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Containment Control for Multiple Euler-Lagrange Systems with Parametric Uncertainties in Directed Networks

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Abstract— **In this paper, we study the distributed containment control problem for networked Lagrangian systems with multiple stationary or dynamic leaders in the presence of parametric uncertainties under a directed graph that characterizes the interaction among the leaders and the followers. When the leaders are stationary, a distributed adaptive control algorithm is proposed. We present a necessary and sufficient condition on the directed graph such that all followers converge to the stationary convex hull spanned by the stationary leaders asymptotically. As a byproduct, we show a necessary and sufficient condition on leaderless consensus for networked Lagrangian systems under a directed graph. When the leaders are dynamic, two cases are considered: i) The leaders have constant vectors of generalized coordinate derivatives; ii) The leaders have varying vectors of generalized coordinate derivatives. In the first case, we propose a distributed continuous estimator and a distributed adaptive control algorithm. In the second case, we propose a distributed adaptive control algorithm combined with distributed sliding-mode estimators. In both cases, a necessary and sufficient condition on the directed graph is presented such that all followers converge to the dynamic convex hull spanned by the dynamic leaders asymptotically.**

I. INTRODUCTION

Recently, distributed coordination of multi-agent systems has gained much attention due to its broad applications, including consensus, flocking, and formation control. Many existing works in distributed coordination focus on the consensus problem when there is no leader. We refer the readers to [1], [2] and references therein for more details. In reality, the presence of a single leader or multiple leaders can broaden the applications as a group objective can be encapsulated by the leader or the leaders. In the case where there exists one leader, [3] studies the coordinated tracking problem with an active leader under the assumption that the leader's acceleration is known by all followers. In [4], the distributed coordinated tracking and swarm tracking problems are studied in the absence of velocity or acceleration measurements. Distributed sliding-mode estimators are proposed in [5] to solve the finite-time formation tracking problem. In the case where there exist multiple leaders, [6] proposes a distributed containment control algorithm for agents with single-integrator dynamics such that a group of followers is driven to the convex hull spanned by multiple leaders under an undirected graph. The work of [6] is extended in [7] to the case of a directed and switching interaction graph and in [8] to the case of double-integrator dynamics. Note that the above results focus on linear systems with single-integrator or double-integrator dynamics.

A class of mechanical systems including autonomous vehicles, robotic manipulators, and walking robots are Lagrangian systems. Therefore, distributed coordination of networked Lagrangian systems has many applications. Unfortunately, the results for single- and double-integrator dynamics cannot be directly applied to Lagrangian systems due to their inherent nonlinearity. Recent work on coordination of networked Lagrangian systems focuses on the leaderless case [9], [10], the case with a single leader [11]–[13], and the case with multiple leaders [14], [15]. In the leaderless case, a controller based on potential functions is proposed in [9] for networked Lagrangian systems to achieve leaderless flocking. In [10], three distributed leaderless consensus algorithms are proposed for networked Lagrangian systems under an undirected graph. In the case of a single leader, output synchronization of networked Lagrangian systems is studied in [11] under a passivity-based framework. Both fixed and switching graphs as well as communication delays are considered. Based on nonlinear contraction analysis, [12] analyzes the stability of cooperative tracking control laws for multiple robotic manipulators. In [13], the distributed coordinated tracking problem for networked Lagrangian systems is solved in the presence of a dynamic leader, where the leader is a neighbor of only a subset of the followers and the followers have only local interaction. In the case of multiple leaders, [14] studies the distributed attitude containment control problem for multiple rigid bodies with multiple stationary leaders under an undirected graph. In [15], the distributed finite-time containment control problem is studied for networked Lagrangian systems under the assumption that the interaction graph associated with the followers is undirected.

In this paper, we study the distributed containment control problem for networked Lagrangian systems with multiple stationary or dynamic leaders in the presence of parametric uncertainties under a directed graph that characterizes the interaction among the leaders and the followers. The objective is that a team of followers modeled by Euler-Lagrange equations converge to the convex hull spanned by multiple stationary or dynamic leaders. When the leaders are stationary, a distributed adaptive control algorithm is proposed. We present a necessary and sufficient condition on the directed graph such that all followers converge to the stationary convex hull spanned by the stationary leaders asymptotically. As a byproduct, we show a necessary and sufficient condition

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on leaderless consensus for networked Lagrangian systems under a directed graph. When the leaders are dynamic, we consider two cases: i) The leaders have constant vectors of generalized coordinate derivatives; ii) The leaders have varying vectors of generalized coordinate derivatives. In the first case, we propose a distributed continuous estimator and a distributed adaptive control algorithm. A necessary and sufficient condition on the directed graph is presented such that all followers converge to the dynamic convex hull spanned by the dynamic leaders. In the second case, we propose a distributed adaptive control algorithm combined with distributed sliding-mode estimators and present a necessary and sufficient condition on the directed graph.

Comparison with existing work in the literature: In contrast to the containment control algorithms for first- and second-order linear dynamics [6]–[8], we study the nonlinear Lagrangian systems. In contrast to the leaderless case or the case with a single leader for networked Lagrangian systems [9]–[13], we consider the containment control problem with multiple leaders. In contrast to the rigid body attitude containment control problem in [14] and the finite-time containment control problem for networked Lagrangian systems in [15] , we deal with the containment control problem for networked Lagrangian systems in the presence of parametric uncertainties under a directed graph.

Notations: Let $\mathbf{1}_m$ and $\mathbf{0}_m$ denote, respectively, the $m \times 1$ column vector of all ones and all zeros. Let $\mathbf{0}_{m\times n}$ denote the $m \times n$ matrix with all zeros and I_m denote the $m \times m$ identity matrix. For a point x and a set M, let $d(x, M) \triangleq \inf_{y\in M} \|x - y\|$ denote the distance between x and M. Throughout the paper, we use $\|\cdot\|$ to denote the Euclidean norm.

II. Background and Problem Statement

Suppose that there exist m followers, labeled as agents 1 to m, and $n - m$ $(n > m)$ leaders labeled as agents $m + 1$ to n, in a team. The m followers are represented by Euler-Lagrange equations of the form

$$M_i(q_i)\ddot{q}_i + C_i(q_i, \dot{q}_i)\dot{q}_i + g_i(q_i) = \tau_i, \quad i = 1, \cdots, m, \tag{1}$$

where $q_i \in \mathbb{R}^p$ is the vector of generalized coordinates, $M_i(q_i) \in \mathbb{R}^{p\times p}$ is the symmetric positive-definite inertia matrix, $C_i(q_i, \dot{q}_i)\dot{q}_i \in \mathbb{R}^p$ is the vector of Coriolis and centrifugal torques, $g_i(q_i)$ is the vector of gravitational torque, and $\tau_i \in \mathbb{R}^p$ is the vector of control torque on the ith agent.

Throughout the subsequent analysis we assume that the following assumptions hold [16], [17]:

(A1) Parametric Boundedness: For any i, there exist positive constants $k_{\underline{m}}$, $k_{\overline{m}}$, k_C, and k_{g_i} such that $0 < k_{\underline{m}} I_p \leq M_i(q_i) \leq k_{\overline{m}} I_p$, $\|C_i(x, y)z\| \leq k_C\|y\| \, \|z\|$ for all vectors $x, y, z \in \mathbb{R}^p$, and $\|g_i(q_i)\| \leq k_{g_i}$.

(A2) Skew symmetric property: $\dot{M}_i(q_i) - 2C_i(q_i, \dot{q}_i)$ is skew symmetric.

(A3) Linearity in the dynamic parameters: $M_i(q_i)x + C_i(q_i, \dot{q})y + g_i(q_i) = Y_i(q_i, x, y, z)\Theta_i$ for all vectors $x, y \in \mathbb{R}^p$, where $Y_i(q_i, x, y)$ is the regressor and Θ_i is the constant parameter vector associated with the ith agent.

We use a directed graph to describe the network topology between the n agents. Let $\mathcal{G} \triangleq (\mathcal{V}, \mathcal{E})$ be a directed graph with the node set $\mathcal{V} \triangleq \{1, ..., n\}$ and the edge set $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$. An edge $(i, j) \in \mathcal{E}$ denotes that agent j can obtain information from agent i, but not vice versa. Here, node i is the parent node while node j is the child node. A directed path from node i to node j is a sequence of edges of the form (i_1, i_2), (i_2, i_3), $\ldots$, in a directed graph. A directed tree is a directed graph, where every node has exactly one parent except for one node, called the root, and the root has directed paths to every other node. A directed spanning tree of a directed graph is a direct tree that contains all nodes of the directed graph. A directed graph has a spanning tree if there exists a directed spanning tree as a subset of the directed graph.

The adjacency matrix $\mathcal{A} = [a_{ij}] \in \mathbb{R}^{n\times n}$ associated with $\mathcal{G}$ is defined as $a_{ij} > 0$ if $(j, i) \in \mathcal{E}$, and $a_{ij} = 0$ otherwise. In this paper, self edges are not allowed, *i.e.,* $a_{ii} = 0$. The (nonsymmetric) Laplacian matrix $\mathcal{L}_A = [l_{ij}] \in \mathbb{R}^{n\times n}$ associated with $\mathcal{A}$ and hence $\mathcal{G}$ is defined as $l_{ii} = \sum_{j=1,j\neq i}^{n} a_{ij}$ and $l_{ij} = -a_{ij}$, $i \neq j$.

Lemma 2.1: [18] Let $\mathcal{G}$ be a directed graph of order n and $\mathcal{L}_A \in \mathbb{R}^{n\times n}$ be the associated (nonsymmetric) Laplacian matrix. The following three statements are equivalent:

1) The matrix $\mathcal{L}_A$ has a single zero eigenvalue and all other eigenvalues have positive real parts;
2) $\mathcal{G}$ has a directed spanning tree;
3) Given a system $\dot{z} = -\mathcal{L}_A z$, where $z = [z_1, \ldots, z_n]^T$, consensus is reached exponentially. In particular, for all $i = 1, \ldots, n$ and all $z_i(0)$, $z_i(t) \to \sum_{i=1}^{n} p_i z_i(0)$ as $t \to \infty$, where $\mathbf{p} = [p_1, \ldots, p_n]^T$ is a nonnegative left eigenvector of $\mathcal{L}_A$ associated with the zero eigenvalue satisfying $\sum_{i=1}^{n} p_i = 1$.

For the n agents with m $(m < n)$ followers and $n - m$ leaders, we use $\mathcal{V}_F \triangleq \{1, \ldots, m\}$ and $\mathcal{V}_L \triangleq \{m+1, \ldots, n\}$ to denote, respectively, the follower set and the leader set. Let q_F and q_L be the column stack vectors of, respectively, q_i, $\forall i \in \mathcal{V}_F$, and q_i, $\forall i \in \mathcal{V}_L$. In this paper, we assume that the directed graph $\mathcal{G}$ satisfies the following assumption.

Assumption 2.1: For each of the m followers, there exists at least one leader that has a directed path to the follower.

Definition 2.2: [19] Let $\mathcal{C}$ be a set in a real vector space $\mathcal{S} \subseteq \mathbb{R}^n$. The set $\mathcal{C}$ is convex if, for any x and y in $\mathcal{C}$, the point $(1-t)x+ty \in \mathcal{C}$ for any $t \in [0, 1]$. The convex hull for a set of points $X = \{x_1, \ldots, x_n\}$ in $\mathcal{S}$ is the minimal convex set containing all points in X. We use $\text{Co}(X)$ to denote the convex hull of X. In particular, $\text{Co}(X) = \{\sum_{i=1}^{n} \alpha_i x_i | x_i \in X, \alpha_i \geq 0, \sum_{i=1}^{n} \alpha_i = 1\}$.

Definition 2.3: [20] Let $Z_n \subset \mathbb{R}^{n\times n}$ denote the set of all square matrices of dimension n with nonpositive off-diagonal entries. A matrix $A \in \mathbb{R}^{n\times n}$ is said to be a nonsingular M-matrix if $A \in Z_n$ and all eigenvalues of A have positive real parts.

Lemma 2.2: [20] A matrix $A \in Z_n$ is a nonsingular M-

matrix if and only if A^{-1} exists and each entry of A^{-1} is nonnegative.

Note that the (nonsymmetric) Laplacian matrix $\mathcal{L}_A$ associated with $\mathcal{A}$ hence $\mathcal{G}$ can be written as

$$\mathcal{L}_A = \begin{bmatrix} L_1 & L_2 \\ \mathbf{0}_{(n-m)\times m} & \mathbf{0}_{(n-m)\times(n-m)} \end{bmatrix}, \quad (2)$$

where $L_1 \in \mathbb{R}^{m\times m}$ and $L_2 \in \mathbb{R}^{m\times(n-m)}$.

Lemma 2.3: The matrix L_1 defined as in (2) is a nonsingular M-matrix if and only if Assumption 2.1 holds. In addition, if Assumption 2.1 holds, then each entry of $-L_1^{-1}L_2$ is nonnegative and all row sums of $-L_1^{-1}L_2$ equal to one.

Proof: See [21]. ■

Lemma 2.4: [22] Consider the system

$$\dot{x} = f(t, x, u), \quad (3)$$

where $f(t, x, u)$ is continuously differentiable and globally Lipschitz in (x, u), uniformly in t. If the unforced system $\dot{x} = f(t, x, 0)$ has a globally exponentially stable equilibrium point at the origin $x = 0$, then the system (3) is input-to-state stable.

III. Distributed Containment Control with Multiple Stationary Leaders

In this section, we consider the case where all leaders are stationary, *i.e.,* $\dot{q}_i = 0, \forall i \in \mathcal{V}_L$. We will design a distributed control algorithm for (1) such that all followers converge to the convex hull spanned by the stationary leaders.

Before moving on, we introduce the following auxiliary variables

$$\dot{q}_{ri} \triangleq -\alpha \sum_{j\in\mathcal{V}_L \bigcup \mathcal{V}_F} a_{ij}(q_i - q_j), \quad (4)$$

$$s_i \triangleq \dot{q}_i - \dot{q}_{ri} = \dot{q}_i + \alpha \sum_{j\in\mathcal{V}_L \bigcup \mathcal{V}_F} a_{ij}(q_i - q_j), \quad i \in \mathcal{V}_F, \quad (5)$$

where α is a positive constant, and a_{ij} is the (i, j)th entry of the adjacency matrix $\mathcal{A}$ associated with $\mathcal{G}$. We propose the following distributed adaptive control algorithm for (1) in the presence of parametric uncertainties

$$\tau_i = -K_i s_i + Y_i(q_i, \ddot{q}_{ri}, \dot{q}_i, \dot{q}_{ri})\widehat{\Theta}_i, \quad (6a)$$

$$\dot{\widehat{\Theta}}_i = -\Lambda_i Y_i^T(q_i, \ddot{q}_{ri}, \dot{q}_i, \dot{q}_{ri})s_i, \quad i \in \mathcal{V}_F, \quad (6b)$$

where K_i is a symmetric positive-definite matrix, $\widehat{\Theta}_i$ is the estimate of Θ_i, and Λ_i is a symmetric positive-definite matrix.

Theorem 3.1: Suppose that all leaders are stationary. Using (6) for (1), $d[q_i(t), \mathrm{Co}(q_L)] \to 0$ and $\dot{q}_i \to \mathbf{0}_p$ as $t \to \infty$, $\forall i \in \mathcal{V}_F$, for arbitrary initial conditions in the presence of parametric uncertainties if and only if Assumption 2.1 holds. More specifically, $q_F(t) \to -(L_1^{-1}L_2 \otimes I_p)q_L$ as $t \to \infty$, that is, the final vectors of generalized coordinates of the followers are given by $-(L_1^{-1}L_2 \otimes I_p)q_L$.

Proof: (*Sufficiency*) Let s_F, q_F, $\dot{q}_r$, and q_L be the column stack vectors of, respectively, s_i, $\forall i \in \mathcal{V}_F$, q_i, $\forall i \in \mathcal{V}_F$, $\dot{q}_{ri}$, $\forall i \in \mathcal{V}_F$, and q_i, $\forall i \in \mathcal{V}_L$. Let $\widetilde{\Theta}_i \triangleq \Theta_i - \widehat{\Theta}_i$. Also let $\widetilde{\Theta}$, Θ, and $\widehat{\Theta}$ be, respectively, the column stack vectors of $\widetilde{\Theta}_i$, Θ_i, and $\widehat{\Theta}_i$, $\forall i \in \mathcal{V}_F$. From Assumption (A3), it follows that

$$M_i(q_i)\ddot{q}_{ri} + C_i(q_i, \dot{q}_i)\dot{q}_{ri} + g_i(q_i) = Y_i(q_i, \ddot{q}_{ri}, \dot{q}_i, \dot{q}_{ri})\Theta_i, \quad i \in \mathcal{V}_F. \quad (7)$$

Hence, using (6) and (7), the closed-loop system (1) can be written in a vector form as

$$M(q_F)\dot{s}_F = -C(q_F, \dot{q}_F)s_F - K_F s_F - Y(q_F, \ddot{q}_r, \dot{q}_F, \dot{q}_r)\widetilde{\Theta}, \quad (8)$$

where $M(q_F)$, $C(q_F, \dot{q}_F)$, $Y(q_F, \ddot{q}_r, \dot{q}_F, \dot{q}_r)$, and K_F are, respectively, the block diagonal matrices of $M_i(q_i)$, $C_i(q_i, \dot{q}_i)$, $Y_i(q_i, \ddot{q}_{ri}, \dot{q}_i, \dot{q}_{ri})$, and K_i, $\forall i \in \mathcal{V}_F$.

Consider the following Lyapunov function candidate

$$V(t) = \frac{1}{2}s_F^T M(q_F)s_F + \frac{1}{2}\widetilde{\Theta}^T \Xi \widetilde{\Theta}, \quad (9)$$

where Ξ is the block diagonal matrix of Λ_i^{-1}, $\forall i \in \mathcal{V}_F$. Taking the derivative of V along (8) gives that

$$\begin{aligned} \dot{V}(t) &= s_F^T M(q_F)\dot{s}_F + \frac{1}{2}s_F^T \dot{M}(q_F)s_F + \tilde{\Theta}^T \Xi \dot{\tilde{\Theta}} \\ &= -s_F^T K_F s_F, \end{aligned} \quad (10)$$

where we have used Assumption (A2) and (6b) to obtain (10). Because K_F is symmetric positive definite, we can get that $\dot{V}(t) \le 0$, which means that s_F and $\widetilde{\Theta}$ are bounded. If Assumption 2.1 holds, it follows from Lemma 2.3 that L_1 is a nonsingular M-matrix, which implies that L_1^{-1} exists. Note that (5) can be written in a vector form as

$$\dot{q}_F = -\alpha(L_1 \otimes I_p)q_F - \alpha(L_2 \otimes I_p)q_L + s_F. \quad (11)$$

Also note that $\dot{q}_L = 0$. Then (11) can be written as

$$\dot{\bar{q}}_F = -\alpha(L_1 \otimes I_p)\bar{q}_F + s_F, \quad (12)$$

where

$$\bar{q}_F \triangleq q_F + (L_1^{-1}L_2 \otimes I_p)q_L. \quad (13)$$

Because L_1 is a nonsingular M-matrix, it follows from Definition 2.3 that all eigenvalues of L_1 have positive real parts. It thus follows that when $s_F = \mathbf{0}_{mp}$, (12) is globally exponentially stable at the origin $\bar{q}_F = \mathbf{0}_{mp}$. We can conclude from Lemma 2.4 that (12) is input-to-state stable with respect to the input s_F and the state $\bar{q}_F$. Because s_F is bounded, so is $\bar{q}_F$. Because q_L is constant, it follows from (13) that q_F is bounded. We can get from (4) that $\dot{q}_{ri}$, $\forall i \in \mathcal{V}_F$, is bounded. It also follows from (11) that $\dot{q}_F$ is bounded. Note from Assumption (A1) that $\|C_i(q_i, \dot{q}_i)\dot{q}_{ri}\| \le k_C\|\dot{q}_i\|\|\dot{q}_{ri}\|$ and $\|g_i(q_i)\| \le k_{g_i}$, $\forall i \in \mathcal{V}_F$. Therefore, both $\|C_i(q_i, \dot{q}_i)\dot{q}_{ri}\|$ and $\|g_i(q_i)\|$ are bounded. Differentiating (4), we can see that $\ddot{q}_{ri}$, $\forall i \in \mathcal{V}_F$, is bounded. Note that in (7), $M_i(q_i)$, $\ddot{q}_{ri}$, $C_i(q_i, \dot{q}_i)\dot{q}_{ri}$ and $g_i(q_i)$, $\forall i \in \mathcal{V}_F$, are all bounded. We conclude from (7) that $Y_i(q_i, \ddot{q}_{ri}, \dot{q}_i, \dot{q}_{ri})$ is bounded. Note again from Assumption (A1) that $\|C_i(q_i, \dot{q}_i)s_i\| \le k_C\|\dot{q}_i\|\|s_i\|$, $\forall i \in \mathcal{V}_F$. From (8), we can get that $\dot{s}_F$ is bounded. By differentiating (10), we can see that $\ddot{V}(t)$ is bounded. Therefore, $\dot{V}(t)$ is uniformly continuous in time.

From Barbalat's Lemma [22], we can conclude that $\dot{V}(t) \to 0$ as $t \to \infty$, *i.e.,* $s_F(t) \to \mathbf{0}_{mp}$ as $t \to \infty$. Because (12) is input-to-state stable with respect to the input s_F and the state $\bar{q}_F$, we have that $\bar{q}_F(t) \to \mathbf{0}_{mp}$ as $t \to \infty$. As a result, it follows that $q_F(t) \to -(L_1^{-1}L_2 \otimes I_p)q_L$ and $\dot{q}_F \to \mathbf{0}_{mp}$ as $t \to \infty$. If Assumption 2.1 holds, it follows from Lemma 2.3 that each entry of $-L_1^{-1}L_2$ is nonnegative and each row of $-L_1^{-1}L_2$ has a sum equal to one. We then get from Definition 2.2 that $-(L_1^{-1}L_2 \otimes I_p)q_L$ is within the convex hull spanned by the stationary leaders. It thus follows that $d[q_i(t), \mathrm{Co}(q_L)] \to 0$ as $t \to \infty$. This conclude the sufficiency part.

(*Necessity*) We prove the necessity part by contraposition. If Assumption 2.1 does not hold, there exists a subset of the followers who cannot receive any information from the leaders directly or indirectly. That is, the motions of these followers are independent of the states of the leaders. Therefore, these followers cannot always converge to the convex hull spanned by the stationary leaders for arbitrary initial conditions. ■

Corollary 3.2: Suppose that $\mathcal{V}_L = \emptyset$.[1] Using (6) for (1), $\|q_i(t) - q_j(t)\| \to 0$ and $\dot{q}_i(t) \to \mathbf{0}_p$ as $t \to \infty$ for arbitrary initial conditions in the presence of parametric uncertainties if and only if the directed graph $\mathcal{G}$ associated with the n agents has a directed spanning tree.

Proof: (*Sufficiency*) Because $\mathcal{V}_L = \emptyset$, (11) can be written as

$$\dot{q} = -\alpha(\mathcal{L}_A \otimes I_p)q + s, \tag{14}$$

where $\mathcal{L}_A \in \mathbb{R}^{n\times n}$ is the (nonsymmetric) Laplacian matrix associated with $\mathcal{G}$, and q and s are column stack vectors of q_i and s_i, $i = 1, \ldots, n$. Following the same steps as in the proof of Theorem 3.1, we can get that $s(t) \to \mathbf{0}_{np}$ as $t \to \infty$. For the linear system $\dot{q} = -\alpha(\mathcal{L}_A \otimes I_p)q$, if $\mathcal{G}$ has a directed spanning tree, then it follows from Lemma 2.1 that consensus is reached exponentially. Thus, there exists $\bar{q} = \sum_{i=1}^n p_i q_i(0)$, where p_i is defined in Lemma 2.1, such that $\mathbf{1}_n \otimes \bar{q}$ is a globally exponentially stable equilibrium point of $\dot{q} = -\alpha(\mathcal{L}_A \otimes I_p)q$. We can conclude from Lemma 2.4 that the system (14) is input-to-state stable with the input s and the state $q - \mathbf{1}_n \otimes \bar{q}$. Note that $s(t) \to \mathbf{0}_{np}$ as $t \to \infty$. We can conclude that $\|q_i(t) - q_j(t)\| \to 0$ and $\dot{q}_i(t) \to \mathbf{0}_p$ as $t \to \infty$.

(*Necessity*) The proof of the necessity part is the same as Theorem 3.1 and is omitted here. ■

Remark 3.3: In Corollary 3.2, we have shown that the adaptive control algorithm (6) can be used to deal with the leaderless consensus problem for networked Lagrangian system. Thus, Corollary 3.2 extends the first algorithm in [10] to a directed graph in the presence of parameter uncertainties.

IV. Distributed Containment Control with Multiple Dynamic Leaders

In this section, we consider the case where the leaders are dynamic. We consider two subcases. In the first subcase, we assume that the leaders have constant vectors of generalized coordinate derivatives. In the second subcase, we assume that the leaders have varying vectors of generalized coordinate derivatives.

A. Leaders with Constant Vectors of Generalized Coordinate Derivatives

In this subsection, we deal with the case where the leaders have constant vectors of generalized coordinate derivatives. Define the following auxiliary variables

$$\dot{q}_{ri} \triangleq \hat{v}_i - \alpha \sum_{j\in\mathcal{V}_L \bigcup \mathcal{V}_F} a_{ij}(q_i - q_j), \tag{15}$$

$$s_i \triangleq \dot{q}_i - \dot{q}_{ri} = \dot{q}_i - \hat{v}_i + \alpha \sum_{j\in\mathcal{V}_L \bigcup \mathcal{V}_F} a_{ij}(q_i - q_j),\ i \in \mathcal{V}_F, \tag{16}$$

where α is a positive constant, a_{ij} is defined as in (4), and $\hat{v}_i$ is the ith follower's estimate of its desired vector of generalized coordinate derivatives in the convex hull spanned by those of the leaders' that will be designed later. We propose the following control algorithm for (1) in the presence of parametric uncertainties

$$\tau_i = -K_i s_i + Y_i(q_i, \ddot{q}_{ri}, \dot{q}_i, \dot{q}_{ri})\widehat{\Theta}_i, \tag{17a}$$

$$\dot{\hat{v}}_i = -\beta\Big[\sum_{j\in\mathcal{V}_F} a_{ij}(\hat{v}_i - \hat{v}_j) + \sum_{j\in\mathcal{V}_L} a_{ij}(\hat{v}_i - \dot{q}_j)\Big], \tag{17b}$$

$$\dot{\widehat{\Theta}}_i = -\Lambda_i Y_i^T(q_i, \ddot{q}_{ri}, \dot{q}_i, \dot{q}_{ri})s_i, \quad i \in \mathcal{V}_F, \tag{17c}$$

where β is a positive constant rather that (4), and K_i and Λ_i are symmetric positive-definite matrices.

We next state the main result of containment control with multiple dynamic leaders that have constant vectors of generalized coordinate derivatives.

Theorem 4.1: Suppose that the leaders have constant vectors of generalized coordinate derivatives. Using (17) for (1), $d\{q_i(t), \mathrm{Co}[q_L(t)]\} \to 0$, $\forall i \in \mathcal{V}_F$, as $t \to \infty$ for arbitrary initial conditions in the presence of parametric uncertainties if and only if Assumption 2.1 holds. More specifically, $\|q_F(t) + (L_1^{-1}L_2 \otimes I_p)q_L(t)\| \to 0$ as $t \to \infty$.

Proof: (*Sufficiency*) Let s_F, q_F, q_L, and $\widetilde{\Theta}_i$ defined as in the proof of Theorem 3.1. Let $\hat{v}_F$ be the column stack vector of $\hat{v}_i$, $\forall i \in \mathcal{V}_F$. Consider the Lyapunov function candidate defined in (9). Following the proof of Theorem 3.1, we get (10), which implies that s_F and $\widetilde{\Theta}$ are bounded. Because Assumption 2.1 holds, we can get from Lemma 2.3 and Definition 2.3 that L_1^{-1} exists and all eigenvalues of L_1 have positive real parts. Note from (16) that

$$\dot{q}_F = -\alpha(L_1 \otimes I_p)q_F - \alpha(L_2 \otimes I_p)q_L + \hat{v}_F + s_F. \tag{18}$$

Then (18) can be written as

$$\dot{\bar{q}}_F = -\alpha(L_1 \otimes I_p)\bar{q}_F + \bar{v}_F + s_F, \tag{19}$$

where $\bar{q}_F \triangleq q_F + (L_1^{-1}L_2 \otimes I_p)q_L$ and $\bar{v}_F \triangleq \hat{v}_F + (L_1^{-1}L_2 \otimes I_p)\dot{q}_L$. It thus follows from Lemma 2.4 that (19) is input-to-state stable with respect to the input $\bar{v}_F + s_F$ and the state

[1] In this case, there does not exist a leader. Therefore, (6) becomes a leaderless consensus algorithm accounting for parametric uncertainties.

$\bar{q}_F$. Note that (17b) can be written in a vector form as

$$\dot{\hat{v}}_F = -\beta(L_1 \otimes I_p)\hat{v}_F - \beta(L_2 \otimes I_p)\dot{q}_L, \tag{20}$$

Because $\dot{q}_L$ is constant and hence $\ddot{q}_L = 0$, (20) can be written as $\dot{\bar{v}}_F = -\beta(L_1 \otimes I_p)\bar{v}_F$. It thus follows that $\bar{v}_F = \mathbf{0}_{mp}$ is globally exponentially stable. Therefore, $\bar{v}_F$ is bounded, which in turn implies that $\hat{v}_F$ is also bounded because $\dot{q}_L$ is constant. Because $\bar{v}_F + s_F$ is bounded, it follows from (19) that $\bar{q}_F$ is bounded. Because (18) can be written as $\dot{q}_F = -\alpha(L_1 \otimes I_p)\bar{q}_F + \hat{v}_F + s_F$, it follows that $\dot{q}_F$ is bounded. Note from (16) that $\dot{q}_{ri} = \dot{q}_i - s_i$, $\forall i \in \mathcal{V}_F$. We can get that q_{ri} is bounded, $\forall i \in \mathcal{V}_F$. Differentiating (15), we can get that $\ddot{q}_{ri}$, $\forall i \in \mathcal{V}_F$, is bounded. A similar statement to that in the proof of Theorem 3.1 show that $Y_i(q_i, \ddot{q}_{ri}, \dot{q}_i, \dot{q}_{ri})$, $\dot{s}_F$, and hence $\ddot{V}(t)$ are bounded. Therefore, $\dot{V}(t)$ is uniformly continuous in time. From Barbalat's Lemma, we can conclude that $\dot{V}(t) \to 0$ as $t \to \infty$, *i.e.,* $s_F \to \mathbf{0}_{mp}$ as $t \to \infty$. Because both $\bar{v}_F(t) \to \mathbf{0}_{mp}$ and $s_F(t) \to \mathbf{0}_{mp}$ as $t \to \infty$ in (19), it thus follows that $\bar{q}_F(t) \to \mathbf{0}_{mp}$, *i.e.,* $\|q_F(t) + (L_1^{-1}L_2 \otimes I_p)q_L(t)\| \to 0$, as $t \to \infty$. A similar statement to that in the proof in Theorem 3.1 shows that $-(L_1^{-1}L_2 \otimes I_p)q_L(t)$ is within the convex hull spanned by the dynamic leaders. It thus follows that $d\{q_i(t), \mathrm{Co}[q_L(t)]\} \to 0$ as $t \to \infty$.

(*Necessity*) The proof of the necessity part is the same as Theorem 3.1 and is omitted here. ∎

B. Leaders with Varying Vectors of Generalized Coordinate Derivatives

In this subsection, we deal with the case where the leaders have varying vectors of generalized coordinate derivatives. Suppose that the leaders' vectors of generalized coordinate derivatives and their first-order and seconde-order derivatives are all bounded. Let $q_d \stackrel{\triangle}{=} [q_{d1}^T, \ldots, q_{dm}^T]^T = -(L_1^{-1}L_2 \otimes I_p)q_L$.

We first introduce the following auxiliary variables

$$\hat{\dot{q}}_{ri} \stackrel{\triangle}{=} \hat{v}_i - \alpha \sum_{j \in \mathcal{V}_L \bigcup \mathcal{V}_F} a_{ij}(q_i - q_j), \tag{21}$$

$$\hat{\ddot{q}}_{ri} \stackrel{\triangle}{=} \hat{a}_i - \alpha \sum_{j \in \mathcal{V}_L \bigcup \mathcal{V}_F} a_{ij}(\dot{q}_i - \dot{q}_j), \tag{22}$$

$$\hat{s}_i \stackrel{\triangle}{=} \dot{q}_i - \hat{\dot{q}}_{ri} = \dot{q}_i - \hat{v}_i + \alpha \sum_{j \in \mathcal{V}_L \bigcup \mathcal{V}_F} a_{ij}(q_i - q_j), \quad i \in \mathcal{V}_F, \tag{23}$$

where α is a positive constant, a_{ij} is defined as in (4), and $\hat{v}_i$ (respectively $\hat{a}_i$) is the ith follower's estimate of its desired vector of generalized coordinate derivatives (respectively, accelerations) in the convex hull spanned by those of the leaders that will be designed later. We then propose the following distributed algorithm combined with distributed sliding-mode estimators

$$\tau_i = -K_i\hat{s}_i + Y_i(q_i, \hat{\ddot{q}}_{ri}, \dot{q}_i, \hat{\dot{q}}_{ri})\widehat{\Theta}_i, \tag{24a}$$

$$\dot{\hat{v}}_i = -\beta_1 \mathrm{sgn}\Big[\sum_{j \in \mathcal{V}_F} a_{ij}(\hat{v}_i - \hat{v}_j) + \sum_{j \in \mathcal{V}_L} a_{ij}(\hat{v}_i - \dot{q}_j)\Big] \tag{24b}$$

$$\dot{\hat{a}}_i = -\beta_2 \mathrm{sgn}\Big[\sum_{j \in \mathcal{V}_F} a_{ij}(\hat{a}_i - \hat{a}_j) + \sum_{j \in \mathcal{V}_L} a_{ij}(\hat{a}_i - \ddot{q}_j)\Big], \tag{24c}$$

$$\dot{\widehat{\Theta}}_i = -\Lambda_i Y_i^T(q_i, \hat{\ddot{q}}_{ri}, \dot{q}_i, \hat{\dot{q}}_{ri})\hat{s}_i, \quad i \in \mathcal{V}_F, \tag{24d}$$

where K_i and Λ_i are symmetric positive-definite matrixes, β_1 and β_2 are positive constants, and $\mathrm{sgn}(\cdot)$ is the signum function defined componentwise.

Lemma 4.1: Suppose that Assumption 2.1 holds. Let $q_d \stackrel{\triangle}{=} [q_{d1}^T, \ldots, q_{dm}^T]^T = -(L_1^{-1}L_2 \otimes I_p)q_L$, where $q_{di} \in \mathbb{R}^p$.[2] If $\beta_1 > \|\ddot{q}_d\|$, then $\|\hat{v}_i(t) - \dot{q}_{di}(t)\| \to 0$, $\forall i \in \mathcal{V}_F$, in finite time. Similarly, if $\beta_2 > \|\dddot{q}_d\|$, then $\|\hat{a}_i(t) - \ddot{q}_{di}(t)\| \to 0$, $\forall i \in \mathcal{V}_F$, in finite time.

Proof: See [21]. ∎

Theorem 4.2: Suppose that the leaders have varying vectors of generalized coordinate derivatives, $\beta_1 > \|\ddot{q}_d\|$, and $\beta_2 > \|\dddot{q}_d\|$. Using (24) for (1), $d\{q_i(t), \mathrm{Co}[q_L(t)]\} \to 0$ as $t \to \infty$, $\forall i \in \mathcal{V}_F$, for arbitrary initial conditions in the presence of parametric uncertainties if and only if Assumption 2.1 holds. More specifically, $\|q_F(t) + (L_1^{-1}L_2 \otimes I_p)q_L(t)\| \to 0$ as $t \to \infty$.

Proof: (*Sufficiency*) First, we show that for bounded initial values $q_i(0)$ and $\dot{q}_i(0)$, using the control algorithm (24) for (1), the states $q_i(t)$ and $\dot{q}_i(t)$, $\forall i \in \mathcal{V}_F$, will remain bounded in finite time. From (24b) and (24c), we can get that $\hat{v}_i(t)$ and $\hat{a}_i(t)$, $\forall i \in \mathcal{V}_F$, are bounded in finite time for bounded initial values $\hat{v}_i(0)$ and $\hat{a}_i(0)$. For bounded states q_i and $\dot{q}_i$, $\forall i \in \mathcal{V}_F$, we can get that $\hat{s}_i$, $\hat{\dot{q}}_{ri}$ and $\hat{\ddot{q}}_{ri}$, $\forall i \in \mathcal{V}_F$, are bounded. From Assumption (A1), we can get that $Y_i(q_i, \hat{\ddot{q}}_{ri}, \dot{q}_i, \hat{\dot{q}}_{ri})$ defined in (24a) is bounded and therefore, $\widehat{\Theta}_i(t)$ is bounded for bounded initial value $\widehat{\Theta}_i(0)$. Thus, we can get from (24a) that τ_i is bounded. Finally, from (1), for bounded q_i, $\dot{q}_i$, and τ_i, under Assumption (A1), we can get that $\ddot{q}_i$ is also bounded. Thus, we can conclude that for bounded initial values $q_i(0)$ and $\dot{q}_i(0)$, $q_i(t)$ and $\dot{q}_i(t)$, $\forall i \in \mathcal{V}_F$, are bounded in finite time.

Let $\dot{q}_{ri} \stackrel{\triangle}{=} \dot{q}_{di} - \alpha \sum_{j \in \mathcal{V}_L \bigcup \mathcal{V}_F}(q_i - q_j)$, and

$$s_i \stackrel{\triangle}{=} \dot{q}_i - \dot{q}_{ri} = \dot{q}_i - \dot{q}_{di} + \alpha \sum_{j \in \mathcal{V}_L \bigcup \mathcal{V}_F}(q_i - q_j), \quad i \in \mathcal{V}_F, \tag{25}$$

where q_{di} is defined in Lemma 4.1. Under the condition of the theorem, using the sliding-mode estimators (24b) and (24c), we can get from Lemma 4.1 that $\hat{v}_i(t) \equiv \dot{q}_{di}(t)$ and $\hat{a}_i(t) \equiv \ddot{q}_{di}(t)$ when $t \geq \max\{T_1, T_2\} \stackrel{\triangle}{=} T_0$. Therefore, $\hat{\dot{q}}_{ri}(t) \equiv \dot{q}_{ri}(t)$, $\hat{\ddot{q}}_{ri}(t) \equiv \ddot{q}_{ri}(t)$, and $\hat{s}_i(t) \equiv s_i(t)$, $\forall i \in \mathcal{V}_F$, when $t \geq T_0$. Note that (25) can be written in a vector

[2] When Assumption 2.1 holds, it follows from Lemma 2.3 that L_1^{-1} exists. Therefore, q_d is well defined.

form as (12), where $\bar{q}_F$ is defined in (13). Let s_F and $\widetilde{\Theta}$ be, respectively, the column state vectors of s_i and $\widetilde{\Theta}_i \stackrel{\triangle}{=} \Theta_i - \widehat{\Theta}_i$, $\forall i \in \mathcal{V}_F$. When $t \geq T_0$, consider the Lyapunov function candidate defined in (9). The rest of the proof follows that of Theorem 3.1.

(*Necessity*) The proof of the necessity part is the same as Theorem 3.1 and is omitted here. ■

Remark 4.3: Note that a single leader is a special case of multiple leaders. Therefore, the distributed coordinated tracking problem of a single leader for networked Lagrangian systems is a special case of the distributed containment control problem. Thus the results in the current paper can be used to deal with the coordinated tracking problem and hence extend the work in [11]–[13] to a directed graph.

V. Conclusions

The distributed containment control problem for networked Lagrangian systems with multiple stationary or dynamic leaders in the presence of parametric uncertainties has been studied under a directed graph that characterizes the interaction among the leaders and the followers. In the case of multiple stationary leaders, we have proposed a distributed adaptive control algorithm and a necessary and sufficient condition on the directed graph such that all followers converge to the stationary convex hull spanned by the stationary leaders asymptotically. As a byproduct, we have presented a necessary and sufficient condition on leaderless consensus algorithm for networked Lagrangian systems under a directed graph. In the case with multiple dynamic leaders, we have considered two subcases where the leaders have constant or varying vectors of generalized coordinate derivatives. In the first subcase, a distributed continuous estimator and a distributed adaptive control algorithm have been proposed. In the second subcase, we have proposed a distributed adaptive control algorithm combined with distributed sliding-mode estimators and presented a necessary and sufficient condition on the directed graph.

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