, \dots, n\}$,

$$V_i \triangleq \big\{\mathbf{z} \in \mathcal{Q} \mid \|\mathbf{z} - \mathbf{p}_i\| \leq \|\mathbf{z} - \mathbf{p}_j\|, \forall\, j \neq i\big\},$$

where $\|\cdot\|$ denotes the standard Euclidean norm. We will refer to V_i as the *Voronoi cell* of $\mathbf{p}_i$. ◇

Given a set $\mathcal{Q} \subset \mathbb{R}^2$, let $\mathbf{q}_k = [q_{kx}, q_{ky}]^T$, $k \in \{1, \dots, N\}$ be the k-th of N *particles* describing the occurrence of some event of interest in $\mathcal{Q}$ and evolving over time according to,

$$\dot{\mathbf{q}} = \boldsymbol{\Upsilon}(\mathbf{q}, t), \tag{6}$$

where $\mathbf{q} = [\mathbf{q}_1^T, \dots, \mathbf{q}_N^T]^T$ and $\boldsymbol{\Upsilon} = [\boldsymbol{\Upsilon}_1^T, \dots, \boldsymbol{\Upsilon}_N^T]^T : \mathcal{Q}^N \times \mathbb{R}_{\geq 0} \to (\mathbb{R}^2)^N$ is a vector field *unknown* to the agents. Since each vehicle is assumed to be equipped with a limited-footprint sensor (e.g., a camera pointing downward), it will be able to measure the x-,y-coordinates of only a subset of the N particles. For the sake of simplicity, in this paper we will abstract from any actual sensor model and assume that agent i processes only the particles lying within the Voronoi cell V_i that it generates, while ignoring those possibly located outside (see Fig. 3): in other words, agent i is only responsible for the particles over its "dominance region" V_i. Note that since the sides $V_i \cap V_j$, $i \neq j$, of the Voronoi cells are of measure zero, each particle will be assigned exactly to one agent, thus avoiding possible double countings. We will denote by N_i, $0 < N_i < N$, the number of particles lying within V_i (note that by construction $\sum_{i=1}^n N_i = N$), and we will assume that agent i computes the following vector from the N_i particles:

$$\mathbf{h}_i = \sum_{\mathbf{q}_k :\, \mathbf{q}_k \in V_i} \phi(\mathbf{q}_k), \quad i \in \{1, \dots, n\}.$$

Note that the Voronoi cells $V_1, \dots, V_n$ can be *locally* computed and maintained by the agents using the distributed asynchronous algorithms presented in [16, Sect. IVB].

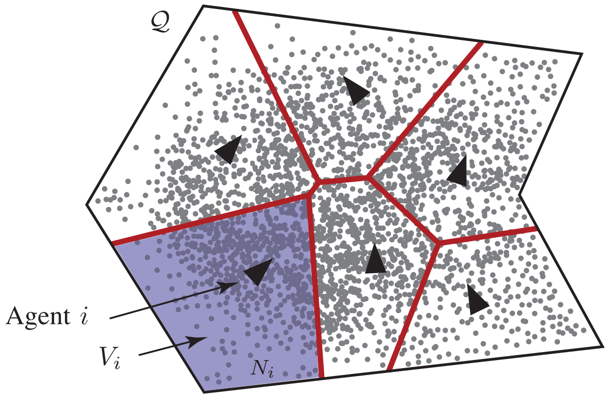

Fig. 3. Voronoi partition of the set $\mathcal{Q}$ generated by six agents (black triangles). Agent i processes only N_i of the overall N particles (gray dots): these are the particles lying within the Voronoi cell V_i of $\mathbf{p}_i$ (purple).

We recall here that our ultimate goal is to match the geometric moments of the swarm with those of the ensemble of particles: this in turn guarantees a suitable coverage of the region spanned by the particles. In order to obtain local estimates of the function $\mathbf{f}(\mathbf{p})$, necessary for a distributed implementation of control (5) (recall Remark 2) and of the *environmental goal vector*,

$$\mathbf{f}^{\star}_{\text{env}} \triangleq \frac{1}{N}\sum_{k=1}^{N}\boldsymbol{\phi}(\mathbf{q}_k) = \frac{1}{N}\sum_{i=1}^{n}\mathbf{h}_i,$$

obtained from the *overall* particles and thus unknown to the vehicles, we will suppose that agent i runs the following *proportional-integral (PI) average consensus estimator* [7], [8]:

$$\begin{aligned}\dot{\boldsymbol{\xi}}_i &= -\gamma\,\boldsymbol{\xi}_i - \sum_{j\neq i}\sigma(\mathbf{p}_i,\mathbf{p}_j)\,(\boldsymbol{\xi}_i-\boldsymbol{\xi}_j)\\ &\quad + \sum_{j\neq i}\tau(\mathbf{p}_i,\mathbf{p}_j)\,(\boldsymbol{\eta}_i-\boldsymbol{\eta}_j) + \gamma\,[\boldsymbol{\phi}(\mathbf{p}_i)^T,\, \mathbf{h}_i^T,\, N_i]^T, \quad (7)\\ \dot{\boldsymbol{\eta}}_i &= -\sum_{j\neq i}\tau(\mathbf{p}_i,\mathbf{p}_j)\,(\boldsymbol{\xi}_i-\boldsymbol{\xi}_j),\\ \boldsymbol{\chi}_i &= \boldsymbol{\xi}_i[1:\ell] - \frac{\boldsymbol{\xi}_i[\ell+1:2\ell]}{\boldsymbol{\xi}_i[2\ell+1]}, \quad (8)\end{aligned}$$

where $[\boldsymbol{\phi}(\mathbf{p}_i)^T,\, \mathbf{h}_i^T,\, N_i]^T \in \mathbb{R}^{2\ell}\times\mathbb{Z}_{>0}$, $i\in\{1,\ldots,n\}$, is agent i's vector input, $\boldsymbol{\xi}_i$ is agent i's estimate of the average of all the agents' input, $\boldsymbol{\eta}_i$ is the internal estimator state, $\gamma>0$ is a global *forgetting factor* governing the rate at which new information replaces old information in the dynamic averaging process, and $\sigma,\,\tau : \mathbb{R}^2\times\mathbb{R}^2 \to \mathbb{R}_{\geq 0}$ are C^1 bounded symmetric gain functions (i.e., $\sigma(\mathbf{p}_i,\,\mathbf{p}_j) = \sigma(\mathbf{p}_j,\,\mathbf{p}_i)$ and $\tau(\mathbf{p}_i,\,\mathbf{p}_j) = \tau(\mathbf{p}_j,\,\mathbf{p}_i)$, $\forall\,\mathbf{p}_i,\mathbf{p}_j$ with $i\neq j$), such that $\sigma(\mathbf{p}_i,\mathbf{p}_j)$ and $\tau(\mathbf{p}_i,\mathbf{p}_j)$ are different from zero only if agents i and j can communicate with each other. We also suppose that $\tau(\cdot,\cdot)$ has bounded first-order partial derivatives. Vector $\boldsymbol{\chi}_i\in\mathbb{R}^{\ell}$ is the *output* of the PI estimator and $\boldsymbol{\xi}_i[1:\ell]$ denotes the vector consisting of the first ℓ components of $\boldsymbol{\xi}_i$. Note that the last entry of the input vector of the PI estimator is necessary for a correct estimation of $\mathbf{f}^{\star}_{\text{env}}$: in fact the *overall* number of particles N is unknown to the agents (in fact, agent i has only knowledge of the number N_i of particles that it processes).

We henceforth assume that each agent is able to measure its pose $[\mathbf{p}_i^T,\,\theta_i]^T$, and that agents i and j can communicate with each other if and only if $\|\mathbf{p}_i-\mathbf{p}_j\|\leq R$, where $R>0$ represents a fixed communication radius (agent j is then said a *neighbor* of agent i). Each configuration $\mathbf{p}\in(\mathbb{R}^2)^n$ then defines the graph of an underlying communication network and we will use $\mathfrak{C}\subset(\mathbb{R}^2)^n$ to denote the set of all such configurations for which this graph is connected. As the agents move with time, the topology of the network can change, but we will assume that $\mathbf{p}(t)\in\mathfrak{C}$, i.e., that the network remains connected in forward time. Each agent transmits its estimate $\boldsymbol{\xi}_i$ and its internal estimator state $\boldsymbol{\eta}_i$ to its neighbors in the network. Each $\boldsymbol{\xi}_i$ will approximately track the true average of the inputs $[\boldsymbol{\phi}(\mathbf{p}_i)^T,\,\mathbf{h}_i^T,\,N_i]^T$, $i\in\{1,\ldots,n\}$. If the input and the topology of the (connected) network were *ideally constant*, each $\boldsymbol{\xi}_i$ would exactly converge to $\frac{1}{n}\sum_{i=1}^{n}[\boldsymbol{\phi}(\mathbf{p}_i)^T,\,\mathbf{h}_i^T,\,N_i]^T = [\mathbf{f}(\mathbf{p})^T,\,\frac{1}{n}\sum_{k=1}^{N}\boldsymbol{\phi}(\mathbf{q}_k)^T,\,N/n]^T$ and the output $\boldsymbol{\chi}_i$ of the PI estimator would converge exactly to $\mathbf{f}(\mathbf{p})-\mathbf{f}^{\star}_{\text{env}}$. However, it has been shown in [8, Th. 3] that the network of n PI estimators (7) is *input-to-state stable*: hence, even if the inputs and the topology of the network are arbitrary fast time-varying, a bound on the norm of the input implies a bound on the norm of the estimation error.

We define the *proportional Laplacian* $\mathbf{L}_P(\mathbf{p})\in\mathbb{R}^{n\times n}$ to be the symmetric matrix whose off-diagonal elements in row i, column j are equal to $-\sigma(\mathbf{p}_i,\mathbf{p}_j)$ and whose diagonal elements are such that $\mathbf{L}_P(\mathbf{p})\mathbb{1} = \mathbf{0}_n$, where $\mathbb{1}$ denotes the vector of n ones and $\mathbf{0}_n$ the vector of n zeros. The *integral Laplacian* $\mathbf{L}_I(\mathbf{p})\in\mathbb{R}^{n\times n}$ is defined in an analogous way, but using function $\tau(\cdot,\cdot)$ instead of $\sigma(\cdot,\cdot)$. Let Orth($\mathbb{1}$) denote the collection of $n\times(n-1)$ matrices $\mathbf{S}$ such that $\mathbf{S}^T\mathbf{S} = \mathbf{I}_{n-1}$, where $\mathbf{I}_{n-1}$ is the $(n-1)\times(n-1)$ identity matrix, and $\mathbf{S}^T\mathbb{1} = \mathbf{0}_{n-1}$. Then, by orthogonal decomposition,

$$\mathbf{I}_n = \mathbf{S}\,\mathbf{S}^T + \frac{1}{n}\mathbb{1}\mathbb{1}^T, \quad (9)$$

and thus $\|\mathbf{A}\mathbf{S}\|_F \leq \|\mathbf{A}\|_F$ for any n-column real matrix $\mathbf{A}$, where $\|\cdot\|_F$ denotes the Frobenius norm. Fixing some $\mathbf{S}\in$ Orth($\mathbb{1}$), we finally define the *reduced proportional and integral Laplacians* to be the $(n-1)\times(n-1)$ symmetric matrices $\mathbf{L}^{\star}_P(\mathbf{p}) \triangleq \mathbf{S}^T\mathbf{L}^{\star}_P(\mathbf{p})\mathbf{S}$ and $\mathbf{L}^{\star}_I(\mathbf{p}) \triangleq \mathbf{S}^T\mathbf{L}^{\star}_I(\mathbf{p})\mathbf{S}$, respectively. We finally assume that there exist constants $\varrho>-\gamma$ and $\vartheta>0$, such that,

$$\varrho\,\mathbf{I}_{n-1}\leq\mathbf{L}^{\star}_P(\mathbf{p})\leq\bar{\varrho}\,\mathbf{I}_{n-1},\ \ \vartheta\,\mathbf{I}_{n-1}\leq\mathbf{L}^{\star}_I(\mathbf{p})\leq\bar{\vartheta}\,\mathbf{I}_{n-1}, \quad (10)$$

along trajectories in forward time (this implies a connected network $\mathbf{p}(t)\in\mathfrak{C}$). The constants $\bar{\varrho},\,\bar{\vartheta}>0$ represent upper bounds on the reduced Laplacians, which exist since the functions $\sigma(\cdot,\cdot)$ and $\tau(\cdot,\cdot)$ are bounded.

IV. Closed-loop stability analysis

In this section we study the closed-loop behavior of the system arising from the combination of the gradient-descent controllers (5) and the PI estimators (7)-(8). Following [7], we will assume that the maximum diameter of a connected swarm of n agents,

$$\mathfrak{d}(n) \triangleq \sup_{\mathbf{p}\,\in\,\mathfrak{C}\,\cap\,(\mathbb{R}^2)^n}\ \max_{i,j\,\in\,\{1,\ldots,n\}}\|\mathbf{p}_i-\mathbf{p}_j\|, \quad (11)$$

is finite for every n. It follows from (11) that there exists a class-$\mathcal{K}$ function $\mathfrak{a}$ and a C^1 function $\varpi : \mathbb{R}^2 \to \mathbb{R}_{\geq 0}$

such that

$$\|[\boldsymbol{\phi}(\mathbf{p}_i)^T, \mathbf{h}_i^T, N_i]^T - [\boldsymbol{\phi}(\mathbf{p}_j)^T, \mathbf{h}_j^T, N_j]^T\|^2 \leq \mathfrak{a}(\mathfrak{d}(n))\varpi(\mathbf{p}_i), \quad (12)$$

for every $\mathbf{p} \in \mathfrak{C}$ and every $i, j \in \{1, \ldots, n\}$. Let us also assume that the following inequality holds,

$$\lambda_{\max}(\text{blkdiag}(\boldsymbol{\Gamma}, \mathbf{B})) < 2\,\delta_1, \quad (13)$$

where $\lambda_{\max}(\text{blkdiag}(\boldsymbol{\Gamma}, \mathbf{B}))$ denotes the largest eigenvalue of the block diagonal matrix $\text{blkdiag}(\boldsymbol{\Gamma}, \mathbf{B})$, and $\mathbf{B}$ is a certain $(\ell+1) \times (\ell+1)$ invertible symmetric matrix (see the proof of Th. 2 for more details). $\delta_1 > 0$ is a scalar constant depending on n, ϱ, $\bar{\varrho}$, ϑ, $\bar{\vartheta}$, γ and the bounds on the partial derivatives of $\tau(\cdot, \cdot)$ (the exact dependencies are given in the proof). Note that inequality (13) represents a *small-gain condition*: in fact, for given estimator gains which determine δ_1, the control gain $\boldsymbol{\Gamma}$ (as well as the matrix $\mathbf{B}$) should be sufficiently small. Given a closed set of swarm configurations $\mathcal{P} \subset (\mathbb{R}^2)^n$ and $\mathbf{f}^\star_{\text{env}}(t) \in \mathbf{f}(\mathcal{P})$, $\forall\, t \geq 0$, let $\bigcap_{t\geq 0} \mathcal{G}(\mathbf{f}^\star_{\text{env}}(t), \mathcal{P})$ denote the convex cone of all symmetric positive-definite matrices $\boldsymbol{\Gamma}$ such that no bad critical points of the potential function,

$$\Pi_{\text{env}}(\mathbf{p}) \triangleq (\mathbf{f}(\mathbf{p}) - \mathbf{f}^\star_{\text{env}})^T\, \boldsymbol{\Gamma}\, (\mathbf{f}(\mathbf{p}) - \mathbf{f}^\star_{\text{env}}),$$

in $\mathcal{P}$ are local minima of Π_{env}.

***Theorem** 2 (Closed-loop stability):* Let us suppose there exists a closed set $\mathcal{P} \subset (\mathbb{R}^2)^n$ such that $\boldsymbol{\Gamma} \in \bigcap_{t\geq 0} \mathcal{G}(\mathbf{f}^\star_{\text{env}}(t), \mathcal{P})$ and $\mathbf{p}(t) \in \mathcal{P}$, $\forall\, t \geq 0$. Suppose that $n \geq 3$ is fixed, that (10) holds for some $\varrho > -\gamma$ and $\vartheta > 0$ (with $\gamma > 0$), and that (13) is satisfied. Let the control input of agent i be of the form (5), with $\alpha_i(t) \triangleq \text{proj}(\arg(\mathbf{g}_i(t)) - \theta_i(t))$ and

$$\mathbf{g}_i(t) = -(\mathcal{J}\boldsymbol{\phi}(\mathbf{p}_i(t)))^T\, \boldsymbol{\Gamma}\, \boldsymbol{\chi}_i(t). \quad (14)$$

Let us finally suppose that the evolution of the N particles is governed by (6) and that $\|\boldsymbol{\Upsilon}_k(\mathbf{q}, t)\|$, $\forall\, k \in \{1, \ldots, N\}$, is sufficiently smaller than v. Then, for almost every initial configuration of the agents, each trajectory of the system (1), (5), (7)-(8), (14) is bounded in forward time. Moreover, for any $\epsilon > 0$, there exists a sufficiently large gain ρ on the angular control such that the error on the desired swarm configuration is uniformly ultimately bounded with an ultimate bound ϵ.

Proof: The proof follows the same lines as that of [7, Th. 4]. Let $\mathbf{e}_i \triangleq \big[\mathbf{f}(\mathbf{p})^T, \frac{1}{n}\sum_{k=1}^N \boldsymbol{\phi}(\mathbf{q}_k)^T, N/n\big]^T - \boldsymbol{\xi}_i$, $i \in \{1, \ldots, n\}$ be the estimation error and let $\boldsymbol{\zeta}_i \triangleq \frac{d}{dt}[\boldsymbol{\phi}(\mathbf{p}_i)^T, \mathbf{h}_i^T, N_i]^T$. Let us choose $\text{V} = n\,\Pi_{\text{env}}(\mathbf{p})$ as storage function: its time derivative along the system's trajectories is given by:

$$\begin{aligned}
\dot{\text{V}} &= 2\,(\mathbf{f}(\mathbf{p}) - \mathbf{f}^\star_{\text{env}})^T\, \boldsymbol{\Gamma}\, \Big(v\sum_{i=1}^n \mathcal{J}\boldsymbol{\phi}(\mathbf{p}_i)\,[\,\cos\theta_i, \sin\theta_i\,]^T \\
&- \frac{n}{N}\sum_{k=1}^N \mathcal{J}\boldsymbol{\phi}(\mathbf{q}_k)\,\boldsymbol{\Upsilon}_k(\mathbf{q}, t)\Big) = -2\,v\sum_{i=1}^n \|\mathbf{g}_i\|\cos\alpha_i \\
&+ 2\,v\sum_{i=1}^n [\,\cos\theta_i, \sin\theta_i\,]\,\mathcal{J}\boldsymbol{\phi}(\mathbf{p}_i)^T\,\boldsymbol{\Gamma}\,\boldsymbol{\varepsilon}_i \\
&- \frac{2\,n}{N}\,(\mathbf{f}(\mathbf{p}) - \mathbf{f}^\star_{\text{env}})^T\, \boldsymbol{\Gamma} \sum_{k=1}^N \mathcal{J}\boldsymbol{\phi}(\mathbf{q}_k)\,\boldsymbol{\Upsilon}_k(\mathbf{q}, t), \quad (15)
\end{aligned}$$

where $\boldsymbol{\varepsilon}_i \triangleq (\mathbf{f}(\mathbf{p}) - \mathbf{f}^\star_{\text{env}}) - \boldsymbol{\chi}_i$ (the output estimation error) and $\alpha_i \triangleq \text{proj}(\arg(\mathbf{g}_i) - \theta_i)$ with $\mathbf{g}_i$ defined in (14). We will use equ. (15) later in the proof. Let us now focus on the estimation part and let us introduce the $(2\ell+1) \times n$ matrices $\boldsymbol{\Xi} \triangleq [\boldsymbol{\xi}_1, \boldsymbol{\xi}_2, \ldots, \boldsymbol{\xi}_n]$, $\mathbf{W} \triangleq [\boldsymbol{\eta}_1, \boldsymbol{\eta}_2, \ldots, \boldsymbol{\eta}_n]$,

$$\mathbf{U} \triangleq \begin{bmatrix} \boldsymbol{\phi}(\mathbf{p}_1) & \boldsymbol{\phi}(\mathbf{p}_2) & \ldots & \boldsymbol{\phi}(\mathbf{p}_n) \\ \mathbf{h}_1 & \mathbf{h}_2 & \ldots & \mathbf{h}_n \\ N_1 & N_2 & \ldots & N_n \end{bmatrix}.$$

We can then write the collection of PI estimators (7)-(8) as:

$$\dot{\boldsymbol{\Xi}} = -\boldsymbol{\Xi}(\gamma\,\mathbf{I} + \mathbf{L}_P(\mathbf{p})) + \mathbf{W}\mathbf{L}_I(\mathbf{p}) + \gamma\,\mathbf{U},\ \dot{\mathbf{W}} = -\boldsymbol{\Xi}\mathbf{L}_I(\mathbf{p}). \quad (16)$$

Let us now introduce the aggregate estimation error $\mathbf{E} = [\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n] \triangleq \mathbf{U}\,\frac{\mathbb{1}\mathbb{1}^T}{n} - \boldsymbol{\Xi}$. We thus obtain,

$$\dot{\mathbf{E}}\mathbb{1} = -\,\gamma\,\mathbf{E}\mathbb{1} + \dot{\mathbf{U}}\mathbb{1}, \quad \dot{\mathbf{W}}\mathbb{1} = \mathbf{0}. \quad (17)$$

Note that by definition $\dot{\mathbf{U}} = [\boldsymbol{\zeta}_1, \boldsymbol{\zeta}_2, \ldots, \boldsymbol{\zeta}_n]$. From (9), we have $\mathbf{L}_P = \mathbf{L}_P\mathbf{S}\mathbf{S}^T$ and $\mathbf{L}_I = \mathbf{L}_I\mathbf{S}\mathbf{S}^T$, which means that we can post-multiply both sides of (16) by $\mathbf{S}$, to obtain,

$$\begin{aligned}
&\dot{\boldsymbol{\Xi}}\,\mathbf{S} = -\boldsymbol{\Xi}\,\mathbf{S}(\gamma\,\mathbf{I} + \mathbf{L}^\star_P(\mathbf{p})) + \mathbf{W}\,\mathbf{S}\,\mathbf{L}^\star_I(\mathbf{p}) + \gamma\,\mathbf{U}\,\mathbf{S}, \\
&\dot{\mathbf{W}}\,\mathbf{S} = -\boldsymbol{\Xi}\,\mathbf{S}\,\mathbf{L}^\star_I(\mathbf{p}).
\end{aligned} \quad (18)$$

Using the change of variables $\mathbf{H} = \mathbf{W}\,\mathbf{S} + \gamma\mathbf{U}\,\mathbf{S}\,[\mathbf{L}^\star_I(\mathbf{p})]^{-1}$, $\boldsymbol{\Omega} = [\boldsymbol{\Xi}\,\mathbf{S} \;\; \mathbf{H}]$, equ. (18) becomes

$$\dot{\boldsymbol{\Omega}} = \boldsymbol{\Omega}\,\mathbf{F}^T + \mathbf{N}\,\mathbf{G}^T, \quad (19)$$

where $\mathbf{F} = \begin{bmatrix} -\gamma\,\mathbf{I} - \mathbf{L}^\star_P(\mathbf{p}) & \mathbf{L}^\star_I(\mathbf{p}) \\ -\mathbf{L}^\star_I(\mathbf{p}) & \mathbf{0} \end{bmatrix}$, $\mathbf{G} = [\mathbf{0} \;\; \mathbf{I}]^T$, $\mathbf{N} = \gamma\,\dot{\mathbf{U}}\,\mathbf{S}\,[\mathbf{L}^\star_I(\mathbf{p})]^{-1} + \gamma\,\mathbf{U}\,\mathbf{S}\frac{d}{dt}[\mathbf{L}^\star_I(\mathbf{p})]^{-1}$. We will write $\mathbf{N}$ as,

$$\mathbf{N} = \gamma\sum_{i=0}^n \mathbf{N}_i, \quad (20)$$

where $\mathbf{N}_0 = \dot{\mathbf{U}}\,\mathbf{S}\,[\mathbf{L}^\star_I(\mathbf{p})]^{-1}$,

$$\begin{aligned}
\mathbf{N}_i &= -v\,\mathbf{U}\,\mathbf{S}\,[\mathbf{L}^\star_I(\mathbf{p})]^{-1}\Big(\frac{\partial\,\mathbf{L}^\star_I(\mathbf{p})}{\partial\,p_{ix}}\,[\mathbf{L}^\star_I(\mathbf{p})]^{-1}\cos\theta_i \\
&+ \frac{\partial\,\mathbf{L}^\star_I(\mathbf{p})}{\partial\,p_{iy}}\,[\mathbf{L}^\star_I(\mathbf{p})]^{-1}\sin\theta_i\Big),\ i \in \{1, \ldots, n\}.
\end{aligned}$$

We now derive bounds on these matrices $\mathbf{N}_i$. First, using the second condition in (10) we obtain,

$$\mathbf{N}_0\,\mathbf{N}_0^T = \dot{\mathbf{U}}\,\mathbf{S}\,[\mathbf{L}^\star_I(\mathbf{p})]^{-2}\,\mathbf{S}^T\,\dot{\mathbf{U}}^T \leq \tfrac{1}{\vartheta^2}\,\dot{\mathbf{U}}\,\mathbf{S}\,\mathbf{S}^T\,\dot{\mathbf{U}}^T. \quad (21)$$

Next, using (12) and the assumption that $\mathbf{p}(t) \in \mathfrak{C}$, $\forall\, t \geq 0$, we get,

$$\begin{aligned}
&\|\mathbf{U}\,\mathbf{S}\|_F^2 = \|(\mathbf{U} - [\boldsymbol{\phi}(\mathbf{p}_i)^T, \mathbf{h}_i^T, N_i]^T\,\mathbb{1}^T)\,\mathbf{S}\|_F^2 \\
&\leq \|\mathbf{U} - [\boldsymbol{\phi}(\mathbf{p}_i)^T, \mathbf{h}_i^T, N_i]^T\mathbb{1}^T\|_F^2 \leq \sum_{j\neq i}\big\|\big[\boldsymbol{\phi}(\mathbf{p}_i)^T, \mathbf{h}_i^T, N_i\big]^T \\
&- \big[\boldsymbol{\phi}(\mathbf{p}_j)^T, \mathbf{h}_j^T, N_j\big]^T\big\|^2 \leq (n-1)\,\mathfrak{a}(\mathfrak{d}(n))\,\varpi(\mathbf{p}_i).
\end{aligned}$$

It follows from the second condition in (10) and the fact that τ has bounded partial derivatives, that there exists a constant $\kappa > 0$ such that $\|\mathbf{N}_i\|_F^2 \leq \kappa\,v^2\,\varpi(\mathbf{p}_i)$, $i \in \{1, 2, \ldots, n\}$, where the constant κ depends on n, ϑ, and the bounds on the partial derivatives of τ. Let $\lambda \in (0, 1)$ be such that

$$\lambda \leq \frac{\vartheta(\gamma + \varrho)}{(\gamma + \bar{\varrho}^2) + 2\vartheta\bar{\vartheta}}. \quad (22)$$

Then the positive-definite matrices $\mathbf{P} = \begin{bmatrix} \mathbf{I} & -\lambda\,\mathbf{I} \\ -\lambda\,\mathbf{I} & \mathbf{I} \end{bmatrix}$, $\mathbf{Q} =$

$\begin{bmatrix} (\gamma+\varrho)\,\mathbf{I} & \mathbf{0} \\ \mathbf{0} & \lambda\,\vartheta\,\mathbf{I} \end{bmatrix}$, satisfy $(1-\lambda)\mathbf{I} \le \mathbf{P} \le (1+\lambda)\mathbf{I}$ and

$$
\begin{aligned}
&\mathbf{P}\,\mathbf{F} + \mathbf{F}^T\mathbf{P} + \mathbf{Q} \\
&= \begin{bmatrix} -2\mathbf{L}_P^\star(\mathbf{p}) + (\varrho-\gamma)\mathbf{I} + 2\lambda\mathbf{L}_I^\star(\mathbf{p}) & \lambda(\gamma\,\mathbf{I} + \mathbf{L}_P^\star(\mathbf{p})) \\ \lambda(\gamma\mathbf{I} + \mathbf{L}_P^\star(\mathbf{p})) & \lambda(-2\mathbf{L}_I^\star(\mathbf{p}) + \vartheta\mathbf{I}) \end{bmatrix} \\
&\le -\lambda \underbrace{\begin{bmatrix} (\frac{1}{\lambda}(\gamma+\varrho) - 2\bar{\vartheta})\mathbf{I} & -\gamma\mathbf{I} - \mathbf{L}_P^\star(\mathbf{p}) \\ -\gamma\mathbf{I} - \mathbf{L}_P^\star(\mathbf{p}) & \vartheta\mathbf{I} \end{bmatrix}}_{\mathbf{R}(\mathbf{p})} \le \mathbf{0},
\end{aligned}
$$

because (22) implies that $\mathbf{R}(\mathbf{p}) \ge \mathbf{0}$. Let $\varkappa > 0$ be such that $\varkappa < \min\{\frac{\gamma+\varrho}{\lambda},\, \lambda\vartheta\}$. Then we have $\mathbf{P}\,\mathbf{G}\,\mathbf{G}^T\mathbf{P} = \mathbf{P} - \begin{bmatrix} (1-\lambda^2)\mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}$ and

$$
\begin{aligned}
&\mathbf{Q} - \tfrac{\varkappa}{1+\lambda}\mathbf{P}\,\mathbf{G}\,\mathbf{G}^T\mathbf{P} = \begin{bmatrix} (\gamma+\varrho+\varkappa(1-\lambda))\mathbf{I} & \mathbf{0} \\ \mathbf{0} & \lambda\,\vartheta\,\mathbf{I} \end{bmatrix} - \tfrac{\varkappa}{1+\lambda}\mathbf{P} \\
&\ge \min\{\gamma+\varrho+\varkappa(1-\lambda),\, \lambda\vartheta\}\mathbf{I} - \varkappa\mathbf{I} = \overline{\alpha}\,\mathbf{I},
\end{aligned}
$$

where $\overline{\alpha} = \min\{\gamma+\varrho-\lambda\varkappa,\, \lambda\vartheta-\varkappa\}$. It now follows that:

$$
\mathbf{P}\,\mathbf{F} + \mathbf{F}^T\mathbf{P} + \tfrac{\varkappa}{1+\lambda}\,\mathbf{P}\,\mathbf{G}\,\mathbf{G}^T\mathbf{P} \le -\overline{\alpha}\,\mathbf{I}. \tag{23}
$$

Let us introduce the matrix $\boldsymbol{\Psi} = \boldsymbol{\Omega}\,\mathbf{P}\,\boldsymbol{\Omega}^T + \beta\,\mathbf{E}\,\mathbb{1}\mathbb{1}^T\,\mathbf{E}^T + \mathbf{W}\,\mathbb{1}\mathbb{1}^T\,\mathbf{W}^T$ where $\beta > 0$ is a constant parameter. Defining $\varsigma = \frac{\gamma^2(n+1)(\lambda+1)}{\varkappa}$, we use (9), (17), (19), (20), (21) and (23) to obtain,

$$
\begin{aligned}
&\dot{\boldsymbol{\Psi}} = \boldsymbol{\Omega}[\mathbf{P}\,\mathbf{F} + \mathbf{F}^T\mathbf{P}]\boldsymbol{\Omega}^T + \gamma\sum_{i=0}^{n}[\mathbf{N}_i\,\mathbf{G}^T\mathbf{P}\,\boldsymbol{\Omega}^T + \boldsymbol{\Omega}\,\mathbf{P}\,\mathbf{G}\,\mathbf{N}_i^T] \\
&- 2\beta\gamma\,\mathbf{E}\mathbb{1}\mathbb{1}^T\,\mathbf{E}^T + \beta\,\dot{\mathbf{U}}\mathbb{1}\mathbb{1}^T\,\mathbf{E}^T + \beta\,\mathbf{E}\mathbb{1}\mathbb{1}^T\dot{\mathbf{U}}^T \\
&\le \boldsymbol{\Omega}\big[\mathbf{P}\,\mathbf{F}^T + \mathbf{F}^T\mathbf{P} + \tfrac{\varkappa}{1+\lambda}\,\mathbf{P}\,\mathbf{G}\,\mathbf{G}^T\mathbf{P}\big]\boldsymbol{\Omega}^T - \beta\gamma\,\mathbf{E}\mathbb{1}\mathbb{1}^T\mathbf{E}^T \\
&+ \varsigma\sum_{i=0}^{n}\mathbf{N}_i\,\mathbf{N}_i^T + \frac{\beta}{\gamma}\,\dot{\mathbf{U}}\,\mathbb{1}\mathbb{1}^T\dot{\mathbf{U}}^T \le -\overline{\alpha}\,\boldsymbol{\Omega}\,\boldsymbol{\Omega}^T - \beta\gamma\,\mathbf{E}\mathbb{1}\mathbb{1}^T\mathbf{E}^T \\
&+ \max\left\{\frac{n\beta}{\gamma},\, \frac{\varsigma}{\vartheta^2}\right\}\dot{\mathbf{U}}\,\dot{\mathbf{U}}^T + \varsigma\sum_{i=1}^{n}\mathbf{N}_i\,\mathbf{N}_i^T.
\end{aligned}
$$

Because $\mathbf{E}\,\mathbf{S} = -\boldsymbol{\Xi}\,\mathbf{S}$ we also have $\boldsymbol{\Omega}\,\boldsymbol{\Omega}^T = \boldsymbol{\Xi}\,\mathbf{S}\,\mathbf{S}^T\boldsymbol{\Xi}^T + \mathbf{H}\,\mathbf{H}^T = \mathbf{E}\,\mathbf{S}\,\mathbf{S}^T\mathbf{E}^T + \mathbf{H}\,\mathbf{H}^T$ and therefore $\dot{\boldsymbol{\Psi}} \le -\nu_1\,\mathbf{E}\,\mathbf{E}^T - \overline{\alpha}\,\mathbf{H}\,\mathbf{H}^T + \nu_2\,\dot{\mathbf{U}}\,\dot{\mathbf{U}}^T + \varsigma\sum_{i=1}^{n}\mathbf{N}_i\,\mathbf{N}_i^T$ where $\nu_1 = \min\{\overline{\alpha},\, n\,\beta\,\gamma\}$ and $\nu_2 = \max\{\frac{n\,\beta}{\gamma},\, \frac{\varsigma}{\vartheta^2}\}$. Defining the storage function $\mathrm{M} = \mathrm{trace}(\boldsymbol{\Psi})$, we see that

$$
\dot{\mathrm{M}} \le -\overline{\alpha}\,\|\mathbf{H}\|_F^2 + \sum_{i=1}^{n}[-\nu_1\|\mathbf{e}_i\|^2 + \nu_2\|\boldsymbol{\zeta}_i\|^2 + \varsigma\,\kappa\,v^2\varpi(\mathbf{p}_i)]. \tag{24}
$$

With reference to equ. (15), note that $2\,v\sum_{i=1}^{n}[\cos\theta_i,\, \sin\theta_i]\,\mathcal{J}\boldsymbol{\phi}(\mathbf{p}_i)^T\,\boldsymbol{\Gamma}\,\boldsymbol{\varepsilon}_i = 2\sum_{i=1}^{n}\boldsymbol{\zeta}_i[1 : \ell]^T\,\boldsymbol{\Gamma}\,\boldsymbol{\varepsilon}_i \le 2\sum_{i=1}^{n}\boldsymbol{\zeta}_i^T\,\mathrm{blkdiag}(\boldsymbol{\Gamma},\mathbf{B})\,\mathbf{e}_i$, for a certain $(\ell+1)\times(\ell+1)$ invertible symmetric matrix $\mathbf{B}$. Let now $\aleph$ be a positive constant such that the inequality $2\,\boldsymbol{\zeta}_i^T\mathrm{blkdiag}(\boldsymbol{\Gamma},\mathbf{B})\,\mathbf{e}_i \le \aleph\,\mathbf{e}_i^T\,\mathrm{blkdiag}(\boldsymbol{\Gamma}^2,\mathbf{B}^2)\,\mathbf{e}_i - \|\boldsymbol{\zeta}_i\|^2$ is satisfied. We can then bound (15) from above as,

$$
\begin{aligned}
\dot{\mathrm{V}} &\le -2\,v\sum_{i=1}^{n}\|\mathbf{g}_i\|\cos\alpha_i + \sum_{i=1}^{n}[\aleph\,\mathbf{e}_i^T\,\mathrm{blkdiag}(\boldsymbol{\Gamma}^2,\mathbf{B}^2)\,\mathbf{e}_i \\
&-\|\boldsymbol{\zeta}_i\|^2] - \frac{2\,n}{N}\,(\mathbf{f}(\mathbf{p}) - \mathbf{f}^\star_{\text{env}})^T\,\boldsymbol{\Gamma}\sum_{k=1}^{N}\mathcal{J}\boldsymbol{\phi}(\mathbf{q}_k)\boldsymbol{\Upsilon}_k(\mathbf{q},\, t).
\end{aligned} \tag{25}
$$

Assume that (13) holds with $\delta_1 = \frac{1}{2\,\aleph}\sqrt{\frac{\nu_1}{\nu_2}}$ and choose $\varphi > 0$ so that

$$
\mathrm{blkdiag}(\boldsymbol{\Gamma},\mathbf{B}) < \tfrac{\varphi\,\nu_1}{2\,\delta_1}\,\mathbf{I} < 2\,\delta_1\mathbf{I}. \tag{26}
$$

Upon taking inverses and then multiplying by $\mathrm{blkdiag}(\boldsymbol{\Gamma},\mathbf{B})$ from the left and the right, we obtain,

$$
\tfrac{1}{2\,\delta_1}\,\mathrm{blkdiag}(\boldsymbol{\Gamma}^2,\mathbf{B}^2) < \mathrm{blkdiag}(\boldsymbol{\Gamma},\mathbf{B}). \tag{27}
$$

In particular, (26) and (27) imply the existence of a scalar constant $\bar{\nu} > 0$ such that $(1-\varphi\,\nu_2)\,\mathbf{I} \ge \bar{\nu}\,\mathbf{I}$, $\varphi\,\nu_1\mathbf{I} - \aleph\,\mathrm{blkdiag}(\boldsymbol{\Gamma}^2,\mathbf{B}^2) \ge \bar{\nu}\,\mathbf{I}$. We then define the combined storage function $\mathrm{Z}(\mathbf{p},\mathbf{E},\mathbf{W}) = \mathrm{V} + \varphi\,\mathrm{M}$ and use (24), (25), and the two previous inequalities to obtain,

$$
\begin{aligned}
\dot{\mathrm{Z}} &\le -\overline{\alpha}\,\varphi\,\|\mathbf{H}\|_F^2 + \sum_{i=1}^{n}[-2\,v\,\|\mathbf{g}_i\|\cos\alpha_i - \bar{\nu}\|\mathbf{e}_i\|^2 - \bar{\nu}\|\boldsymbol{\zeta}_i\|^2 \\
&+ \varsigma\varphi\kappa v^2\varpi(\mathbf{p}_i)] - \frac{2\,n}{N}(\mathbf{f}(\mathbf{p}) - \mathbf{f}^\star_{\text{env}})^T\,\boldsymbol{\Gamma}\sum_{k=1}^{N}\mathcal{J}\boldsymbol{\phi}(\mathbf{q}_k)\boldsymbol{\Upsilon}_k(\mathbf{q},t).
\end{aligned} \tag{28}
$$

If we now use our assumption on the norm of the vector field $\boldsymbol{\Upsilon}_k(\mathbf{q},\, t)$, $k \in \{1,\ldots,N\}$ and observe that we have some freedom in the choice of the C^1 function $\varpi(\mathbf{p}_i)$ (recall inequality (12)), from (28) we can conclude that for almost every initial configuration of the agents, each trajectory of the swarm system (1), (5), (14), (7)-(8) is bounded in forward time. Furthermore, by acting on the angular control gain ρ, it is possible to make the error on the desired swarm configuration arbitrarily small. ■

V. Simulation results

Simulation experiments have been conducted to illustrate the proposed theory. Fig. 4 shows the closed-loop behavior of the gradient-descent controllers and PI estimators for a swarm of $n = 4$ agents. The plots were generated using $v =$ 1 m/s, $\rho = 3$ and $\boldsymbol{\Gamma} = \mathrm{diag}(100, 100, 0.1, 0.1, 0.1)$. As far as the PI estimators are concerned, we set $\gamma = 7$ and chose the gain functions according to an equal weighting scheme with a communication radius $R = 27$ m: $\sigma(\mathbf{p}_i,\mathbf{p}_j) = 25$ and $\tau(\mathbf{p}_i,\mathbf{p}_j) = 0.8$ when $\|\mathbf{p}_i - \mathbf{p}_j\| \le R$ and $\sigma(\mathbf{p}_i,\mathbf{p}_j) = \tau(\mathbf{p}_i,\mathbf{p}_j) = 0$, otherwise. The PI estimators have been initialized with $\boldsymbol{\xi}_i(0) = [0,\, 0,\, 80,\, 80, 0, 0, 0, 0, 0, 0, 50]^T$ and $\boldsymbol{\eta}_i(0) = \mathbf{0}$, for all i. The particles $\mathbf{q}_k$, $k \in \{1,\ldots,N\}$, $N = 200$, have been drawn from a bivariate normal distribution $\mathscr{N}(\boldsymbol{\mu},\,\boldsymbol{\Sigma})$ with mean $\boldsymbol{\mu} = [10,\, 5]^T$ and variance $\boldsymbol{\Sigma} = \begin{bmatrix} 70 & 1 \\ 1 & 70 \end{bmatrix}$, and lie within a rectangular domain $\mathcal{Q}$ with vertices $(-30,\, -22)$, $(32,\, -22)$, $(32,\, 30)$, $(-30,\, 30)$. Fig. 4(a) shows the cluster of particles and the corresponding ellipse of desired geometric moments of the swarm. Fig. 4(b) reports the initial random pose of the four agents and the corresponding ellipse of geometric moments (red). The initial Voronoi partition $\{V_1, V_2, V_3, V_4\}$ of $\mathcal{Q}$ and the graph of the underlying communication network are represented with solid and dashed lines in the figure. Fig. 4(c) shows the trajectory of the four agents and the ellipse of geometric moments of the swarm at the final time instant: note that the red and black ellipses are almost exactly superimposed and the agents rotate around four points corresponding to a minimum of $\Pi_{\text{env}}(\mathbf{p})$. Finally, Fig. 4(d) shows the time history of $\log_{10}\Pi_{\text{env}}(\mathbf{p}(t))$, and Figs. 4(e) and 4(f) the time evolution of $\mathbf{f}^\star_{\text{env}}(t)$ (dashed) and $\mathbf{f}(\mathbf{p}(t))$ (solid), respectively: the symbols CMx, CMy and Ixx, Iyy, Ixy refer to the first- and second-order moment statistics, respectively.

VI. Conclusions and future work

In this paper we have proposed a new estimation and control strategy for distributed monitoring tasks. The geometric moments of a team of UAVs modeled as constant-speed

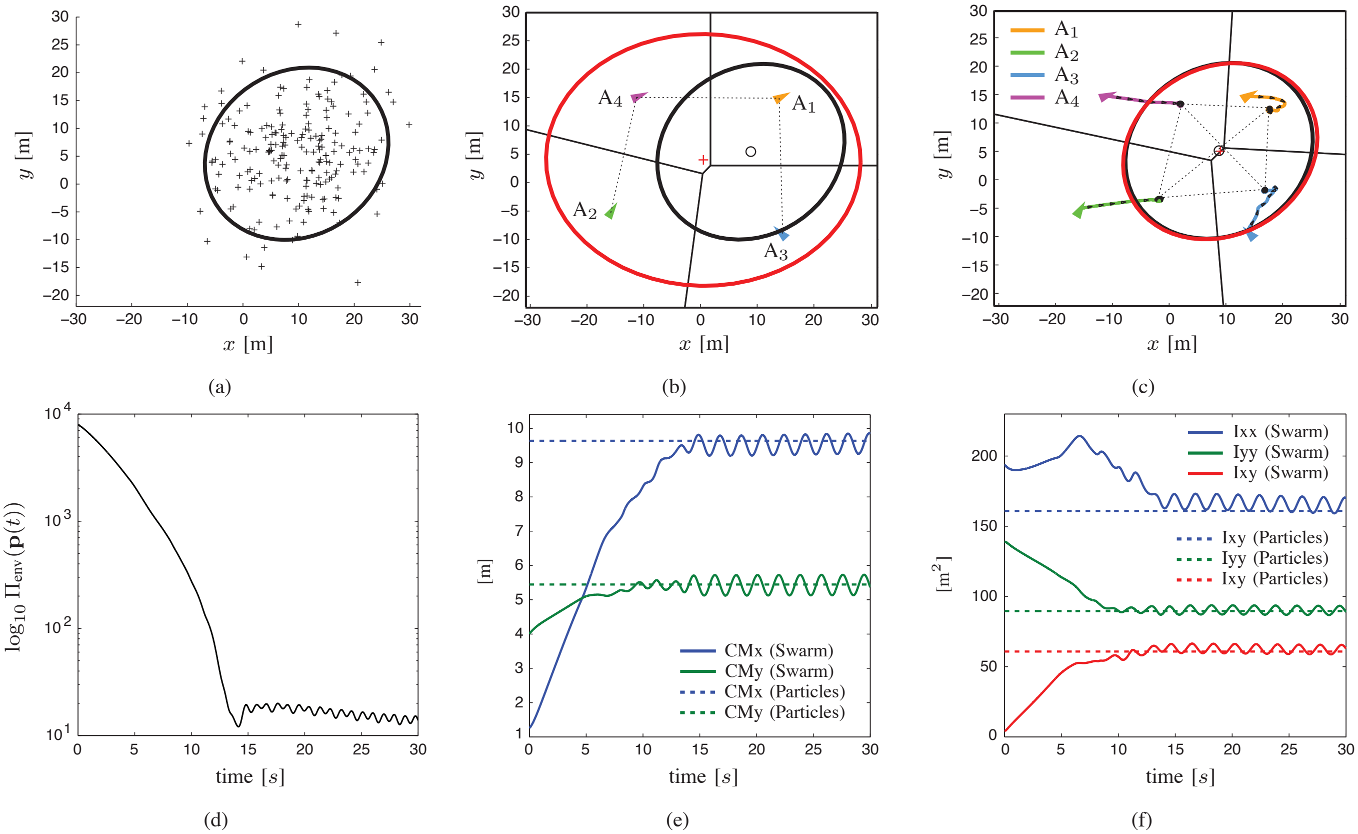

Fig. 4. *Simulation results*: (a) Cluster of $N = 200$ particles and corresponding ellipse of desired geometric moments of the swarm (black); (b) Initial pose of the four agents and corresponding ellipse of geometric moments (red). The initial Voronoi partition of $\mathcal{Q}$ and the graph of underlying communication network are represented with solid and dashed lines; (c) Trajectory of the four agents and ellipse of geometric moments at the final time instant (red); (d) Time history of $\log_{10} \Pi_{\text{env}}(\mathbf{p}(t))$; (e) First-order moments and (f) second-order moments of $\mathbf{f}^{\star}_{\text{env}}(t)$ (dashed, "Particles") and $\mathbf{f}(\mathbf{p}(t))$ (solid, "Swarm").

unicycles are controlled via a nonlinear gradient descent to match those of an ensemble of discrete particles describing the occurrence of some event of interest to be monitored. A PI average consensus estimator is run by each agent to locally estimate the desired moments of the swarm from the environmental data. The closed-loop stability of the system has been studied and simulation results have been presented to support the theoretical analysis. The extension of our strategy to SE(3) and to vehicles with *non-constant* positive forward velocity are subjects of ongoing research. In future investigations, we also aim to use second-order *central moments* in order to have a translation-invariant description of the desired swarm configuration, and plan to test our estimation and control algorithm on field data (e.g., on recorded trajectories of marine oil spills).

References

[1] N.E. Leonard, D.A. Paley, F. Lekien, R. Sepulchre, D.M. Fratantoni, and R.E. Davis. Collective Motion, Sensor Networks, and Ocean Sampling. *Proc. IEEE*, 95(1):48–74, 2007.

[2] R.N Smith, Y. Chao, P.P Li, D.A. Caron, B.H. Jones, and G.S. Sukhatme. Planning and Implementing Trajectories for Autonomous Underwater Vehicles to Track Evolving Ocean Processes Based on Predictions from a Regional Ocean Model. *Int. J. Robot. Res.*, 29(12):1475–1497, 2010.

[3] D. Marthaler and A.L. Bertozzi. Tracking environmental level sets with autonomous vehicles. In S. Butenko, R. Murphey, and P.M. Pardalos, editors, *Recent Developments in Cooperative Control and Optimization*, chapter 17, pages 317–332. Kluwer, 2004.

[4] J. Clark and R. Fierro. Mobile robotic sensors for perimeter detection and tracking. *ISA Trans.*, 46(1):3–13, 2007.

[5] D.W. Casbeer, D.B. Kingston, R.W. Beard, and T.W. Mclain. Cooperative forest fire surveillance using a team of small unmanned air vehicles. *Int. J. Syst. Sci.*, 37(6):351–360, 2006.

[6] S. Susca, F. Bullo, and S. Martínez. Monitoring Environmental Boundaries with a Robotic Sensor Network. *IEEE Trans. Contr. Syst. T.*, 16(2):288–296, 2008.

[7] P. Yang, R.A. Freeman, and K.M. Lynch. Multi-Agent Coordination by Decentralized Estimation and Control. *IEEE Trans. Automat. Contr.*, 53(11):2480–2496, 2008.

[8] K.M. Lynch, I.B. Schwartz, P. Yang, and R.A. Freeman. Decentralized Environmental Modeling by Mobile Sensor Networks. *IEEE Trans. Robot.*, 24(3):710–724, 2008.

[9] J. Choi, S. Oh, and R. Horowitz. Distributed learning and cooperative control for multi-agent systems. *Automatica*, 45(12):2802–2814, 2009.

[10] P. Ögren, E. Fiorelli, and N.E. Leonard. Cooperative control of mobile sensor networks: Adaptive gradient climbing in a distributed environment. *IEEE Trans. Automat. Contr.*, 49(8):1292–1302, 2004.

[11] F. Zhang and N.E. Leonard. Cooperative Filters and Control for Cooperative Exploration. *IEEE Trans. Automat. Contr.*, 55(3):650–663, 2010.

[12] S. Martínez. Distributed interpolation schemes for field estimation by mobile sensor networks. *IEEE Trans. Contr. Syst. T.*, 18(2):491–500, 2010.

[13] C. Belta and V. Kumar. Abstraction and Control for Groups of Robots. *IEEE Trans. Robot. Autom.*, 20(5):865–875, 2004.

[14] N. Michael and V. Kumar. Planning and Control of Ensembles of Robots with Non-holonomic Constraints. *Int. J. Robot. Res.*, 28(8):962–975, 2009.

[15] E. Lalish, K.A. Morgansen, and T. Tsukamaki. Oscillatory Control for Constant-Speed Unicycle-Type Vehicles. In *Proc. 46th IEEE Conf. Dec. Contr*, pages 5246–5251, 2007.

[16] J. Cortés, S. Martínez, T. Karatas, and F. Bullo. Coverage Control for Mobile Sensing Networks. *IEEE Trans. Robot. Autom.*, 20(2):243–255, 2004.

[17] A. Kwok and S. Martínez. Unicycle Coverage Control via Hybrid Modeling. *IEEE Trans. Automat. Contr.*, 55(2):528–532, 2010.

[18] M. Schwager, D. Rus, and J.-J. Slotine. Decentralized, Adaptive Coverage Control for Networked Robots. *Int. J. Robot. Res.*, 28(3):357–375, 2009.

[19] S.H. Strogatz. From Kuramoto to Crawford: exploring the onset of synchronization in populations of coupled oscillators. *Physica D*, 143(1–4):1–20, 2000.