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Model predictive control of switched nonlinear systems under average dwell-time

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***Abstract*— In this paper, we propose a model predictive control (MPC) algorithm for switched nonlinear systems under average dwell-time switching signals. Assuming that a stabilizing MPC controller exists for each of the subsystems, we show that recursive feasibility of the repeatedly solved optimal control problem and asymptotic stability of the closed-loop switched system can be established if a certain average dwell-time condition is satisfied. If the switching times are unknown a priori and cannot be detected instantly, further application of the MPC controller calculated for the previously active subsystem might result in instability of the newly activated subsystem. It is shown that if the switches can be detected fast enough, then still ultimate boundedness in an arbitrarily small region around the origin can be ensured, i.e., the proposed MPC algorithm is inherently practically robust with respect to small errors in the detection of the switching times.**

I. Introduction

Over the last two decades, the concept of model predictive control (MPC) has gained more and more attention, both from a theoretical point of view as well as in various applications. In MPC, the control action is computed by minimizing a certain cost functional and the obtained optimal input is then applied until the next sampling instant, where the procedure is repeated again [1, 2]. One of the key advantages of MPC is that with this control scheme, input and state constraints can be explicitly taken into account. A crucial problem is to guarantee closed-loop stability for the controlled system, which e.g. can be achieved using a terminal region and terminal cost function approach [1–3]. Recently, MPC algorithms have been developed for more complex system classes, like hybrid systems (see, e.g., [4]), time-delay systems (see, e.g., [5]), or distributed systems (see, e.g., [6]).

Another important systems class which has gained a lot of attention recently is the class of switched systems (see, e.g., [7], and the references therein). Switched systems consist of a family of dynamical subsystems together with a switching signal specifying at each time the active subsystem dynamics. It is well known that a switched system does not necessarily inherit the properties of its constituent subsystems. For example, asymptotic stability of a switched system is not necessarily established for arbitrary switching signals even if all of the subsystems exhibit this property [7]. Thus when controlling a switched system, it is not necessarily enough to design a stabilizing controller for each of the subsystems. In order to ensure stability, different concepts in constraining the switching have been proposed, like multiple Lyapunov functions (introduced in [8]) with dwell-time [9] or average dwell-time [10] switching signals. We will make use of the latter approach in this paper.

Stabilization of switched nonlinear systems using MPC was considered in [11], where Lyapunov-based predictive controllers were developed for a switched system whose switching signal has to be an a priori known, prescribed schedule. Stability was established by using a variable prediction horizon (up to the next switching time) and by adding a transition constraint to the optimization problem, which is assumed to be feasible for a reasonably chosen switching schedule. These results were extended in [12] to the case where the switching times are not exactly known, but only to lie within certain a priori known small intervals. In [13], the authors developed a robust model predictive controller for discrete-time switched linear systems where the switching signal is a design parameter which can be optimized in order to ensure closed-loop stability. In [4], model predictive control of discrete-time piecewise affine systems was considered, which can be seen as a special case of switched systems where the switching signal is state-dependent.

In this paper, we consider the stabilization of switched nonlinear systems with a general, time-dependent switching signal via MPC. In contrast to [11, 12], the switching signal does not have to be known a priori, and cannot necessarily be used as a design parameter as in [13]. In particular, we design an MPC controller for each of the subsystems, and asymptotic stability of the closed-loop is established for all switching signals satisfying a certain average dwell-time condition. To this end, we will exploit some of the ideas presented in [14], where an MPC algorithm for continuous-time (non-switched) systems was developed which included switches between different cost functionals in order to improve performance. When considering switching signals with a priori unknown switching times, a major problem is how recursive feasibility of the repeatedly solved optimization problem can be ensured. The problem is that feasibility for all times cannot necessarily be guaranteed from initial feasibility as in the standard case, because the feasibility regions for the different subsystems are in general not identical. Nevertheless, we show how recursive feasibility can be established for a certain set of initial conditions, if again the switching signal satisfies a certain average dwell-

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time condidion. Furthermore, we show that the proposed MPC algorithm for switched systems is inherently practically robust with respect to small errors in the detection of the switching times. This means that recursive feasibility and ultimate boundedness of the closed-loop trajectory in a region around the origin can still be ensured if the switching times cannot be detected instantly, but only within a small interval after their occurrence.

The remainder of this paper is structured as follows. In Section II, preliminaries and the considered setup are described. Section III contains the main results, including the proposed MPC algorithm for switched systems as well as feasibility and stability considerations. Section IV shows the robustness property of the proposed algorithm with respect to delays in the switch detection times. Some concluding remarks are given in Section V.

II. PRELIMINARIES AND SETUP

Let $\mathbb{R}$ denote the field of real numbers. For any vector $x \in \mathbb{R}^n$, let $\|x\|$ denote an arbitrary p-norm. For a set $A \in \mathbb{R}^n$, denote its boundary by ∂A. A function α: $[0, \infty) \rightarrow [0, \infty)$ is of *class* $\mathcal{K}$ if α is continuous, strictly increasing, and $\alpha(0) = 0$. If α is also unbounded, it is of *class* $\mathcal{K}_\infty$.

Consider a family of subsystems

$$\dot{x} = f_p(x, u) \qquad p \in \mathcal{P} \tag{1}$$

where the state $x \in \mathbb{R}^n$, the input $u \in \mathbb{R}^m$ and $\mathcal{P}$ is a finite index set. For every $p \in \mathcal{P}$, $f_p(\cdot, \cdot)$ is locally Lipschitz and $f_p(0, 0) = 0$, i.e., the origin is an equilibrium of the undriven system. A *switched system*

$$\dot{x} = f_\sigma(x, u) \tag{2}$$

is generated by the family of subsystems (1) and a switching signal $\sigma(\cdot)$, where $\sigma : [0, \infty) \rightarrow \mathcal{P}$ is a piecewise constant, right continuous function which specifies at each time t the index of the active subsystem.

According to [10] we say that a switching signal has *average dwell-time* τ_a if there exist numbers $N_0, \tau_a > 0$ such that

$$\forall T \geq t \geq 0 : \qquad N_\sigma(T, t) \leq N_0 + \frac{T - t}{\tau_a}, \tag{3}$$

where $N_\sigma(T, t)$ is the number of switches occurring in the interval $(t, T]$.

Denote the switching times in the interval $(0, t]$ by $\tau_1, \tau_2, ..., \tau_{N_\sigma(t,0)}$ (by convention, $\tau_0 := 0$) and the index of the system that is active in the interval $[\tau_i, \tau_{i+1})$ by p_i.

Our goal is to asymptotically stabilize the origin of the switched system (2) via MPC. We will do this under the assumption that there exists a stabilizing MPC controller for each of the subsystems. For this purpose, let each of the subsystems of the family (1) be associated with a certain performance criterion J_p to be minimized. In order to design a stabilizing MPC controller for the subsystem with index p, consider the following finite horizon open-loop optimal control problem:

Problem 1: *At time t_k with $x := x(t_k) \in \mathcal{X}$, solve the optimization problem*

$$\underset{\overline{u}(\cdot)}{\text{minimize}} \; J_p(x, \overline{u}(\cdot)) = \int_{t_k}^{t_k+T} L_p(\overline{x}(\tau; t_k), \overline{u}(\tau)) d\tau + F_p(\overline{x}(t + T; t_k)) \tag{4}$$

subject to

$$\begin{aligned} &\dot{\overline{x}}(\tau; t_k) = f_p(\overline{x}(\tau; t_k), \overline{u}(\tau)) \\ &\overline{x}(t_k; t_k) = x \\ &\overline{u}(\tau) \in \mathcal{U} \qquad \forall\, \tau \in [t_k, t_k + T] \\ &\overline{x}(\tau; t_k) \in \mathcal{X} \qquad \forall\, \tau \in [t_k, t_k + T] \\ &\overline{x}(t_k + T; t_k) \in \mathcal{X}_p^f. \end{aligned} \tag{5}$$

In Problem 1, $\overline{x}(\cdot; t_k)$ denotes the predicted state trajectory over the prediction horizon T, with initial condition x at time t_k, i.e., $\overline{x}(t_k; t_k) = x$. The sets $\mathcal{U} \subset \mathbb{R}^m$ and $\mathcal{X} \subseteq \mathbb{R}^n$ are the input and state constraint sets, respectively, and contain the origin in their interior. Furthermore, $\mathcal{U}$ is assumed to be compact. The terminal region $\mathcal{X}_p^f \subseteq \mathcal{X}$ is a compact set which contains the origin in its interior. The terminal predicted state $\overline{x}(t_k + T; t_k)$ is required to lie inside this set, which is used to establish asymptotic stability. The stage cost L_p is assumed to be continuous in (x, u) and positive definite, i.e., there exists a function $\alpha_p \in \mathcal{K}_\infty$ such that $L_p(x, u) \geq \alpha_p(\|x\|)$ for all $u \in \mathcal{U}$. The terminal cost F_p is assumed to be a continuously differentiable, positive definite function.

Denote the optimal input and state trajectories obtained by solving Problem 1 by $\overline{u}_p^*(\tau; t_k)$ and $\overline{x}_p^*(\tau; t_k)$, respectively, for $t_k \leq \tau \leq t_k + T$, where the subscript p indicates that the optimal input and state trajectories were obtained by minimizing the cost functional J_p for the subsystem with index p. Furthermore, denote the optimal value of the cost functional J_p by $V_p(x) := J_p(x, \overline{u}_p^*)$, and the set of all the states for which Problem 1 has a solution, i.e., is feasible, by $\mathcal{X}_{p,T} \subseteq \mathcal{X}$. In the MPC setup, the control input is defined in the usual receding horizon fashion: only the first part of the computed input trajectory $\overline{u}_p^*$ up to the next sampling instant t_{k+1} is applied to the subsystem, i.e.,

$$u(\tau) = \overline{u}_p^*(\tau; t_k) \qquad t_k \leq \tau < t_{k+1}, \tag{6}$$

and then the procedure is repeated again.

In order to guarantee asymptotic stability of the (non-switched) subsystem with index p in closed loop with the implicit feedback controller defined by (6), the following assumption is usually made (see, e.g. [1],[3]):

Assumption 1: Suppose there exists an auxiliary local control law $u = k_p^{loc}(x)$, such that the terminal region $\mathcal{X}_p^f$ is invariant with respect to the closed loop $\dot{x} = f_p(x, k_p^{loc}(x))$, and the following holds:

$$k_p^{loc}(x) \in \mathcal{U} \qquad \forall\, x \in \mathcal{X}_p^f \tag{7}$$

$$(\dot{F}_p + L_p)(x, k_p^{loc}(x)) \leq 0 \qquad \forall\, x \in \mathcal{X}_p^f. \tag{8}$$

Note that (8) implies the invariance condition for $\mathcal{X}_p^f$, if this set is chosen as a level set of the terminal cost function F_p.

Remark 1: In the literature, many procedures have been proposed on how to design the auxiliary control law $k_p^{loc}(x)$ as well as the terminal region $\mathcal{X}_p^f$ and the terminal cost function F_p such that Assumption 1 is satisfied; see, e.g., [2, 3, 15, 16]. □

III. MODEL PREDICTIVE CONTROL OF SWITCHED SYSTEMS

In this section, we propose a model predictive control algorithm for switched systems and show under what conditions it is feasible for all times and asymptotically stabilizes the origin of the considered switched system.

The proposed MPC algorithm for switched systems is specified as follows:

Algorithm 1: *MPC for switched systems.*

0) *Initialization: Set* $i = k = 0$ *and* $\tau_0 = t_0 = 0$. *Determine* $p_0 = \sigma(0)$.
1) *At time instant* t_k*, measure the state* $x(t_k)$.
2) *Solve Problem 1 using the subsystem dynamics and the cost functional with index* p_i.
3) *Apply the first part of the computed optimal control input* $u(\tau) = \overline{u}^*_{p_i}(\tau; t_k)$, $t_k \leq \tau < t_{k+1}$. *The next sampling instant* t_{k+1} *is determined as* $t_{k+1} := \min\{t_k + \delta, \tau_{i+1}\}$.
4) *If* $t_{k+1} = \tau_{i+1}$*, let* $i := i + 1$.
5) *Let* $k := k + 1$ *and go to 1).*

Remark 2: In Algorithm 1, the switching times τ_i don't have to be known a priori, but it is assumed that they can be detected instantly. Namely, according to Step 3 of the algorithm, the computed optimal control input at time t_k is applied until either δ time units have passed or a switch occurs, and then Problem 1 is resolved. This means that the time inbetween two sampling instances t_k and t_{k+1} is smaller than the "nominal" sampling interval length δ, whenever a switch occurs. We will show in Section IV how the assumption that switches can be detected instantly can be relaxed. □

Algorithm 1 yields the control law

$$u(\tau) = k_\sigma^{MPC}(\tau) := \overline{u}^*_{p_i}(\tau; t_k), \qquad \tau_i \leq t_k \leq \tau < t_{k+1} \leq \tau_{i+1},$$

and hence the switched closed-loop system

$$\dot{x} = f_\sigma(x, k_\sigma^{MPC}). \tag{9}$$

In the following, we will show that the proposed MPC algorithm is recursively feasible for a certain set of initial conditions and that the closed-loop system (9) is asymptotically stable if σ is a switching signal with average dwell-time larger than some constant. We will start in Section III-A with proving asymptotic stability of the closed-loop system (9), which is done under the assumption that at every sampling instant t_k, Problem 1 in Step 2 of Algorithm 1 is feasible. We will then discuss in Section III-B how this assumption can be satisfied for a certain set of initial states. For proving asymptotic stability, we make a compatibility assumption on the optimal value functions V_p, and we require that in between switching times, the optimal value function of the active subsystem decays exponentially. These two assumptions are common in the switched systems literature in the setting of multiple Lyapunov functions and average dwell-time switching signals [7]. We will comment in Subsection III-C on how these two assumptions can be satisfied in the MPC context.

A. Stability

The following Theorem considers closed-loop stability when applying Algorithm 1.

Theorem 1: *Suppose that Assumption 1 holds for all* $p \in \mathcal{P}$*, and that at every sampling instant* t_k*, Problem 1 in Step 3 of Algorithm 1 is feasible. Furthermore, assume that there exist constants* $\mu \geq 1$ *and* $\lambda_s > 0$ *such that for all* $x \in \mathcal{X}_T$ *and all* $p, q \in \mathcal{P}$ *we have*

$$V_p(x) \leq \mu V_q(x), \tag{10}$$

and along the closed-loop trajectories of (9) *it holds that*

$$V_{p_i}(x(t_2)) - V_{p_i}(x(t_1)) \leq -\lambda_s \int_{t_1}^{t_2} V_{p_i}(x(s))ds \qquad \tau_i \leq t_1 \leq t_2 < \tau_{i+1}, \quad \forall i = 0, 1, \ldots \tag{11}$$

Then the closed-loop system (9)*, obtained by applying Algorithm 1, is asymptotically stable, if* $\sigma(\cdot)$ *is a switching signal with average dwell-time*

$$\tau_a > \frac{\ln \mu}{\lambda_s}. \tag{12}$$

□

Before proving Theorem 1, we will shortly comment on the assumptions made herein.

Remark 3: Assumption (10) implies that the optimal value functions somehow have to be compatible; this condition is quite common in the considered setting of multiple Lyapunov functions for a switched system (see, e.g., [10]), and ensures that the ratio of the new value function to the old one at the switching times is bounded by μ. We will show in Section III-C how this assumption can be satisfied in the MPC context. □

Remark 4: For the case where the optimal value function $V_p(x)$ is continuously differentiable in x, condition (11) transforms into

$$\dot{V}_{p_i}(x(t)) \leq -\lambda_0 V_{p_i}(x(t)), \qquad \forall\ \tau_i \leq t < \tau_{i+1}, \quad \forall i = 0, 1, \ldots \tag{13}$$

which is usually used in the switched systems literature (cf. [10]). However, as the optimal value function is often not continuously differentiable or even discontinuous due to the input and state constraints, we use the more general form (11). Either way, the decay rate of V_{p_i} during the interval $[\tau_i, \tau_{i+1})$ is less than or equal to an exponential decay rate, and thus it holds that

$$V_{p_i}(x(\tau_{i+1})) \leq e^{-\lambda_s(\tau_{i+1} - \tau_i)} V_{p_i}(x(\tau_i)). \tag{14}$$

□

Remark 5: Note that (11) only has to hold for the subsystem with index p_i which is active during the time interval $[\tau_i, \tau_{i+1})$. For all other $p \in \mathcal{P}$, V_p also can increase in this time interval. We will show in Section III-C how this assumption can be satisfied if Assumption 1 holds for all $p \in \mathcal{P}$. □

Proof of Theorem 1: To prove asymptotic stability of the closed-loop system (9), obtained by applying Algorithm 1, consider the function $V_{\sigma(t)}(x(t))$. According to Remark 4, for any two switching times τ_i, τ_{i+1}, equation (14) is satisfied. Thus, together with (10) we obtain

$$\begin{aligned} V_{\sigma(\tau_{i+1})}(x(\tau_{i+1})) &\leq \mu V_{\sigma(\tau_i)}(x(\tau_{i+1})) \\ &\leq \mu e^{-\lambda_0(\tau_{i+1}-\tau_i)} V_{\sigma(t_i)}(x(\tau_i)) \end{aligned}$$

Iterating this inequality from $i = 0$ to $i = N_\sigma(t, 0)$ and using (3) yields [10]

$$\begin{aligned} V_{\sigma(t)}(x(t)) &\leq \mu^{N_\sigma(t,0)} e^{-\lambda_0 t} V_{\sigma(0)}(x_0) \\ &= e^{N_\sigma(t,0)\ln(\mu) - \lambda_0 t} V_{\sigma(0)}(x_0) \\ &\leq \mu^{N_0} e^{(\ln(\mu)/\tau_a - \lambda_0)t} V_{\sigma(0)}(x_0) \\ &\leq \mu^{N_0} e^{-\lambda t} V_{\sigma(0)}(x_0) =: c e^{-\lambda t} V_{\sigma(0)}(x_0) \end{aligned} \quad (15)$$

for some $\lambda \in (0, \lambda_s)$ if the average dwell-time τ_a satisfies the bound (12). As $V_p(0) = 0$ and $L_p(x, u) \geq \alpha_p(\|x\|)$ for all $p \in \mathcal{P}$, $x \in \mathcal{X}_T$ and $u \in \mathcal{U}$, there exists a function $\alpha_1 \in \mathcal{K}_\infty$ such that $V_p(x) \geq \alpha_1(\|x\|)$ for all $p \in \mathcal{P}$ and $x \in \mathcal{X}_T$. Thus, (15) translates into

$$\|x(t)\| \leq \alpha_1^{-1}\big(c e^{-\lambda t} V_{\sigma(0)}(x_0)\big), \quad (16)$$

which proves that $x(t)$ asymptotically converges to the origin. Asymptotic stability can then be concluded from this and the fact that for all $p \in \mathcal{P}$, V_p is continuous at the origin [3], which implies that for any $\varepsilon > 0$ we can find a $\delta > 0$ such that $\|x_0\| \leq \delta \;\Rightarrow\; \|x(t)\| \leq \varepsilon$ for all $t \geq 0$ according to (16). □

B. Feasibility

Theorem 1 proves asymptotic stability of the closed-loop system (9) under the assumption that at every sampling instant t_k, Problem 1 in Step 2 of Algorithm 1 is feasible. In the following, we will show how to calculate a set of initial states for which this assumption holds. First, note that inbetween two switches, standard arguments can be used to prove recursive feasibility, i.e., by showing that the endpiece of the optimal input calculated at the previous sampling instant concatenated with the auxiliary local control law is a feasible input [3]. However, in general this is not true anymore at the switching times τ_i, as Problem 1 has to be recalculated for a different subsystem and a different performance criterion than at the previous sampling instant. Hence the major problem is that the regions $\mathcal{X}_{p,T}$, for which Problem 1 for the subsystems with index $p \in \mathcal{P}$ are feasible, are not necessarily identical.

However, recursive feasibility of Algorithm 1 can be established by other means. Namely, denote by $\mathcal{X}_T$ the set of all initial states for which Algorithm 1 is feasible for all times. Clearly, $\mathcal{X}_T \subseteq \cup_{p\in\mathcal{P}} \mathcal{X}_{p,T}$. Define V_{max} as

$$V_{max}(x_0) := c V_{\sigma(0)}(x_0). \quad (17)$$

According to (15), it holds that

$$V_{\sigma(t)}(x(t)) \leq V_{max}(x_0) \qquad \forall t \geq 0, \qquad \forall x_0 \in \mathcal{X}_T. \quad (18)$$

Thus recursive feasibility of Algorithm 1 can be established if for all $p \in \mathcal{P}$, the sublevel set $\Omega_p(x_0) := \{x \in \mathcal{X} : V_p(x) \leq V_{max}(x_0)\}$ is contained in $\mathcal{X}_{p,T}$, as then the closed-loop trajectory (9) is inside the feasible set $\mathcal{X}_{\sigma(t),T}$ for all $t \geq 0$. We thus arrive at the following result:

Lemma 1: *Suppose Assumption 1 holds and* (10)–(12) *are satisfied. Then*

$$\Psi_T := \{x_0 \in \mathcal{X} : \Omega_p(x_0) \subseteq \mathcal{X}_{p,T},\ \forall p \in \mathcal{P}\} \subseteq \mathcal{X}_T. \quad (19)$$

□

Remark 6: The estimate Ψ_T, obtained via (19), might be quite conservative. A larger set $\bar{\Psi}_T \supseteq \Psi_T$ of initial states for which Algorithm 1 is feasible for all times can be found as follows. Suppose that in the interval $[0, \tau_s]$, no switches occur, i.e., $N_\sigma(t, 0) = N_\sigma(t, \tau_s)$ for all $t \geq \tau_s$. Then, by following the same steps as in (15), we obtain

$$\begin{aligned} V_{\sigma(t)}(x(t)) &\leq e^{-\lambda_s t} V_{\sigma(0)}(x_0) \qquad t \in [0, \tau_s) \\ V_{\sigma(t)}(x(t)) &\leq \bar{c} e^{-\lambda t} V_{\sigma(0)}(x_0) \qquad t \geq \tau_s \end{aligned} \quad (20)$$

where

$$\bar{c} := \mu^{(N_0 - \frac{\tau_s}{\tau_a})} = c\mu^{-\frac{\tau_s}{\tau_a}} < c.$$

Now define $\bar{V}_{max}(x_0) := \bar{c} V_{\sigma(0)}(x_0)$ and $\bar{\Omega}_p(x_0) := \{x \in \mathcal{X} : V_p(x) \leq \bar{V}_{max}(x_0)\}$. Furthermore, analogously to (19), let

$$\tilde{\Psi}_T = \{x_0 \in \mathcal{X} : \bar{\Omega}_p(x_0) \subseteq \mathcal{X}_{p,T},\ \forall p \in \mathcal{P}\}.$$

However, the set $\tilde{\Psi}_T$ is not necessarily contained in $\mathcal{X}_T$, as it is not necessarily contained in the union of the sets $\mathcal{X}_{p,T}$. This is the case because $\bar{c}$, in contrast to c, is possibly smaller than 1, and $V_p(x(t)) \leq \bar{c} V_{max}(x_0)$ is only valid for $t \geq \tau_s$, according to (20). However, as for $t \in [0, \tau_s]$ no switches occur, $x(t)$ stays in the set $\mathcal{X}_{\sigma(0)}$ during this time interval. Thus an estimate $\bar{\Psi}_T$ of the set of initial states for which Algorithm 1 is feasible for all times is given by the intersection of the set $\tilde{\Psi}_T$ with the union of the sets $\mathcal{X}_{p,T}$, i.e.,

$$\bar{\Psi}_T = \tilde{\Psi}_T \cap \{\cup_{p\in\mathcal{P}} \mathcal{X}_{p,T}\} \supseteq \Psi_T. \quad (21)$$

□

Remark 7: The set Ψ_T in (19), and analogously also $\bar{\Psi}_T$ in (21), might be difficult to calculate. However, determining whether a given initial condition x_0 together with an initial index $p_0 = \sigma(0)$ is feasible can be done as follows. Namely, compute $V_{max}(x_0)$ according to (17). Then, if for all points $x \in \partial\mathcal{X}_{p,T}$ we have $V_p(x) \leq V_{max}(x_0)$, it follows that $\Omega_p(x_0) \subseteq \mathcal{X}_p$. If this holds for all $p \in \mathcal{P}$, the given initial contition is feasible and hence belongs to the set Ψ_T. □

Remark 8: There are other situations than the one described in Remark 6, in which a less conservative estimate for the set $\mathcal{X}_T$ can be found. Namely, if the sets $\mathcal{X}_{p,T}$ coincide for all $p \in \mathcal{P}$, which e.g. is the case if all states in the state constraint set $\mathcal{X}$ are feasible for all $p \in \mathcal{P}$, i.e., $\mathcal{X}_{p,T} = \mathcal{X}$, recursive feasibility follows from initial feasibility by standard arguments. Also, if there is some freedom in the design of the switching signal as in [13], this can be used to construct a larger estimate of the set of initially feasible states. □

C. Matching value functions with exponential decay rates

In this section, we show how the constant μ in the matching condition (10) for the Lyapunov functions as well as the exponential decay rate λ_s for the optimal value functions in (11) can be found in the MPC context. Due to space restrictions, only the ideas are presented, but some of the details in the calculation are left out.

A procedure on how to calculate λ_s was presented in [14]. The idea is that inbetween switching times, by following standard MPC stability proofs (see, e.g., [3]), one obtains that along the closed-loop trajectory of (9), the following is valid:

$$V_{p_i}(x(t_2)) - V_{p_i}(x(t_1)) \leq -\int_{t_1}^{t_2} L_{p_i}\big(x(s), k_\sigma^{MPC}(s)\big)ds$$
$$\forall\ \tau_i \leq t_1 \leq t_2 < \tau_{i+1}, \qquad \forall i = 0, 1, \ldots$$

Thus, if one can show that for all $x \in \mathcal{X}_p$ and all $u \in \mathcal{U}$ it holds that

$$L_p(x, u) \geq \lambda_{0,p} V_p(x) \tag{22}$$

for all $p \in \mathcal{P}$ and some $\lambda_{0,p} > 0$, we obtain the desired result (11) with

$$\lambda_0 = \min_p \lambda_{0,p}. \tag{23}$$

Equation (22) is established by finding some function $T_p(x)$ which is both an upper bound for $\lambda_{0,p} V_p(x)$ and a lower bound for $L_p(x, u)$, i.e., $L_p(x, u) \geq T_p(x) \geq \lambda_{0,p} V_p(x)$. In [14], two possibilities of how to determine such a function $T_p(x)$ are described, either through worst case estimates, or, less conservative, if some feasible input trajectory is known such that the stage cost L_p can be upper bounded along the corresponding predicted trajectory. For further details, the reader is referred to [14].

A similar idea can be used to find the constant μ in (10). Namely, if for all $p \in \mathcal{P}$ it holds that $V_p \leq \mu_{p,2} W(x)$ and $V_p \geq \mu_{p,1} W(x)$ for some function $W(x)$, then for any $p, q \in \mathcal{P}$ we obtain

$$V_p(x) \leq \mu_{p,2} W(x) \leq \frac{\mu_{p,2}}{\mu_{q,1}} V_q(x). \tag{24}$$

Thus (10) is satisfied with

$$\mu := \max_{p,q \in \mathcal{P},\ p \neq q} \frac{\mu_{p,2}}{\mu_{q,1}}.$$

The function $W(x)$ and the constants $\mu_{p,2}$ and $\mu_{p,1}$ can be found as follows. An upper bound for V_p can be found as described above. For the lower bound $V_p \geq \mu_{p,1} W(x)$, note that $V_p(x)$, i.e., the optimal value of the cost functional for the constrained finite horizon optimal control problem (4)–(5), is greater or equal than the optimal value $V_p^{uc}(x)$ of the cost functional for an unconstrained finite horizon optimization problem. Thus V_p^{uc} is a lower bound for V_p. For linear systems, for example, V_p^{uc} is given by $V_p^{uc} = x' P_p(0) x$, where P_p is the solution to the corresponding Riccati differential equation [17]. If an expression of $V_p^{uc}(x)$ cannot be found, a more conservative lower bound for V_p can be found via worst-case estimates, similar to the procedure described in [14] for worst-case estimates for an upper bound.

Remark 9: Note that in order to satisfy the compatibility condition (10) via the above described procedure, the function $W(x)$ in (24) has to be the same for all $p \in \mathcal{P}$. This can be fulfilled if the cost functionals J_p are of the same type and thus compatible, i.e., if they are e.g. all quadratic or quartic. □

IV. DELAY IN SWITCH DETECTION

As mentioned above, it might be unrealistic to assume that a switch between two subsystems occurring at time τ_i can be detected instantly, if the switching signal is not known a priori. Rather, the time τ_i' when this switch can be detected, lies within a certain interval $[\tau_i, \tau_i + \varepsilon]$, for some $\varepsilon > 0$.

Remark 10: For clarity of presentation, we assume that all switches can be detected, and furthermore in order of their occurrence. However, the subsequent results also hold for the case where some switches of a fast switching sequence, where several switches occur within an interval of length ε, cannot be detected at all. □

In the following, we will show that Algorithm 1 still works if the true switching times τ_i are replaced by the switch detection times τ_i', i.e., Problem 1 is not resolved as soon as a switch occurs, but only as soon as it is detected. However, in this case not asymptotic stability of the closed-loop can be established, but only ultimate boundedness in an (arbitrarily small) region around the origin. This means that Algorithm 1 is inherently practically robust with respect to errors in the detection of the switching times. The modified algorithm is thus given as follows:

Algorithm 2: *Execute Algorithm 1 with τ_{i+1} in Step 3 and 4 of Algorithm 1 substituted by τ_{i+1}'.*

When applying Algorithm 2, during the time intervals $[\tau_i, \tau_i')$ where the switch at time τ_i has not been detected yet, the optimal input trajectory computed for the previously active system is applied to the newly active system, which might result in an unstable closed-loop system. According to the above considerations, for each switch, the length of this time interval, given by $\kappa_i := \tau_i' - \tau_i$, is bounded above by $\kappa_i \leq \varepsilon$. With this it holds that for any $t \geq 0$, the fraction of the interval $[0, t]$ where the closed-loop system (9) is unstable, denoted by $T^u(t, 0)$, is bounded above by

$$T^u(t, 0) \leq \sum_{i=0}^{N_\sigma(t,0)} \kappa_i \leq N_\sigma(t, 0)\varepsilon. \tag{25}$$

The following Theorem considers feasibility and stability issues of the closed-loop system (2) when applying Algorithm 2, the proof of which is omitted in this conference paper for space reasons.

Theorem 2: *Suppose that Assumption 1 holds for all $p \in \mathcal{P}$. Furthermore, assume that there exist constants $\mu \geq 1$ and $\lambda_s, \lambda_u > 0$ such that for all $x \in \mathcal{X}_T$ and all $p, q \in \mathcal{P}$,* (10) *is satisfied, and along the closed loop trajectory of* (9) *it holds that*

$$V_{p_i}(x(t_2)) - V_{p_i}(x(t_1)) \leq -\lambda_s \int_{t_1}^{t_2} V_{p_i}(x(s))ds \qquad (26)$$
$$\tau_i' \leq t_1 \leq t_2 < \tau_{i+1}, \qquad \forall i = 0, 1, \ldots$$

and

$$V_{p_i}(x(t_2)) - V_{p_i}(x(t_1)) \leq \lambda_u \int_{t_1}^{t_2} V_{p_i}(x(s))ds \qquad (27)$$
$$\tau_i \leq t_1 \leq t_2 < \tau_i', \qquad \forall i = 0, 1, \ldots$$

If $\sigma(\cdot)$ is a switching signal with average dwell-time

$$\tau_a > \frac{\ln \mu + (\lambda_s + \lambda_u)\varepsilon}{\lambda_s}, \qquad (28)$$

then the closed-loop system (9)*, obtained by applying Algorithm 2, is asymptotically stable. If* (26)–(27) *only hold for all $\|x\| \geq \nu$ for some $\nu > 0$, then there exists a function $\gamma \in \mathcal{K}_\infty$ such that the closed-loop system* (9) *is ultimately bounded in $B_{\gamma(\nu)}(0)$*[1]*. Furthermore, an estimate of the set of initial states for which Algorithm 2 is feasible for all times, can be calculated as proposed in Lemma 1.* □

Remark 11: For the case when $\varepsilon = 0$, i.e., the switches can be detected instantly, (28) reduces to $\tau_a > \ln \mu / \lambda_s$, which is the result obtained in Theorem 1. Furthermore, for any switching signal with average dwell-time satisfying (12), we can find an $\varepsilon > 0$ such that also (28) is satisfied. One can show that for each $\nu > 0$, we can find λ_s and λ_u such that (26)–(27) hold for all $\|x\| \geq \nu$. This means that the closed-loop system can be ultimately bounded in an arbitrarily small region around the origin for small enough ε, i.e., the proposed MPC algorithm for switched systems is inherently practically robust with respect to errors in the detection of the switching times. □

Remark 12: The setup of Theorem 2 can be viewed as a special case of the setting in [18], where asymptotic stability for switched systems including unstable subsystems was considered. There, asymptotic stability was proven for the case that the average dwell-time satisfies the inequality

$$\tau_a > \frac{\ln \mu}{\lambda_s(1-\rho) - \lambda_u \rho}, \qquad (29)$$

where $\rho > 0$ is a constant such that $\rho < \lambda_s/(\lambda_s + \lambda_u)$ and such that $T^u(t, 0) \leq \tau_0 + \rho t$ for some $\tau_0 > 0$. For the setting considered in this paper, we obtain from (25) that $T^u(t, 0) \leq N_\sigma(t, 0)\varepsilon \leq N_0\varepsilon + \frac{\varepsilon}{\tau_a} t =: \tau_0 + \rho t$. Plugging this into (29) and rearranging terms yields condition (28). □

[1]For $y \in \mathbb{R}^n$ and $a > 0$, denote by $B_a(y)$ the ball of radius a centered at y, i.e., $B_a(y) := \{x \in \mathbb{R}^n : \|x - y\| \leq a\}$.

V. Conclusion

In this paper, we proposed a novel MPC algorithm for switched nonlinear systems. We showed that if the switching signal satisfies a certain average dwell-time condition, asymptotic stability of the closed loop can be established. Furthermore, we illustrated how an estimate of the set of initially feasible states can be obtained. Finally, we showed that the proposed MPC algorithm is inherently practically robust with respect to small errors in the detection of the switching times.

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