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Adaptive estimation in nonlinearly parameterized nonlinear dynamical systems

Veronica Adetola, Devon Lehrer and Martin Guay

Abstract— **The paper presents a new technique for the adaptive parameter estimation in nonlinear parameterized dynamical systems. The technique proposes an uncertainty set-update approach that guarantees forward invariance of the true value of the parameters. In addition, it is shown that in the presence of sufficiently exciting state trajectories, the parameter estimates converge to the true values and the uncertainty set vanishes around the true value of the parameters. Two simulation examples are presented that demonstrate the effectiveness of the technique.**

I. INTRODUCTION

Parameter estimation in dynamical systems has been a central theme in systems research for decades. Starting from nonlinear regression techniques to the advent of nonlinear adaptive control, the problem has received considerable attention. Adaptive estimation of nonlinear systems remains a relatively unexplored field. Most existing design techniques are restricted to systems that are linear in the unknown (constant) parameters. Representative techniques are discussed in several references such as [9],[10], [11]. Work on nonlinearly parameterized systems remains scarce. The most significant approach to solve this problem can be found in several papers by Annaswamy and co-workers ([6], [4], [8],[12]). Their approach exploits convexity of the system dynamics with respect to the parameters to develop a class of min-max adaptive estimation routines. A gradient-based approach is proposed subject to a worst-case parameter set. Several authors have also studied this class of problems for specific applications. One such application is the study of microbial growth kinetics where most models, due to the importance of classical enzyme kinetics models, are nonlinearly parameterized ([5],[16]). The nonlinearity of these models prevents one from using normal techniques to establish parameter convergence. For Monod models, one can show that parameter convergence can be achieved subject to a conservative persistence of excitation condition that can only be derived using highly tailored Lyapunov based arguments. Another leading approach consists of approximating the nonlinearity using neural networks ([15],[13],[7]). The main drawbacks of these techniques is that such approximations cannot be used to uniquely reconstruct the unknown parameter vector.

A parameter estimation scheme that allows exact reconstruction of the unknown parameters in finite-time was developed in [1]. The finite-time (FT) identification method has two distinguishing features. The drawback of the identification algorithm is the requirement to check the invertibility of a matrix online and compute the inverse matrix when appropriate. To avoid these concerns and enhance the applicability of the FT method in practical situations, the procedure was exploited in [2] to develop a novel adaptive compensator that (almost) recovers the performance of the FT identifier. The compensator guarantees exponential convergence of the parameter estimation error at a rate dictated by the closed-loop system's excitation.

The ability to guarantee the rate of convergence in finite-time depends largely on the good knowledge of the process model. That is, one needs to have a good handle on how the unknown parameters enter the model. Finite-time convergence can only be ensured in the absence of exogenous disturbances and modelling error, if the parameters enter the model equation linearly. In the presence of exogenous variables, one must provide some mechanism to compensate for the effect of disturbances on parameter estimates and the ability to locate the unknown parameter values. In [3], a novel parameter estimate routine was developed for linearly parameterized systems in the presence of exogenous disturbances. The parameter estimation routine are used to update the parameter uncertainty set, at certain time instants, in a manner that guarantees non-expansion of the set leading to a gradual reduction in the conservativeness or computational demands of the algorithms.

In this paper, we propose an extension of the estimation technique proposed in [3] for nonlinearly parameterized nonlinear dynamical systems based on a similar set-update approach. The paper is organized as follows. The problem statement is given in Section 2. Section 3 presents the parameter and uncertainty set estimation routines. Two simulation examples are given in Section 4 followed by short conclusions in Section 5.

II. PROBLEM STATEMENT

Consider the uncertain nonlinear system

$$\dot{x} = f(x, u, \theta) \tag{1}$$

where $x \in \mathbb{R}^n$ is the vector of state variables, $u \in \mathbb{R}^m$ is the vector of input variables, $\theta \in \mathbb{R}^p$ is the vector of unknown parameters. The objective of this study is to develop a set-based parameter identification scheme for this class of systems. The parameter identifier simultaneously estimates the parameters and the parameter uncertainty set. The approach provides a new persistency of excitation condition that guarantees the convergence of the parameter identifier

M. Guay and D. Lehrer are with the Department of Chemical Engineering, Queen's University, Kingston, Ont., Canada K7L 3N6. *E-mail:* martin.guay@chee.queensu.ca, devon.lehrer@chee.queensu.ca.

V. Adetola is with United Technologies Research Centre, Silver Lane, Hartford, CT

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to the true value of the unknown parameters. It is assumed that θ is uniquely identifiable and lie within an initially known compact set $\Theta^0 = B(\theta_0, z_{\theta 0})$ where θ_0 is a nominal parameter value, $z_{\theta 0}$ is the radius of the initial parameter uncertainty set.

III. Parameter and Uncertainty Set Estimation

Let the parameter estimate is given by:

$$\theta = \theta_0 + \delta.$$

In the first stage of the identifier, one attempts to estimate δ. Let us consider the following state prediction system based on the estimate of δ about the nominal parameter value θ_0:

$$\dot{\hat{x}} = f(x, u, \theta_0 + \hat{\delta}) - \Delta\Psi(\theta_0, \hat{\delta})\hat{\delta} + k(x - \hat{x}) + c(t)^T \dot{\hat{\delta}} \tag{2}$$

where

$$\Delta\Psi(x, u, \theta_0, \hat{\delta}) = \int_0^1 \left(\frac{\partial f(x, u, \theta_0 + \lambda\hat{\delta})}{\partial\theta} \right) d\lambda - \frac{\partial f(x, u, \theta_0)}{\partial\theta},$$

$c(t)$ is a filter parameter whose dynamics is defined later and $k \succ 0$ is an $n \times n$ gain matrix.

The error dynamics are given by:

$$\dot{e} = \tilde{f}(x, u, \theta_0) + \Delta\Psi(\theta_0, \hat{\delta})\hat{\delta} - k_w e - c(t)^T \dot{\hat{\delta}}$$

The term $\tilde{f}(x, u, \theta_0) = f(x, u, \theta_0 + \delta) - f(x, u, \theta_0 + \hat{\delta})$ can be written as

$$\begin{aligned} \tilde{f}(x, u, \theta_0) &= f(x, u, \theta_0 + \delta) - f(x, u, \theta_0) \\ &\quad + f(x, u, \theta_0) - f(x, u, \theta_0 + \hat{\delta}) \end{aligned}$$

By the mean-value theorem, we get

$$\begin{aligned} \tilde{f}(x, u, \theta_0) &= \int_0^1 \left(\frac{\partial f(x, u, \theta_0 + \lambda\delta)}{\partial\theta} \right) d\lambda\delta \\ &- \int_0^1 \left(\frac{\partial f(x, u, \theta_0 + \lambda\hat{\delta})}{\partial\theta} \right) d\lambda\hat{\delta} \end{aligned}$$

Let,

$$\Psi(x, u, \theta_0, \delta) = \int_0^1 \left(\frac{\partial f(x, u, \theta_0 + \lambda\delta)}{\partial\theta} \right) d\lambda$$

and

$$\Psi(x, u, \theta_0, \hat{\delta}) = \int_0^1 \left(\frac{\partial f(x, u, \theta_0 + \lambda\hat{\delta})}{\partial\theta} \right) d\lambda$$

then we obtain,

$$\tilde{f}(x, u, \theta_0) = \Psi(x, u, \theta_0, \delta)\delta - \Psi(x, u, \theta_0, \hat{\delta})\hat{\delta} \tag{3}$$

One can define

$$\Psi(x, u, \theta_0, 0) = \int_0^1 \left(\frac{\partial f(x, u, \theta_0)}{\partial\theta} \right) d\lambda$$

and write (3) as

$$\begin{aligned} \tilde{f}(x, u, \theta_0) &= \Psi(x, u, \theta_0, \delta)\delta - \Psi(x, u, \theta_0, 0)\tilde{\delta} \\ &\quad + \Psi(x, u, \theta_0, 0)\tilde{\delta} - \Psi(x, u, \theta_0, \hat{\delta})\hat{\delta} \end{aligned} \tag{4}$$

We finally come to the following form (3):

$$\begin{aligned} \tilde{f}(x, u, \theta_0) &= \Delta\Psi(x, u, \theta_0, \delta)\delta + \Psi(x, u, \theta_0, 0)\tilde{\delta} \\ &\quad - \Delta\Psi(x, u, \theta_0, \hat{\delta})\hat{\delta} \end{aligned} \tag{5}$$

where

$$\Delta\Psi(x, u, \theta_0, \delta) = \int_0^1 \left(\frac{\partial f(x, u, \theta_0 + \lambda\delta)}{\partial\theta} \right) d\lambda - \frac{\partial f(x, u, \theta_0)}{\partial\theta}.$$

As a result the error dynamics are given by:

$$\dot{e} = \Delta\Psi(x, u, \theta_0, \delta)\delta + \Psi(x, u, \theta_0, 0)\tilde{\delta} - ke - c(t)^T \dot{\hat{\delta}}. \tag{6}$$

Let the filter dynamics be selected as

$$\dot{c} = \Psi(x, u, \theta_0, 0) - k_w c \qquad c(t_0) = 0. \tag{7}$$

resulting in state prediction error $e = x - \hat{x}$ and auxiliary variable $\eta = e - c^T \tilde{\delta}$ dynamics:

$$\dot{e} = \Delta\Psi(x, u, \theta_0, \delta)\delta + \Psi(x, u, \theta_0, 0)\tilde{\delta} - k_w e - c^T \dot{\hat{\delta}}$$

$$e(t_0) = x(t_0) - \hat{x}(t_0) \tag{8}$$

$$\dot{\eta} = -k_w \eta + \Delta\Psi(x, u, \theta_0, \delta)\delta \tag{9}$$

$$\eta(t_0) = e(t_0). \tag{10}$$

Since $\Delta\Psi(x, u, \theta_0, \delta)\delta$ is not known, an estimate of η is generated from

$$\dot{\hat{\eta}} = -k_w \hat{\eta}, \qquad \hat{\eta}(t_0) = e(t_0). \tag{11}$$

with resulting estimation error $\tilde{\eta} = \eta - \hat{\eta}$ dynamics

$$\dot{\tilde{\eta}} = -k_w \tilde{\eta} + \vartheta, \qquad \tilde{\eta}(t_0) = 0. \tag{12}$$

Let $\Sigma \in \mathbb{R}^{n_\theta \times n_\theta}$ be generated from

$$\dot{\Sigma} = cc^T, \qquad \Sigma(t_0) = \alpha I \succ 0, \tag{13}$$

based on equations (8), (7) and (11), the preferred parameter update law is given by

$$\dot{\Sigma}^{-1} = -\Sigma^{-1} cc^T \Sigma^{-1}, \qquad \Sigma^{-1}(t_0) = \frac{1}{\alpha} I \tag{14a}$$

$$\dot{\hat{\delta}} = \text{Proj}\left\{ \Sigma^{-1} c(e - \hat{\eta}),\ \|\hat{\delta}\| \leq z_{\theta 0} \right\}, \qquad \hat{\delta}(t_0) = 0 \tag{14b}$$

where $\text{Proj}\{\phi,\ \|\hat{\delta}\| \leq z_{\theta 0}\}$ denotes a Lipschitz projection operator such that

$$-\text{Proj}\{\phi,\ \|\hat{\delta}\| \leq z_{\theta 0}\}^T \tilde{\delta} \leq -\phi^T \tilde{\delta}, \tag{15}$$

$$\|\hat{\delta}(t_0)\| \leq z_{\theta 0} \Rightarrow \|\hat{\delta}(t)\| \leq z_{\theta 0},\ \forall t \geq t_0. \tag{16}$$

More details on parameter projection can be found in [10]. We first need the following assumptions.

Assumption 1: The state and input variables of the dynamical system evolve over a compact set $\mathbb{X} \times \mathbb{U} \in \mathbb{R}^n \times \mathbb{R}^p$.

Assumption 2: The map $f : \mathbb{X} \times \mathbb{U} \times \Theta_0 \to \mathbb{R}^n$ is continuously differentiable and the elements of the jacobian matrix $\frac{\partial f}{\partial\theta}$ is Lipschitz in θ uniformly on $\mathbb{X} \times \mathbb{U}$. That is, there exists a constant $L > 0$ such that

$$\left\| \frac{\partial f(x, u, \theta_1)}{\partial\theta} - \frac{\partial f(x, u, \theta_0)}{\partial\theta} \right\| \leq L\|\theta_1 - \theta_0\|,$$

$$\forall\theta \in \Theta_0, \quad \forall(x, u) \in \mathbb{X} \times \mathbb{U}.$$

Note that by Assumptions 1 and 2, we get that the uncertain term in (8) is such that

$$\|\Delta\Psi(x,u,\theta_0,\delta)\delta\| \le L\|\delta\|^2 \le Lz_{\theta 0}^2 = \vartheta. \qquad (17)$$

Lemma 1: The identifier (14) is such that, for every $\theta_0 \in \Theta_0$, the estimation error $\tilde{\delta} = \delta - \hat{\delta}$ and the auxiliary filter error $\tilde{\eta} = \eta - \hat{\eta}$ are bounded.

Proof: Consider the dynamics of $\tilde{\eta}$ and pose the Lyapunov function, $V_{\tilde{\eta}} = \frac{1}{2}\tilde{\eta}^T\tilde{\eta}$. The rate of change of $V_{\tilde{\eta}}$ gives,

$$\dot{V}_{\tilde{\eta}} = -k_w\tilde{\eta}^T\tilde{\eta} + \tilde{\eta}^T\Delta\Psi(x,u,\theta_0,\delta)\delta$$

such that

$$\dot{V}_{\tilde{\eta}} \le -k_w\|\tilde{\eta}\|^2 + \|\tilde{\eta}\|\vartheta.$$

As result we guarantee that $\dot{V}_{\tilde{\eta}} < 0\, \forall\tilde{\eta} \in \mathbb{R}^n$ such that $\|\tilde{\eta}\| > \frac{\vartheta}{k_w}$. Now since, $\tilde{\eta}(0) = 0$ by construction, then $\|\tilde{\eta}(t)\| \le \frac{\vartheta}{k_w}$, $\forall t$.

Let $V_{\tilde{\delta}} = \tilde{\delta}^T\Sigma\,\tilde{\delta}$, it follows from (14) and the relationship $c\tilde{\theta} = e - \hat{\eta} - \tilde{\eta}$ that

$$\begin{aligned} \dot{V}_{\tilde{\delta}} &\le -2\tilde{\delta}^T c(e-\hat{\eta}) + \tilde{\delta}^T cc^T\tilde{\delta} \\ &= -(e-\hat{\eta})^T(e-\hat{\eta}) + \|\tilde{\eta}\|^2 \\ &\le -(e-\hat{\eta})^T(e-\hat{\eta}) + \left(\frac{Lz_{\theta 0}^2}{k_w}\right)^2. \end{aligned} \qquad (18)$$

Since, $\|\tilde{\eta}\|^2$ is bounded it follows that $\|\tilde{\delta}\|^2$ is bounded. By the projection algorithm, we can always guarantee that

$$\|\tilde{\delta}\|^2 \le \|\delta\|^2 + \|\hat{\delta}\|^2 \le 4z_{\theta 0}^2.$$

Thus, keeping $\|\hat{\delta}\| < z_{\theta 0}$ ensures that $\|\tilde{\delta}\|^2 \le 4z_{\theta 0}^2$ as long as one can guarantee that $\|\delta\| < z_{\theta 0}$. ■

Using standard arguments, it is possible to provide a statement of convergence of $\tilde{\delta}$ to a neighbourhood of the origin. The following persistency of excitation assumption will be required.

Assumption 3: There exists positive constants $T > 0$ and $k_N > 0$ such that

$$\int_t^{t+T} c(\tau,\theta_0)^T c(\tau,\theta_0)d\tau \ge k_N(\theta_0), \quad \forall t \ge 0 I_N, \qquad (19)$$

$\forall\theta_0 \in \Theta_0$ where $k_N(\theta_0) > 0$, $c(t)$ is the solution of eq.(7), I_N is the N-dimensional identity matrix.

Remark 1: Assumption 3 considers the dependence of the filter parameter on the current value of the centre of the uncertainty set, θ_0. This is to highlight the fact that the regressor matrix $\Psi(x,u,\theta_0,0)$ depends on θ_0. Any update of the centre value θ_0 will cause a change or regressor vector and, consequently, a change of the filter parameter dynamics.

First, we consider the convergence properties of the estimates $\hat{\delta}$ at a fixed value θ_0. This can be viewed as a standard result in the design of robust adaptive observers which establishes convergence of the parameter estimates to a neighbourhood of the true value δ.

Lemma 2: Assume that the signals of the system (1) fulfill the persistency of excitation condition as stated in Assumption 3.Then, the parameter estimation scheme (2), (7), (11) and (14) is such that the parameter estimation error converges exponentially to a neighbourhood of the origin.

Proof: Note that the parameter estimation error dynamics is given by,

$$\dot{\tilde{\delta}} \;=\; -\mathrm{Proj}\left\{\Sigma^{-1}c(c^T\tilde{\delta} + \tilde{\eta}),\ \|\hat{\delta}\| \le z_{\theta 0}\right\}.$$

Consider the candidate Lyapunov function, $V_{\tilde{\delta}} = \tilde{\delta}^T\Sigma\tilde{\delta}$. It follows by the property of the projection algorithm that the rate of change of $V_{\tilde{\delta}}$ along the trajectories of the closed-loop system about the nominal parameter value θ_0 is

$$\begin{aligned} \dot{V}_{\tilde{\delta}} &\le -2\tilde{\delta}^T cc^T\tilde{\delta} - 2\tilde{\delta}^T c\tilde{\eta} + \tilde{\delta}^T cc^T\tilde{\delta} \\ &\le -\tilde{\delta}^T cc^T\tilde{\delta} - 2\tilde{\delta}^T c\tilde{\eta} \end{aligned} \qquad (20)$$

Consider the Lyapunov function, $W = V_{\tilde{\delta}} + \tilde{\eta}^T\tilde{\eta}$. Its rate of change is given by

$$\dot{W} \le -\tilde{\delta}^T cc^T\tilde{\delta} - 2\tilde{\delta}^T c\tilde{\eta} - k_w\tilde{\eta}^T\tilde{\eta} + \tilde{\eta}^T\Delta\Psi(x,\theta_0,\delta)\delta$$

Completing the squares, there exist positive constants k and k_w, $k_w > k > 1$, such that,

$$\begin{aligned} \dot{W} &\le -\left(1 - \frac{1}{k}\right)\tilde{\delta}^T cc^T\tilde{\delta} - (k_w - k)\tilde{\eta}^T\tilde{\eta} \\ &\quad + \tilde{\eta}^T\Delta\Psi(x,\theta_0,\delta) \\ &\le -k_1\tilde{\delta}^T cc^T\tilde{\delta} - k_2\tilde{\eta}^T\tilde{\eta} + k_3\tilde{\eta}^T\tilde{\eta} + \frac{1}{k_3}(Lz_{\theta 0}^2)^2 \end{aligned}$$

Thus for $k_2 > k_3$, one obtains

$$\dot{W} \le -k_1\tilde{\delta}^T cc^T\tilde{\delta} - k_4\tilde{\eta}^T\tilde{\eta} + \frac{1}{k_3}(Lz_{\theta 0}^2)^2.$$

As a consequence of Assumption (3), it follows that

$$\begin{aligned} \dot{W} &\le -\gamma_1 c_1 V_{\tilde{\delta}} - k_4\tilde{\eta}^T\tilde{\eta} + \frac{1}{k_3}(Lz_{\theta 0}^2)^2 \\ &\le -k_5 W + \frac{1}{k_3}(Lz_{\theta 0}^2)^2 \end{aligned}$$

which confirms that the parameter estimation error $\tilde{\delta}$ and the η-estimation error $\tilde{\eta}$ converges exponentially to a neighbourhood of the origin for any value of $\theta_0 \in \Theta_0$. ■

The analysis above guarantees the boundedness of the estimation error $\tilde{\delta}$ and its convergence to the origin subject to a persistency of excitation condition. This convergence result applies for any given value of the parameters θ_0. Thus, it remains to show that one can update the position of the centre, θ_0, of the uncertainty set Θ_0 to the true values of the parameters. The strategy considered in this paper is to provide an update mechanism for the uncertainty set $\Theta(\theta_0, z_{\theta 0})$ such that, as the centre of the uncertainty set, θ_0, and the radius of the uncertainty ball, $z_{\theta 0}$, are updated, the uncertainty set is guaranteed to contain the true value of the parameters, θ. The update mechanism monitors the shrinking of the uncertainty set using an upper bound on the Lyapunov function $V_{\tilde{\delta}}$ denoted by $V_{z\theta_0}$. When the shrinking of the set can be done in a way that guarantees containment of the unknown value of the parameters θ, the value of the centre is moved to the current parameter estimate (i.e., $\theta_0 + \hat{\delta}$) and the radius of the uncertainty set (or ball) is updated according to the decrease of the upper bound $V_{z\theta 0}$.

An update law that measures the worst-case progress of the parameter identifier in the presence of disturbance is given by:

$$
\begin{aligned}
z_\theta(t) &= \sqrt{\frac{V_{z\theta}(t)}{4\lambda_{\min}(\Sigma(t))}} && (21a)\\
V_{z\theta}(t_0) &= 4\lambda_{\max}\left[\Sigma(t_0)\right]\,(z_{\theta 0})^2 && (21b)\\
V_\eta(0) &= \|\tilde{\eta}(0)\|^2 = 0 && (21c)\\
\dot{V}_{z\theta} &= \begin{cases} 0 & \text{if } (e-\hat{\eta})^T(e-\hat{\eta}) \le V_\eta(t)\\ -(e-\hat{\eta})^T(e-\hat{\eta}) + V_\eta(t) & \text{otherwise}\end{cases} && (21d)\\
\dot{V}_\eta &= -k_w V_\eta + \left(\frac{L z_{\theta 0}^2}{k_w}\right)^2 && (21e)
\end{aligned}
$$

Using the parameter estimator (14) and its error bound z_θ (21), the uncertain ball $\Theta \triangleq B(\theta_0, z_\theta)$ is adapted online according to the following algorithm:

Algorithm 1: Beginning from time $t_{i-1} = t_0$, the parameter and set adaptation is implemented iteratively as follows:

1 **Initialize** $z_\theta(t_{i-1}) = z_{\theta 0}$, $\hat{\delta}(t_{i-1}) = 0$, $\hat{\theta}(t_{i-1}) = \theta_{i-1}$, $\hat{\eta}(t_{i-1}) = e(t_{i-1})$, $c(t_{i-1}) = 0$ and $\Theta(t_{i-1}) = B(\hat{\theta}(t_{i-1}), z_\theta(t_{i-1}))$.

2 At time t_i, using equations (14) and (21) **perform** the update

$$
(\theta_i,\ \Theta) = \begin{cases} \left(\theta_{i-1} + \hat{\delta}(t_i),\ \Theta(t_i)\right), & \text{if } z_\theta(t_i) \le z_\theta(t_{i-1}) - \|\hat{\delta}(t_i)\|\\ \left(\theta_{i-1},\ \Theta(t_{i-1})\right), & \text{otherwise}\end{cases} \tag{22}
$$

3 **Iterate back** to step 2, **incrementing** $i = i + 1$.

The algorithm ensure that Θ is only updated when z_θ value has decreased by an amount which guarantees a contraction of the set. Moreover z_θ evolution as given in (21) ensures non-exclusion of θ as shown below.

Lemma 3: The evolution of $\Theta = B(\hat{\theta}, z_\theta)$ under (14), (21) and algorithm 1 is such that

i) $\Theta(t_2) \subseteq \Theta(t_1),\ t_0 \le t_1 \le t_2$

ii) $\theta \in \Theta(t_0) \Rightarrow \theta \in \Theta(t),\ \forall t \ge t_0$

Proof:

i) If $\Theta(t_{i+1}) \nsubseteq \Theta(t_i)$, then

$$
\sup_{s\in\Theta(t_{i+1})} \|s - \hat{\theta}(t_i)\| \ge z_\theta(t_i). \tag{23}
$$

However, it follows from triangle inequality and algorithm 1 that Θ, at update times, obeys

$$
\begin{aligned}
&\sup_{s\in\,\Theta(t_{i+1})} \|s - \hat{\theta}(t_i)\|\\
&\quad\le \sup_{s\in\,\Theta(t_{i+1})} \|s - \hat{\theta}(t_{i+1})\| + \|\hat{\theta}(t_{i+1}) - \hat{\theta}(t_i)\|\\
&\quad\le z_\theta(t_{i+1}) + \|\hat{\theta}(t_{i+1}) - \hat{\theta}(t_i)\| \ \le\ z_\theta(t_i),
\end{aligned}
$$

which contradicts (23). Hence, Θ update guarantees $\Theta(t_{i+1}) \subseteq \Theta(t_i)$ and the strict contraction claim follows from the fact that Θ is held constant over update intervals $\tau \in (t_i, t_{i+1})$.

ii) We know that $V_{\tilde{\delta}}(t_0) \le V_{z\theta}(t_0)$ (by definition). The value of $V_{\tilde{\delta}}(t)$ at some $t > t_0$ is given by

$$
V_{\tilde{\delta}}(t) = V_{\tilde{\delta}}(t_0) + \int_{t_0}^{t} \dot{V}_{\tilde{\delta}}(\tau)d\tau
$$

Let t be such that the value of the uncertainty radius remains at $z_{\theta 0}$.

This means that, by projection algorithm, $\|\tilde{\delta}(t)\|^2 \le 4z_{\theta 0}^2$. As a result, it follows that

$$
V_{\tilde{\delta}}(t) = \tilde{\delta}(t)^T\Sigma(t)\tilde{\delta}(t) \ge \lambda_{\min}\left[\Sigma(t)\right]\|\tilde{\delta}(t)\|^2 \le 4z_{\theta 0}^2
$$

or

$$
\|\tilde{\delta}(t)\|^2 \le \frac{V_{\tilde{\delta}}(t_0)}{\lambda_{\min}\left[\Sigma(t)\right]} + \frac{1}{\lambda_{\min}\left[\Sigma(t)\right]}\int_{t_0}^{t}\dot{V}_{\tilde{\delta}}(\tau)d\tau.
$$

Since, it can always be shown that $\lambda_{\min}\left[\Sigma(t_0)\right] \le \lambda_{\min}\left[\Sigma(t)\right]$ for any $t \ge t_0$, it follows that

$$
\begin{aligned}
\|\tilde{\delta}(t)\|^2 &\le \frac{V_{\tilde{\delta}}(t_0)}{\lambda_{\min}\left[\Sigma(t_0)\right]} + \frac{1}{\lambda_{\min}\left[\Sigma(t)\right]}\int_{t_0}^{t}\dot{V}_{\tilde{\delta}}(\tau)d\tau\\
&\le 4z_{\theta 0}^2 + \frac{1}{\lambda_{\min}\left[\Sigma(t)\right]}\int_{t_0}^{t}\dot{V}_{\tilde{\delta}}(\tau)d\tau.
\end{aligned}
$$

Thus, the expansion of the uncertainty set that would occur as a result of $\int_{t_0}^{t}\dot{V}_{\tilde{\delta}}(\tau)d\tau > 0$ would not be violate the fact that $\theta \in \Theta(t_0)$. It is therefore equivalent to write the last inequality using the update (18) as follows,

$$
\begin{aligned}
\|\tilde{\delta}(t)\|^2 &\le 4z_{\theta 0}^2 + \frac{1}{\lambda_{\min}\left[\Sigma(t)\right]}\int_{t_0}^{t}\dot{V}_{z\theta}(\tau)d\tau\\
&= 4z_{\theta 0}^2 + \begin{cases} 0 & \text{if } \dot{V}_{\tilde{\delta}}(\tau)d\tau \ge 0\\ \dot{V}_{\tilde{\delta}}(\tau) & \text{otherwise}\end{cases}
\end{aligned} \tag{24}
$$

As a result, one can guarantee that the set update algorithm is such that

$$
\|\tilde{\delta}(t)\|^2 \le \min\left[4z_{\theta 0}^2, \frac{V_{z\theta}(t)}{\lambda_{\min}(\Sigma(t))}(= 4z_\theta^2(t))\right], \tag{25}
$$

$\forall t \ge t_0$. Hence, if $\theta \in \Theta(t_0)$, then $\theta \in B(\hat{\theta}(t), z_\theta(t))$, $\forall t \ge t_0$.

■

Thus the update mechanism guarantees that the unknown value of the parameters are always contained within the uncertainty set. It also ensures that the uncertainty set is completely contained within its previous estimate, ensuring that the magnitude of the certainty can be be effectively reduced without including unlikely parameter values. It remains to show that the set update algorithm can guarantee that the uncertainty ball radius can reduce to zero if the process dynamics are sufficiently exciting.

This requires a modified notion of persistency of excitation which we state as follows.

Assumption 4: The trajectories of the system are such that

$$
\lim_{t\to\infty}\lambda_{\min}\left[\Sigma(t)\right] = \infty. \tag{26}
$$

Remark 2: Note that Assumption 4 can be seen as an alternative to Assumption 3.

The final result of this paper can now be stated.

Theorem 1: Let the trajectories of the system be such that Assumption 4 holds. Let the system be such that Assumptions 1 and 2 hold then the set update (21), algorithm 1 and the parameter estimation routine ((2),(7),(11) and (14)) guarantee that the parameter estimates $\hat{\theta}$ converge asymptotically to their true values, θ.

Proof: As a result of the proof of Lemma 3 one can always write

$$\|\tilde{\delta}(t)\|^2 \leq \min\left[4z_{\theta 0}^2, 4z_\theta(t)^2\right]$$

where $z_\theta(t)^2 = \frac{V_{z\theta}(t)}{4\lambda_{\min}[\Sigma(t)]}$. Since $V_{z\theta}$ is non-increasing and thus bounded, it follows that

$$\lim_{t\to\infty} \frac{V_{z\theta}(t)}{2\lambda_{\min}\left[\Sigma(t)\right]} = 0.$$

and $\lim_{t\to\infty} z_\theta(t)^2 = 0$ or $\lim_{t\to\infty} \|\tilde{\delta}\|^2 = 0$. By Lemma 3, the set update guarantees that the centre of the uncertainty set must converge to the true value of the parameters as $\lim_{t\to\infty} z_\theta(t)^2 = 0$. ■

IV. Simulation Examples

A. Monod Kinetics

Consider the following system representing a chemostat operating under Monod kinetics

$$\begin{aligned} \dot{x}_1 &= \frac{\theta_1 x_1 x_2}{\theta_2 + x_2} - Dx_1 \\ \dot{x}_2 &= \frac{-\theta_3 x_1 x_2}{\theta_2 + x_2} + S_0 - Dx_2 \end{aligned}$$

Where $\theta = [\mu_{max}, K_s, \frac{1}{Y_{x/s}}]^T$, with values $\theta = [0.33, 0.5, 0.66]^T$. The constant $S_0 = 5$ and the control input D is oscillated such that $0.05 \leq D \leq 0.15$

The results shown in Figure 1 show that the parameter estimates converge to their true values. The parameter uncertainty set update is shown in Figure 2. As expected, the uncertainty set update guarantees forward invariance of each new uncertainty set which contain the true value of the parameters. Figure 3 shows that the magnitude of the δ variable error is always less than the radius of the set. Since the δ variable error is equal to the parameter estimation error, it is possible to verify that the true parameters remain within the uncertainty set throughout the simulation. It is important in this case that the parameter converge to their true values faster than one is allowed to shrink the radius of the uncertainty set. This property shows that the performance of the parameter estimation scheme is far superior to the worst case estimate monitored by the set update mechanism.

B. Nonisothermal CSTR

Next, we consider the estimation of the chemical kinetics in a nonisothermal CSTR where a first order reaction is taking place. A model describing its dynamics is given by:

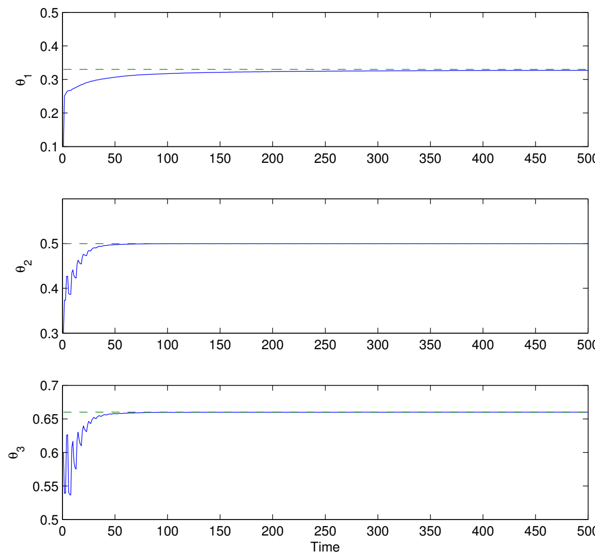

Fig. 1. Time course plot of the parameter estimates and true values under the parameter uncertainty set algorithm, the dashed lines (- -) represent the true parameter values, the solid lines (–) represent the parameter estimates

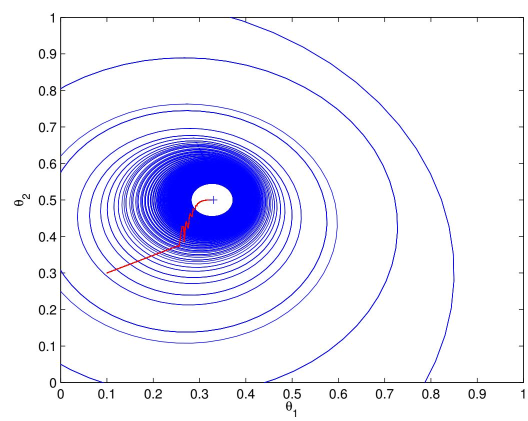

Fig. 2. Phase portrait of the update of θ_1 and θ_2. The circles represent the progression of the uncertainty sets

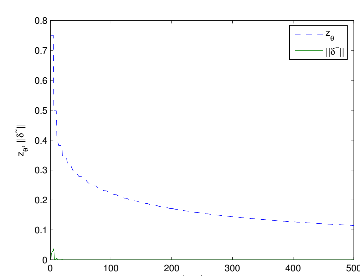

Fig. 3. The progression of the radius of the parameter uncertainty set and the magnitude of the variable $\tilde{\delta}$

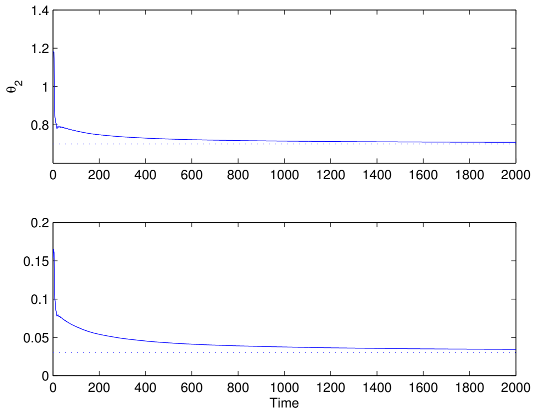

Fig. 4. Time course plot of the parameter estimates and true values under the parameter uncertainty set algorithm, the dashed lines (- -) represent the true parameter values, the solid lines (–) represent the parameter estimates

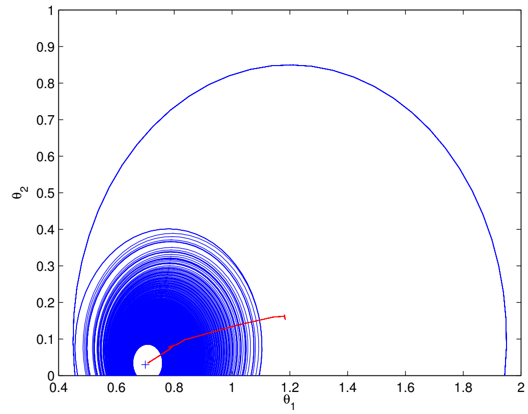

Fig. 5. Phase portrait of the update of θ_1 and θ_2. The circles represent the progression of the uncertainty sets

$$
\begin{aligned}
\dot{x}_1 &= \frac{F}{V}(C_{a0}-x_1) - \theta_1 e^{\left(-\theta_2\left(\frac{1}{x_2}-\frac{1}{T_{ref}}\right)\right)} x_1 \\
\dot{x}_2 &= \frac{F}{V}(T_0-x_2) + \frac{UA}{\rho C_p V}(T_{cin}-x_2) \\
&\qquad -\frac{\Delta H_r \theta_1}{\rho C_p} e^{\left(-\theta_2\left(\frac{1}{x_2}-\frac{1}{T_{ref}}\right)\right)} x_1
\end{aligned}
$$

where x_1 is the concentration of chemical species A (mol/min), x_2 is the reactor temperature (Kelvins), θ_1 and θ_2 are the so-called Arrhenius law parameters. The model parameters are: the reactor flowrate F (m^3/min), the reactor volume V (m^3), the inlet reactor temperature T_0 (K), the inlet concentration of A (mol/min), the inlet cooling jacket temperature T_{cIn} (K), the heat of reaction ΔH_r (cal/mol), the heat transfer coefficient UA and the reference temperature T_{ref} (K). The reactor is initially at $x(0) = [0.265, 393]$. The true value of the parameters is $\theta = [0.7, 0.03]$. The initial estimate is $\hat{\theta}(0) = [1.20.1]$. It is assumed that the true value belongs to a ball of radius $z_{\theta 0} = 0.75$. A sinusoidal signal is injected in the inlet concentration C_{A0} and the inlet temperature T_0.

The estimation of Arrhenius parameters is generally recognized as a very difficult estimation problem (see, for example [14]). Figure 4 shows the resulting estimates of the parameters. As expected, the parameters converge to their unknown true values. The corresponding progression of the uncertainty set is shown in Figure 5. The method is shown to perform well in this case.

V. Conclusions

The paper presents a new technique for the adaptive parameter estimation in nonlinear parameterized dynamical systems. The technique proposes an uncertainty set-update approach that guarantees forward invariance of the true value of the parameters. In addition, it is shown that in the presence of sufficiently exciting state trajectories, the parameter estimates converge to the true values and the uncertainty set vanishes around the true value of the parameters. Future work will be focussed on the generalization of this technique to discrete-time nonlinear systems and adaptive extremum-seeking control.

References

[1] V. Adetola and M. Guay. Finite-time parameter estimation in adaptive control of nonlinear systems. *IEEE Transactions on Automatic Control*, 53(3):807–811, 2008.

[2] V. Adetola and M. Guay. Performance improvement in adaptive control of linearly parameterized nonlinear systems. *IEEE Transactions on Automatic Control*, 55(9), 2010.

[3] V. Adetola and M. Guay. Robust adaptive mpc for constrained uncertain nonlinear systems. *International Journal of Adaptive Control and Signal Processing*, 25, 2010.

[4] A. Annaswamy, A. Loh, and F. P. Skantze. Adaptive control of continuous time systems with convex/concave parametrization. *Automatica*, 34:33–49, 1998.

[5] J. Boskovic. Adaptive control of a class of nonlinearly parameterized plants. *IEEE Transactions on Automatic Control*, 44:930–934, 1998.

[6] C. Cao, A. Annaswamy, and A. Kojic. Parameter convergence in nonlinearly parameterized systems. *IEEETAC*, 48:397–412, 2003.

[7] S. S. Ge. *Stable Adaptive Neural Network Control*. Kluwer Academic Publishers, Norwell, MA, USA, 2001. ISBN 0792375971.

[8] A. Kojic, A. Annaswamy, A.-P. Loh, and R. Lozano. Adaptive control of a class of nonlinear systems with convex/concave parameterization. *Systems and Control Letters*, 37:267–274, 1999.

[9] M. Krstic and H. Deng. *Stabilization of Nonlinear Uncertain Systems*. Springer-Verlag, 1998.

[10] M. Krstic, I. Kanellakopoulos, and P. Kokotovic. *Nonlinear and Adaptive Control Design*. John Wiley and Sons Inc, Toronto, 1995.

[11] R. Marino and P. Tomei. *Nonlinear Control Design*. Prentice Hall, 1995.

[12] M. Netto, A. Annaswamy, R. Ortega, and P. Moya. *Adaptive control of a class of nonlinearly parametrized systems using convexification*, 73:1312–1321, 2000.

[13] J. Wang. Robust adaptive control of a class of nonlinearly parameterized time-varying uncertain systems. In *Proceedings of the2009 American Control Conference*, pages 1940–1945. St-Louis, Missouri, 2009.

[14] D. Watts. Estimating parameters in nonlinear rate equations. *Canadian Journal of Chemical Engineering*, 72(4):701–710, 1994.

[15] T. Zhang, S. Ge, C. Hang, and T. Chai. Adaptive control of first-order systems with nonlinear parameterization. *IEEE Transactions on Automatic Control*, 45:1512–1516, 2000.

[16] T. Zhang and M. Guay. Adaptive parameter estimation for microbial growth kinetics. *AIChE Journal*, 48(3):607–616, 2002.