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Simultaneous Stabilization and Synchronization for Multiple Systems of Non-identical Agents

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Abstract— **We address a simultaneous stabilization and synchronization problem for one class of non-identical multi-agent systems. The agent dynamics can be different and the orders of agents are not necessarily equal. One single control loop is designed for each agent to enable some agent states, named as internal states, to be stabilized and some other states, named as external states, to be synchronized. A distributed control law is designed based on local measurements and information exchanged from neighboring agents to achieve simultaneous stabilization and synchronization. The necessary and sufficient conditions to achieve simultaneous stabilization and synchronization have been obtained using the closed-loop form of the system, followed by specific approaches of designing the control gain matrices that make the sufficient conditions satisfied. A precise form of the synchronized trajectory has also been determined. Simulation results have been provided to demonstrate the performance of the proposed method.**

I. INTRODUCTION

Cooperative control of multi-agent systems has attracted substantial attention over the past decade. The control goal is to design distributed control laws such that a team of agents achieve a desired group behavior. One relevant topic is the synchronization problem of dynamic systems (see, e.g., [1]–[7], just name a few) where the agent trajectories in a network converge to each other through distributed local coupling. Specifically, for linear time invariant (LTI) multi-agent systems, Tuna studied the output synchronization problem of identical agents [1] and investigated the synchronizability conditions for coupled linear systems [2] where the number of inputs is equal to the number of states. Scardovi and Sepulchre [3] investigated the synchronization problem of a network of identical linear state-space models using a dynamic output feedback coupling. Li *et al.* [4] introduced a new framework to address the output feedback synchronization problem of a group of LTI systems by introducing the notion of consensus region. Seo *et al.* [5] presented a low gain synchronization approach for designing an output feedback compensator which only used the local output information. Chopra and Spong [6] investigated the output synchronization problem for a class of passive nonlinear systems. Nair and Leonard [7] solved the stable synchronization problem for a network of under-actuated mechanical systems.

In multi-agent systems, the overall behavior of each agent can be determined by its internal dynamics and external dynamics. The internal dynamics govern the behavior of the agent as an individual system while the external dynamics are related to the coordination with the other agents. In some systems (see, e.g., [8], [9]), the internal dynamics is much faster than the external dynamics, so that the internal dynamics can be ignored and the agents are modeled as first order integrators. For multi-agent systems that need to be represented using more general models, one idea is to use an inner control loop (see, e.g., [10], [11]). Specifically, Fax and Murray [10] proposed an idea to stabilize each agent by closing an inner control loop around its internal dynamics and then closing an outer control loop to achieve the desired formation performance. Arcak [11] assumed that an inner control loop is designed so that the resulting system becomes passive with respect to the external feedback and then proposed a passivity-based method for the coordination purpose. However, as will be shown in the motivating example introduced in Section II, there exist cases where separate control loops are not available for both stabilization of internal dynamics and synchronization of external dynamics.

If no internal control loop is available and the agents have unstable open-loop internal dynamics and/or dynamically coupled internal and external states (see the motivating example in Section II), then the decentralized controller of each agent should perform two tasks simultaneously: 1) stabilize the agent's internal dynamics, and 2) coordinate with other agents to achieve a group behavior. Nair and Leonard [7] developed a new framework for stable synchronization of under-actuated mechanical systems, distinguishing between actuated and under-actuated states. They used an energy shaping method to stabilize the under-actuated states while rendering the actuated states synchronized. It looks encouraging to distinguish the states that are supposed to be stabilized from those that are synchronized through dynamic coupling. This distinction leads to generalizing the existing results for identical LTI systems to non-identical ones in the current work.

In this paper, we consider a *simultaneous stabilization and synchronization (SSS)* problem for a group of non-identical linear agents with potentially unstable open-loop dynamics. A single control loop is designed for each agent to enable the internal states to be stabilized and the external states to be synchronized. Regarding the system model, not only the agent dynamics are non-identical but also the dimensions of internal agent states are not required to be equal. A distributed SSS protocol is designed based on local measurements and information exchanged from neighboring agents to enable simultaneous stabilization and synchronization. The

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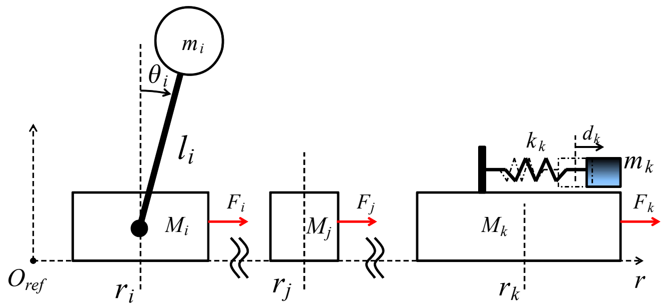

Fig. 1. Schematic of the motivating example.

necessary and sufficient conditions to enable SSS have been obtained using the closed-loop form of the system, followed by specific approaches of designing the control gain matrices that make the sufficient conditions satisfied. A precise form of the synchronized trajectory has also been determined.

II. Motivating Example

The motivating example described in this section was inspired from an example addressed in [7], where Nair and Leonard tested their developed controller to solve a formation problem for systems of identical carts with inverted pendulums. In the following motivating example, formation of a group of different types of cart systems (i.e., non-identical systems) is considered.

Consider a multi-agent system which is consisted of N_i carts with inverted pendulums, N_j simple carts, and N_k carts with mounted mass spring systems. As depicted in Fig. 1, the i-th cart with inverted pendulum is consisted of a point mass m_i connected to a bar of length l_i which is hinged to a cart of mass M_i. The inverted pendulum has an angle θ_i from the vertical axis and the constant g denotes the acceleration of gravity. This system is an under-actuated system with unstable open-loop internal dynamics. The j-th simple cart has the mass M_j. This system is fully-actuated. The k-th cart with spring mass system is consisted of a point mass m_k which is connected to the cart of mass M_k through a linear spring with stiffness coefficient k_k and a displacement of d_k from the equilibrium point. This system is an under-actuated system with marginally stable open-loop internal dynamics. The l-th ($l = i, j, k$) cart is located at a distance of r_l from the origin of the global reference frame O_{ref} and it can move freely along the horizontal axis. In each subsystem, a horizontal control force F_l with the corresponding index i, j, or k is applied to the cart.

The objective is to design a control law for F_l that stabilizes the internal dynamics while enabling a specific formation for all the cart systems. Specifically, the carts are required to move at a desired constant velocity v_d, and all the carts achieve a formation with a pre-specified separation distance Δ_{ij} between cart i and cart j ($i, j = 1, ..., N$). The interesting characteristics of this example are two-fold. First, the internal agent dynamics is unstable, or marginally stable. Second, only one control input is available to stabilize the internal dynamics while enabling the cart to achieve a formation with its neighboring carts.

If there is an additional input, for example in the cart pendulum systems, a torque input is exerted on the inverted pendulum, then it can be used to stabilize the inverted pendulum while the force input F_i enables synchronization. However, when an additional input is not available, it is challenging to achieve the control objective for this motivation example. In this paper, we will formulate a simultaneous stabilization and synchronization problem in Section IV and design a control law to achieve this control objective. In comparison with the motivating example in [7], two generalizations are made here. 1) The agent dynamics and the order of dynamics for the agents can be different. 2) The connection graph can be a directed graph containing a spanning tree instead of an undirected graph.

III. Notation and Preliminaries

Graph theory [12]**:** Let $\mathcal{G} = \{\mathcal{V}, \mathcal{E}\}$ represent a directed graph and V$=\{1, ..., N\}$ denote the set of vertices. Every agent is represented by a vertex. The set of edges is denoted as $\mathcal{E} \subset \mathcal{V} \times \mathcal{V}$. An edge is an ordered pair $(i, j) \in \mathcal{E}$ if agent j can be directly supplied with information from agent i. In this paper, we assume that there is no self loop in the graph, that is, $(i, i) \notin \mathcal{E}$. $\mathcal{N}_i = \{j \in \mathcal{V} \,|(j, i) \in \mathcal{E}\}$ denotes the neighborhood set of vertex i. Graph $\mathcal{G}$ is said to be undirected if for any edge $(i, j) \in \mathcal{E}$, edge $(j, i) \in \mathcal{E}$. Hence, an undirected graph is a special case of a directed graph. A path is referred by the sequence of its vertices. Path $\mathcal{P}$ between two vertices v_0 and v_k is the sequence $\{v_0, ..., v_k\}$ where $(v_{i-1}, v_i) \in \mathcal{E}$ for $i = 1, ..., k$ and the vertices are distinct. The number k is defined as the length of path $\mathcal{P}$. Graph $\mathcal{G}$ is strongly connected if any two vertices are linked with a path in $\mathcal{G}$. Graph $\mathcal{G}$ contains a directed spanning tree if there is a vertex which can reach all the other vertices through a directed path. $\mathcal{A} = [a_{ij}] \in \mathbb{R}^{N \times N}$ denotes the adjacency matrix of $\mathcal{G}$, where $a_{ij} > 0$ if and only if $(j, i) \in \mathcal{E}$ else $a_{ij} = 0$. $L = D - \mathcal{A}$ is called Laplacian matrix of $\mathcal{G}$, where $D = [d_{ii}] \in \mathbb{R}^{N \times N}$ is a diagonal matrix with $d_{ii} = \sum_{j=1}^{N} a_{ij}$.

Lemma 1: [12]–[14] Zero is an eigenvalue of L for both directed and undirected graphs. Zero is a simple eigenvalue of L and the associated eigenvector is $\mathbf{1}$ where $\mathbf{1} \in \mathbb{R}^N$ is a unitary column vector, if and only if the undirected graph is connected or if the directed graph has a spanning tree. All of the nonzero eigenvalues of L are positive for an undirected graph or have positive real parts for a directed graph.

Kronecker Product: Some properties of the Kronecker product are recalled as below [15]

$$
\begin{aligned}
(A \otimes B)(C \otimes D) &= AC \otimes BD \\
A \otimes (B + C) &= A \otimes B + A \otimes C \\
(A \otimes B)^T &= A^T \otimes B^T \\
(A \otimes B)^{-1} &= A^{-1} \otimes B^{-1}.
\end{aligned} \tag{1}
$$

Assume that β_{1i} are the eigenvalues of $A_1 \in \mathbb{R}^{n_1 \times n_1}$ and β_{2j} are the eigenvalues of $A_2 \in \mathbb{R}^{n_2 \times n_2}$. Eigenvalues of

$(I_{n_2} \otimes A_1 + A_2 \otimes I_{n_1})$ are $\beta_{1i} + \beta_{2j}$ with $i \in \{1, ..., n_1\}$ and $j \in \{1, ..., n_2\}$.

Right Inverse: Matrix $B_{m\times r}$ is called the right inverse of matrix $A_{r\times m}$ if $AB = I_{r\times r}$. The necessary condition for the existence of such a matrix B is that $rank(A) = r$.

Lemma 2: [16] The algebraic matrix equation

$$MX + XN = Q, \tag{2}$$

where $M \in \mathbb{R}^{m\times m}$ and $N \in \mathbb{R}^{n\times n}$ are square matrices, has a unique solution if and only if M and $(-N)$ does not have any common eigenvalues.

Remark 1: This algebraic matrix equation (2) is very similar to the Sylvester equation. However, in the Sylvester equation, the matrices M and N are square matrices of the same order.

IV. Problem Formulation

Consider a multi-agent system of N agents with the following dynamics:

$$\begin{aligned} \dot{x}_i &= A_i x_i + B_i u_i \\ \dot{z}_i &= E_i x_i + F z_i, \end{aligned} \tag{3}$$

where $x_i \in \mathbb{R}^{n_i}$ and $z_i \in \mathbb{R}^r$ are the states of agent i, $u_i \in \mathbb{R}^{m_i}$ is the control input of agent i, and $A_i \in \mathbb{R}^{n_i\times n_i}$, $B_i \in \mathbb{R}^{n_i\times m_i}$, $E_i \in \mathbb{R}^{r\times n_i}$, and $F \in \mathbb{R}^{r\times r}$ are constant matrices. In (3), the matrices A_i, B_i, and E_i are not required to be identical, the dimensions n_i and m_i can also be different for different agents. However, the dimensions of the state z_i and the matrix F are the same for all the agents since the state z_i of the agents reaches synchronization as will be discussed later.

The objective of *simultaneous stabilization and synchronization (SSS)* is to design a control law of the form

$$u_i = -K_i x_i + G_i \sum_{j\in\mathcal{N}_i} a_{ij}[z_{ij} + (C_i x_i - C_j x_j)], \tag{4}$$

so that the states x_i are stabilized while the states z_i are synchronized, i.e.,

$$\begin{aligned} x_i &\rightarrow 0 \text{ (stabilization)} \\ z_{ij} &= z_i - z_j \rightarrow 0 \text{ (synchronization)}, \end{aligned} \tag{5}$$

as $t \rightarrow \infty$ for $i, j \in \{1, ..., N\}$. In (4), a_{ij}'s are the elements of the adjacency matrix of the connection graph, $K_i \in \mathbb{R}^{m_i\times n_i}$, $C_i \in \mathbb{R}^{r\times n_i}$, and $G_i \in \mathbb{R}^{m_i\times r}$ are constant gain matrices to be designed, and $C_j \in \mathbb{R}^{r\times n_j}$ determines the portion of the internal states that agent j provides to the other agents. We name the states for synchronization (i.e., z_i) as *external states* and the states for stabilization (i.e., x_i) as *internal states*.

Under control law (4), the closed-loop dynamics of system (3) are given by

$$\dot{x}_i = A_{ci} x_i + B_i G_i \sum_{j\in\mathcal{N}_i} a_{ij}[z_{ij} + (C_i x_i - C_j x_j)] \tag{6}$$

$$\dot{z}_i = E_i x_i + F z_i, \tag{7}$$

where $A_{ci} \triangleq A_i - B_i K_i \in \mathbb{R}^{n_i\times n_i}$.

To facilitate the subsequent design and analysis, we make the following assumptions.

Assumption 1: The connection network has a fixed directed graph $\mathcal{G}$ that contains a directed spanning tree.

Assumption 2: The pair $\{A_i, B_i\}$ for $i \in \{1, ..., N\}$ is stabilizable.

Let $\alpha_i \in \mathbb{C}, i \in \{1, \cdots, r\}$ represent the eigenvalues of F and let $\lambda_2(\mathcal{G}) \in \mathbb{C}$ represent the non-zero eigenvalue of the Laplacian matrix of $\mathcal{G}$ with the smallest real part.

Assumption 3: The biggest real part of the eigenvalues of F (i.e., $\max\{\mathrm{Re}(\alpha_i)\}$) is less than the smallest real part of the nonzero eigenvalues of the Laplacian matrix of $\mathcal{G}$ (i.e., $\sigma \triangleq \mathrm{Re}(\lambda_2(\mathcal{G}))$).

V. Main Results

A. Necessary and Sufficient Conditions for SSS

In order to facilitate the subsequent analysis, we use **bold** font to represent the block diagonal matrices used in the collective forms. For example, $\mathbf{A}$, defined as $\mathbf{A} \triangleq \bigoplus_{i=1}^N A_i$, represents a block diagonal matrix with A_i's as the diagonal elements. The concatenated vectors X and Z are defined as

$$X \triangleq [x_1^T, ..., x_N^T]^T, \quad Z \triangleq [z_1^T, ..., z_N^T]^T.$$

The closed-loop system determined by (6) and (7) can be rewritten in terms of the concatenated vectors X and Z as

$$\begin{bmatrix} \dot{X} \\ \dot{Z} \end{bmatrix} = \begin{bmatrix} \mathbf{A}_c + \mathbf{BG}(L\otimes I_r)\mathbf{C} & \mathbf{BG}(L\otimes I_r) \\ \mathbf{E} & (I_N \otimes F) \end{bmatrix} \begin{bmatrix} X \\ Z \end{bmatrix}, \tag{8}$$

where $\mathbf{A}_c = \mathbf{A} - \mathbf{BK}$.

Lemma 3: Under Assumption 1, there exists a Schur transformation of the form [2], [4]

$$T \triangleq [b, \ M_L]^T, \qquad T^{-1} \triangleq [\mathbf{1}, \ N_L] \tag{9}$$

so that

$$\Lambda \triangleq TLT^{-1} = diag\{0, \Lambda_p\}, \tag{10}$$

where Λ_p is an upper triangular matrix with the nonzero eigenvalues of L along the diagonal. In (9), $b^T \in \mathbb{R}^{1\times N}$ is the normalized left eigenvector of L corresponding to the zero eigenvalue (i.e., $b^T L = 0$ and $\mathbf{1}^T b = 1$), $\mathbf{1} \in \mathbb{R}^N$ is a unitary column vector, and $M_L, \ N_L \in \mathbb{C}^{N\times(N-1)}$.

Lemma 4: Under Assumption 1, the states $z_1, ..., z_N \in \mathbb{R}^r$ are synchronized in the sense that [17] $z_1 = \cdots = z_N$ if and only if $\bar{Z} = 0$ where

$$\bar{Z} \triangleq (L \otimes I_r)Z. \tag{11}$$

Remark 2: Lemma 4 indicates that if $\bar{Z}$ is stabilized, then $z_1, ..., z_N$ are synchronized.

Based on the coordinate transformation (11), (8) can be rewritten as

$$\begin{bmatrix} \dot{X} \\ \dot{\bar{Z}} \end{bmatrix} = \mathcal{H} \begin{bmatrix} X \\ \bar{Z} \end{bmatrix}, \tag{12}$$

where

$$\mathcal{H} \triangleq \begin{bmatrix} \mathbf{A}_c + \mathbf{BG}(L\otimes I_r)\mathbf{C} & \mathbf{BG} \\ (L\otimes I_r)\mathbf{E} & (I_N \otimes F) \end{bmatrix}. \tag{13}$$

Remark 3: The matrix $\mathcal{H}$ has r eigenvalues equal to the eigenvalues of F (see proof of Theorem 2).

Proposition 1: 1) The eigenvalues of a block upper triangular matrix are the eigenvalues of all the diagonal blocks.

2) The stability of a block upper triangular system can be determined by the stability of the subsystems possessing the diagonal blocks.

Theorem 2: Under Assumption 1, the matrix $\mathcal{H}$ has r eigenvalues equal to those of F. In addition, SSS is enabled for system (3) using the controller (4) if and only if the other $(N-1)\,r$ eigenvalues of $\mathcal{H}$ have negative real parts.

Proof: Make the following coordinate transformation:

$$\tilde{Z} = (T \otimes I_r)\,\bar{Z}, \tag{14}$$

where T is defined in (9). The system (12) in the new coordinates $(X, \tilde{Z})$ can be written as

$$\begin{bmatrix} \dot{X} \\ \dot{\tilde{Z}} \end{bmatrix} = \begin{bmatrix} \mathbf{A}_c + \mathbf{BG}\,(L \otimes I_r)\,\mathbf{C} & \mathbf{BG}\left(T^{-1} \otimes I_r\right) \\ (TL \otimes I_r)\,\mathbf{E} & (I_N \otimes F) \end{bmatrix} \begin{bmatrix} X \\ \tilde{Z} \end{bmatrix}. \tag{15}$$

The first r rows of $(TL \otimes I_r)\,\mathbf{E}$ is zero because $\mathbf{E}$ is a block diagonal matrix and $TL = \begin{bmatrix} b^T L \\ M_L^T L \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ M_L^T L \end{bmatrix}$ according to the fact that b is a normalized left eigenvector of L corresponding to the zero eigenvalue (i.e., $b^T L = 0$). The governing equation of $\tilde{z}_1$ (i.e., the first component of $\tilde{Z} \triangleq [\tilde{z}_1^T, ..., \tilde{z}_N^T]^T$) is given by

$$\dot{\tilde{z}}_1 = F\tilde{z}_1. \tag{16}$$

Therefore, the matrix $\mathcal{H}$ in (13) has r eigenvalues which are equal to the eigenvalues of F.

According to the coordinate transformations (11) and (14),

$$\begin{aligned} \tilde{z}_1 &= (b^T \otimes I_r)\bar{Z} = (b^T \otimes I_r)\,(L \otimes I_r)\,Z \\ &= \left(b^T L \otimes I_r\right) Z = \mathbf{0}_{r\times 1}. \end{aligned}$$

This means $\tilde{z}_1(t)$ is always equal to zero based on (11) and (14) even though it is governed by the dynamic equation (16). Since $\mathcal{H}$ has r eigenvalues of F, the system (15) is exponentially stable if and only if the remaining eigenvalues have negative real parts. The stability of system (15) is equivalent to the stability of system (12), which implies that SSS can be enabled for system (3) with control law (4) under the provided conditions in this Theorem. ■

Remark 4: If all the agents have the same dynamics, i.e.,

$$\begin{aligned} A_i &= A,\ B_i = B,\ E_i = E \\ K_i &= K,\ G_i = G,\ C_i = C, \end{aligned} \tag{17}$$

it is possible to use a coordinate transformation to express the stability condition in another way. Specifically, the stability condition will be stated using the eigenvalues of L as follows.

Theorem 3: Under Assumption 1, SSS is enabled for system (3) using the controller (4) if and only if $A_c = A - BK$ is Hurwitz and the matrices H_i's defined as

$$H_i \triangleq \begin{bmatrix} A_c + \lambda_i BGC & BG \\ \lambda_i E & F \end{bmatrix} \tag{18}$$

are Hurwitz for all nonzero eigenvalues λ_i of the Laplacian matrix L.

Proof: For identical agents, the concatenated dynamics (12) of N agents can be written as

$$\begin{bmatrix} \dot{X} \\ \dot{\bar{Z}} \end{bmatrix} = \begin{bmatrix} I_N \otimes A_c + (L \otimes BGC) & (I_N \otimes BG) \\ (L \otimes E) & (I_N \otimes F) \end{bmatrix} \begin{bmatrix} X \\ \bar{Z} \end{bmatrix}. \tag{19}$$

The stability analysis can be performed similar to the method in [10]. As shown in Lemma 3, there exists a Schur transformation T that transforms L into an upper triangular matrix $\Lambda = TLT^{-1} = diag\{0, \Lambda_p\}$. Define new variables $\bar{x}_i'$ and $\bar{z}_i'$ such that $X' \triangleq [x_1'^T, ..., x_N'^T]^T = (T \otimes I_n)X$ and $\bar{Z}' \triangleq [\bar{z}_1'^T, ..., \bar{z}_N'^T]^T = (T \otimes I_r)\bar{Z}$. Based on (19), the derivatives of X' and $\bar{Z}'$ can be determined as

$$\begin{bmatrix} \dot{X}' \\ \dot{\bar{Z}}' \end{bmatrix} = \begin{bmatrix} I_N \otimes A_c + (\Lambda \otimes BGC) & (I_N \otimes BG) \\ (\Lambda \otimes E) & (I_N \otimes F) \end{bmatrix} \begin{bmatrix} X' \\ \bar{Z}' \end{bmatrix}. \tag{20}$$

Since the block components of the system matrix in (20) are block diagonal or block upper triangular, the stability of (20) can be investigated through the stability of the N subsystems defined by

$$\begin{bmatrix} \dot{x}_i' \\ \dot{\bar{z}}_i' \end{bmatrix} = \begin{bmatrix} A_c + \lambda_i BGC & BG \\ \lambda_i E & F \end{bmatrix} \begin{bmatrix} x_i' \\ z_i' \end{bmatrix}, \tag{21}$$

where $\lambda_i, i = 1, \cdots, N.$ are the eigenvalues of L.

Without loss of generality, assume $\lambda_1 = 0$, then (21) for $i = 1$ can be written as

$$\begin{bmatrix} \dot{x}_1' \\ \dot{\bar{z}}_1' \end{bmatrix} = \begin{bmatrix} A_c & BG \\ \mathbf{0} & F \end{bmatrix} \begin{bmatrix} x_1' \\ z_1' \end{bmatrix}. \tag{22}$$

Based on (22), the matrix H_1 has r eigenvalues equal to those of F. Therefore, based on Theorem 2, SSS is enabled for system (3) using the controller (4) if and only if A_c is Hurwitz and H_i in (18) is Hurwitz for $i = 2, ..., N$. ■

Remark 5: Theorem 3 is a special case of Theorem 2 when the agents are identical. The necessary and sufficient conditions in Theorem 3 can be proved to be equivalent to those in Theorem 2 under (17). The condition that A_c is Hurwitz and H_i in (18) is Hurwitz for $i = 2, ..., N$ in Theorem 3 is equivalent to the condition that the other $(N-1)\,r$ eigenvalues of $\mathcal{H}$ have negative real parts in Theorem 2 when all the agents have the same dynamics (see (17)).

B. SSS Protocol Gain Matrix Design and Stability Analysis

Lemma 5: Under Assumptions 1 and 3, the N subsystems

$$\dot{w}_i = Fw_i - \sum a_{ij}(w_i - w_j), \quad i \in \{1, \cdots, N\} \tag{23}$$

synchronize, and the states $w_i\,(t)$ are *synchronized to the trajectory* given by

$$\begin{aligned} \dot{w}_0 &= Fw_0 \\ w_0(t_0) &= (b^T \otimes I_r)W(t_0), \end{aligned} \tag{24}$$

where $W \triangleq [w_1^T, ..., w_N^T]^T$.

Proof: The N subsystems (23) can be written as

$$\dot{W} = (I_N \otimes F - L \otimes I_r)W. \tag{25}$$

Make the following coordinate transformation:

$$W' = (T \otimes I_r)W, \tag{26}$$

where T is defined in (9) and $W' \triangleq [w_1'^T, ..., w_N'^T]^T$. Rewrite the vector W' as

$$W' = [w_1'^T,\ W_*'^T]^T. \tag{27}$$

System (25) can be rewritten in the new coordinate $W'(t)$ as

$$\dot{W}' = (I_N \otimes F - \Lambda \otimes I_r)W'. \tag{28}$$

Based on (26) and the definition of Λ in (10), (28) can be decomposed into the following two subsystems:

$$\dot{w}_1' = F w_1' \tag{29}$$

and

$$\dot{W}_*' = (I_{N-1} \otimes F - \Lambda_p \otimes I_r)W_*'. \tag{30}$$

The eigenvalues of $(I_{N-1} \otimes F - \Lambda_p \otimes I_r)$ are $\alpha_j - \lambda_i,\ j \in \{1, ..., r\},\ i \in \{2, ..., N\}$. According to the assumptions in the lemma, $\mathrm{Re}(\alpha_i) < \sigma$. Therefore, the real parts of $(\alpha_j - \lambda_i)$ are negative so that the matrix $(I_{N-1} \otimes F - \Lambda_p \otimes I_r)$ is Hurwitz. This proves that $W_*'(t)$ in (30) is exponentially stable.

Based on (26) and (27), $W = ([\mathbf{1},\ N_L] \otimes I_r)[w_1'^T,\ W_*'^T]^T$. Since $W_*'(t)$ in (30) is exponentially stable,

$$W(t) \to ([\mathbf{1},\ N_L] \otimes I_r)[w_1'^T(t),\ \mathbf{0}]^T = \mathbf{1} \otimes w_1'(t)$$

as $t \to \infty$. This shows that w_i's are synchronized to w_1'. The synchronized trajectory can be characterized by (29) and the following initial condition:

$$\begin{aligned} w_1'(t_0) &= \left([1,\ 0_{1\times(N-1)}] \otimes I_r\right) W'(t_0) \\ &= \left([1,\ 0_{1\times(N-1)}] \otimes I_r\right) (T \otimes I_r) W(t_0) \\ &= \left(\left([1,\ 0_{1\times(N-1)}][b,\ M_L]^T\right) \otimes I_r\right) W(t_0) \\ &= (b^T \otimes I_r) W(t_0). \end{aligned}$$

■

Lemma 6: For N agents with the following closed-loop dynamics:

$$\begin{aligned} \dot{x}_i &= A_{ci} x_i + B_i G_i \sum a_{ij}(w_i - w_j) \quad &(31) \\ \dot{w}_i &= F w_i - \sum a_{ij}(w_i - w_j), \quad &(32) \end{aligned}$$

suppose that Assumptions 1, 2, and 3 are satisfied and the gains K_i's are selected such that the matrices A_{ci}'s are Hurwitz. Then SSS is enabled exponentially for (31-32), i.e. the $x-$states are stabilized while the $w-$states are synchronized exponentially.

Proof: The proof can be obtained using Lemma 5 and the fact that A_{ci} is Hurwitz. ■

The system (31-32) can be rewritten in the collective form as

$$\begin{bmatrix} \dot{X} \\ \dot{W} \end{bmatrix} = \begin{bmatrix} \mathbf{A}_c & \mathbf{BG}(L \otimes I_r) \\ \mathbf{0} & (I_N \otimes F) - (L \otimes I_r) \end{bmatrix} \begin{bmatrix} X \\ W \end{bmatrix}. \tag{33}$$

In order to design a control law of the form (4) for (3), we make use of the system (31-32). More specifically, the gain matrices K_i, C_i, and G_i will be properly designed so that (8) can be related to (33). According to (6) and (31), a transformation is chosen as

$$W = Z + \mathbf{C}X, \tag{34}$$

where the gain matrix $\mathbf{C} = \bigoplus_{i=1}^N C_i$ will be designed later. Note that this transformation is invertible. Equation (33) in the coordinate (X, Z) becomes

$$\begin{bmatrix} \dot{X} \\ \dot{Z} \end{bmatrix} = \begin{bmatrix} \mathbf{A}_c + \mathbf{BG}(L \otimes I_r)\mathbf{C} & \mathbf{BG}(L \otimes I_r) \\ \mathbf{\Delta}_1 + \mathbf{\Delta}_2 \mathbf{C} & \mathbf{\Delta}_2 + (I_N \otimes F) \end{bmatrix} \begin{bmatrix} X \\ Z \end{bmatrix}, \tag{35}$$

where $\mathbf{\Delta}_1 = -\mathbf{C}\mathbf{A}_c + (I_N \otimes F)\mathbf{C}$ and $\mathbf{\Delta}_2 = -(\mathbf{CBG} + I)(L \otimes I_r)$.

According to (34), SSS is enabled for (33) if and only if SSS is enabled for (35). Under proper design of gain matrices, we will show that the SSS problem of (8) can be transformed to that of (35). The following theorem demonstrates how to design the gain matrices for the SSS purpose.

Theorem 4: Under Assumptions 1, 2, and 3, the gain matrices K_i, C_i, and G_i will be designed according to the following criterions:

A1. $A_{ci} = (A_i - B_i K_i)$ is Hurwitz and A_{ci} does not have common eigenvalues with F.

A2. C_i is the solution of

$$-C_i A_{ci} + F C_i = E_i. \tag{36}$$

A3. G_i is the right inverse of $-C_i B_i$ (i.e., $-C_i B_i G_i = I$).

Then, the control law (4) exponentially enables SSS for (3). Furthermore, the synchronized trajectory that z_i's converge to is characterized as

$$\begin{aligned} \dot{z}_0 &= F z_0 \\ z_0(t_0) &= (b^T \otimes I_r)\left[Z(t_0) + \mathbf{C}X(t_0)\right]. \end{aligned} \tag{37}$$

Proof: Under Assumption 2, there exists K_i such that $A_{ci} = (A_i - B_i K_i)$ is Hurwitz. During the design of K_i through pole placement, we can also tune K_i so that A_{ci} does not have common eigenvalues with F. Thus, *A1* can be easily satisfied. Based on *A1*, A_{ci} and F have no common eigenvalues. According to Lemma 2, there exists C_i such that (36) in *A2* is satisfied. This further indicates that $\mathbf{\Delta}_1 \equiv -\mathbf{C}\mathbf{A}_c + (I_N \otimes F)\mathbf{C} = \mathbf{E}$ based on (36). Selecting G_i as the right inverse of $-C_i B_i$ as stated in *A3* will make $(\mathbf{CBG} + I) = 0$. This indicates that $\mathbf{\Delta}_2 = -(\mathbf{CBG} + I)(L \otimes I_r) = \mathbf{0}$. Thus, the transformed system (35) will be the same as system (8). Based on Lemma 6, under Assumptions 1, 2, and 3, SSS is enabled for (31-32) (i.e., (33)) and therefore (35) according to the transformation (34). Thus, SSS is enabled for the closed-loop system (8). The synchronized trajectory that z_i's converge to can be formulated as (37) based on (24) and (34). ■

Remark 6: In Theorem 4, G_i is the right inverse of $-C_i B_i$. This requires that the rank of $-C_i B_i$ is equal to r. To fulfil this requirement, m_i must be greater than or equal to r, which requires that the number of inputs are not less than the dimension of the states that are required to synchronize.

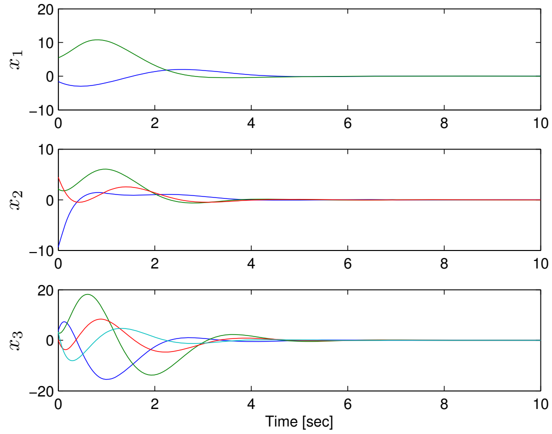

Fig. 2. The internal states are stabilized for all the agents.

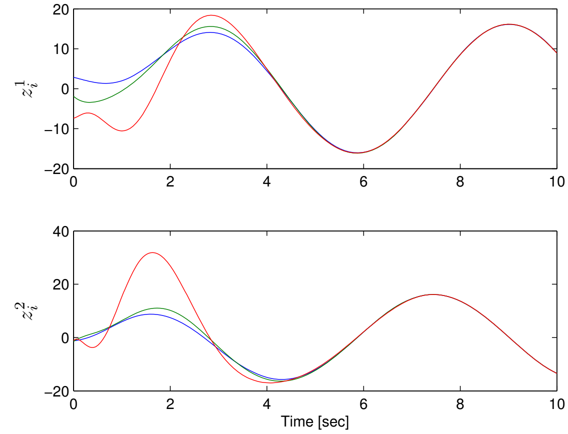

Fig. 3. The external states are synchronized for all the agents.

VI. Example and Simulations

Consider a group of non-identical agents with $N = 3$. The system matrices of the $i-$th agent's dynamics (i.e., (3)) are

$$A_i = \begin{bmatrix} 0 & 1^2 & \cdots & i^2 \\ -1 & 1 & \cdots & 0 \\ \vdots & 0 & \ddots & 0 \\ -i & 0 & 0 & i \end{bmatrix} \in \mathbb{R}^{(i+1)\times(i+1)},$$

$$B_i = \begin{bmatrix} 0 & i \\ 1 & \vdots \\ \vdots & 1 \\ i & 0 \end{bmatrix} \in \mathbb{R}^{(i+1)\times 2}, \ F = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix},$$

$$E_i = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & i \end{bmatrix} \in \mathbb{R}^{2\times(i+1)}.$$

The control gain K_i is selected so that the dominant eigenvalues of $A_{ci} = A_i - B_i K_i$ are $-1 \pm 2j$. The connection graph is a fixed directed graph with the Laplacian $L = [1, -1, 0; 0, 1, -1; 0, -1, 1]$. The initial conditions are selected randomly in $(-5, 5)$. The simulation results are presented in Fig. 2 and Fig. 3, which show that the internal states are stabilized and the external states are synchronized.

VII. Conclusions

We studied the simultaneous stabilization and synchronization (SSS) problem for one class of non-identical multi-agent systems. The agent dynamics are different and the dimensions of agent states are not necessarily equal. A distributed control law is designed based on local measurements and information exchanged from neighboring agents to enable SSS. The necessary and sufficient conditions to achieve this goal have been obtained using the closed-loop form of the system. Specific approaches of designing the control gain matrices are provided such that the sufficient conditions can be satisfied. A precise form of the synchronized trajectory has also been determined.

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