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Coprime factor anti-windup for systems with sensor saturation

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Abstract— **This paper considers the design of anti-windup compensators for linear systems with saturated sensor measurements. The architecture used for the anti-windup (AW) compensators resembles that commonly used in fault-detection and high performance control, rather than the traditional anti-windup approach. Stability of the system is examined and it transpires that the design problem reduces to choosing appropriately a coprime factorisation of the plant, and its associated Bezout complement. In turn, this new problem has a state-space interpretation which requires the choice of appropriate state-feedback and observer gains such that a certain nonlinear matrix inequality (NLMI) is feasible. Although this NLMI is not easily linearised, it is shown that, providing the plant under consideration is detectable and *controllable*, there always exists a choice of parameters such that this inequality is satisfied and therefore, there always exists an anti-windup compensator (of this particular form) such that the overall closed-loop system with sensor saturation is asymptotically stable.**

Keywords: Anti-windup, $\mathscr{L}_2$ gain, sensor saturation.

I. INTRODUCTION

Unlike actuator saturation, sensor saturation problems tend to be rather uncommon as sensors are typically chosen such that they can provide reasonably accurate measurements throughout the range of the measured variable. However, for reasons of economy and availability, some systems are equipped with sensors which provide limited measurements and, thus, they "saturate" when the variable which they are due to measure exceeds this range. In a similar fashion to the actuator saturation problems, these constraints may compromise performance and stability of the closed-loop system, and therefore compensation schemes are of interest when such faulty, saturated measurements are provided to the feedback loop.

Due to the limited amount of well-documented (practical) problems in the area of sensor saturation, the research community has devoted relatively little attention to its study, although several papers have appeared in recent years. Such systems were initially studied in [11], where the concept of observability of linear systems with sensor saturation was investigated. Following this, [12] suggested a globally stabilising control strategy for single-input-single-output (SISO) systems with sensor saturation; this was recently extended to multiple-input-multiple-output (MIMO) systems in [6]. These papers were important contributions since they established that systems with sensor saturation could be globally asymptotically stabilised under much weaker conditions than those with actuator saturation; crucially, no condition was placed upon the open-loop plant's poles. However, the control constructions advocated in [12], [6] are of a complex, nonlinear form and are not attractive for implementation. Simpler approaches have been advocated in [9], [1], [16], [5], [20], but they do not provide the generality of the approach stated in [6]. In particular, these approaches either provide globally stabilising control laws for stable systems [9], [20], or they provide control laws which only guarantee local results [16], [5]. Another interesting observer-based design was proposed in [8], where semi-global stability guarantees were obtained for minimum-phase SISO systems; the results do not appear straightforward to extend to MIMO systems.

One approach which seems particularly promising for implementation, is the anti-windup approach. This approach, of course, was initially developed for control problems involving actuator saturation and basically involves the augmentation of a nominal linear controller with a so-called *anti-windup compensator* which assists the linear controller in maintaining stability and performance during periods of saturation; otherwise the anti-windup controller is inactive. In the actuator saturation case, the anti-windup architecture is simple and unambiguous: the compensator is driven by the difference between the saturated and un-saturated control signals - both of which are known or can be estimated very accurately. In the sensor saturation case, a "dual" anti-windup architecture is not available: the anti-windup compensator cannot be driven by the difference between the "real" output and the saturated output because the real output is not known (this is why it is being sensed!). Hence there are a variety of different pseudo-anti-windup architectures which could be used to generate an estimate of the real output in order to generate a signal to drive the compensator. Some possible choices are discussed in [20], but essentially they involve the use of an observer to estimate the output, as initially proposed in [16]. In this paper we shall use an anti-windup architecture which corresponds to the favoured architecture in [20], but which is similar to coprime-factor based residual generation found in fault detection and high-performance control schemes [25], [2]. The resulting architecture is based on fault detection ideas and standard anti-windup applications, in which a coprime factorisation is used to generate a (residual) signal that drives the anti-windup compensator. As a result, if no sensor saturation occurs, then the driving signal becomes zero; if sensor saturation occurs, then the signal becomes active and hence non-zero. Coprime factorization can be exploited to derive conditions under which the overall control system is stable. A noteworthy feature of this approach is that the main results are obtained as a consequence of a straightforward application of the Circle Criterion, and do not involve the intricacies of the proofs given in [12], [6], [8].

A. Notation

Notation throughout the paper is standard. Linear operators and their transfer functions are indicated by bold-faced characters; other vectors or matrices are not emboldened. The space of real rational linear operators with finite $\mathscr{H}_\infty$ norm is denoted $\mathscr{R}\mathscr{H}_\infty$. As usual, I and 0 represent the identity and null matrices of appropriate dimensions, respectively.

A factorisation $\mathbf{P} = \mathbf{M}_0^{-1}\mathbf{N}_0$ is said to be a left coprime

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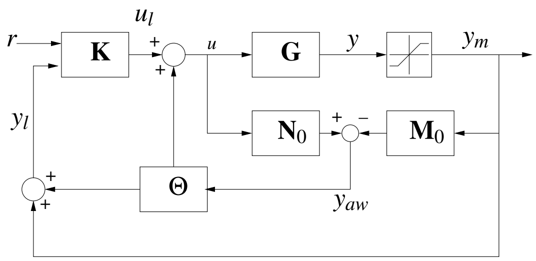

Fig. 1. System with sensor saturation and anti-windup

factorisation if $\mathbf{M}_0, \mathbf{N}_0 \in \mathscr{RH}_\infty$, and there exist $\mathbf{X}_l, \mathbf{Y}_l \in \mathscr{RH}_\infty$ such that the Bezout identity

$$\mathbf{M}_0\mathbf{X}_l + \mathbf{N}_0\mathbf{Y}_l = I$$

is satisfied [24]. We call $\mathbf{X}_l$ and $\mathbf{Y}_l$ the *Bezout complements* of $\mathbf{M}_0, \mathbf{N}_0 \in \mathscr{RH}_\infty$. To simplify matrix notation we often use $\star$ to signify the appropriate term to make the matrix symmetric; following [4] we sometimes use $\mathrm{He}(X)$ to denote Hermitian of X, i.e. $\mathrm{He}(X) = X + X'$.

For a state-space system

$$\dot{x}(t) = f(x(t)) \quad x \in \mathbb{R}^n$$

we say that the origin $x = 0$ is *locally asymptotically stable* if $\lim_{t\to\infty} x(t) = 0$ for all $x \in \mathscr{X} \subset \mathbb{R}^n$. If $\mathscr{X} = \mathbb{R}^n$, we say the system is *globally asymptotically stable*. For the state-space system

$$\dot{x} = f(x,u) \quad x \in \mathbb{R}^n, \quad u \in \mathbb{R}^m$$

we say that the origin is *semi-globally asymptotically stabilisable* if there exists a $u(x) \in \mathbb{R}^m$ such that the origin of the above system is locally asymptotically stable for any arbitrarily large set $\mathscr{X} \subset \mathbb{R}^n$.

II. ANTI-WINDUP ARCHITECTURE

A. The basic architecture

Consider Figure 1 where a system with sensor saturation and anti-windup compensation is depicted. $\mathbf{K} = [\mathbf{K}_1 \quad \mathbf{K}_2]$ denotes the nominal linear controller and $\mathbf{G}$ the nominal plant. A left coprime factorisation of $\mathbf{G}$ is given by $\mathbf{G} = \mathbf{M}_0^{-1}\mathbf{N}_0$ and thus, in the diagram, $\mathbf{M}_0$ and $\mathbf{N}_0$ represent stable filters which generate the "residual" y_{aw} which drives the compensator

$$\Theta = \begin{bmatrix}\Theta_1' & \Theta_2'\end{bmatrix}'$$

The signals y, y_m, y_{aw}, $y_l \in \mathbb{R}^p$ are the real output, the saturated output (sensor measurement), the residual, and the so-called linear output (see later); the signals u, $u_l \in \mathbb{R}^m$ are the control input and the linear control input (see later); $r \in \mathbb{R}^{n_r}$ is the reference signal, which is of no importance in this paper. The anti-windup compensator generates two signals, which are injected to the controller output and controller input respectively. y_m is the saturated version of y, viz $y_m = \mathrm{sat}(y)$, where $\mathrm{sat}(.) : \mathbb{R}^p \mapsto \mathscr{U} \subset \mathbb{R}^p$ is defined as

$$\mathrm{sat}(y) = [\mathrm{sat}_1(y_1) \quad \dots \quad \mathrm{sat}_p(y_p)]' \tag{1}$$

where $\mathrm{sat}_i(y_i) = \mathrm{sign}(y_i)\min\{\bar{y}_i, |y_i|\}$ and $\bar{y}_i > 0$ for all $i \in \{1,\dots,p\}$. Frequent use will be made use of the identity

$$\mathrm{sat}(y) = y - \mathrm{Dz}(y)$$

where $\mathrm{Dz}(.) : \mathbb{R}^p \mapsto \mathbb{R}^p$ is the deadzone function.

Throughout this work it is assumed, as in standard AW compensation for input saturation, that the controller $\mathbf{K}$ has been designed such that in the absence of saturation, the unconstrained closed-loop system in Figure 1 is well-posed, internally stable and yields satisfactory performance levels. Note that in the absence of sensor saturation, i.e. $y_m \equiv y$, the residual signal reduces to

$$y_{aw} = \mathbf{N}_0 u - \mathbf{M}_0\mathbf{G}u = (\mathbf{N}_0 - \mathbf{M}_0\mathbf{M}_0^{-1}\mathbf{N}_0 u) = 0 \tag{2}$$

and hence, if no saturation event occurs, the anti-windup compensator Θ will remain inactive. However, when sensor saturation occurs, $y_{aw} \neq 0$, and the anti-windup compensator is activated. In reality, as $\mathbf{G}$ must be interpreted as the real, probably uncertain, plant, and the filters $\mathbf{N}_0$ $\mathbf{M}_0$ are actually derived from a coprime factorisation of a nominal plant model, y_{aw} will never actually be zero and hence the anti-windup compensator will always be active - in a sense, the anti-windup compensator would be a *weakened* AW compensator ([4]). Nonetheless, using fault detection techniques, it is possible to tune the residual generator in such a manner that disturbances and uncertainties are attenuated. For the purposes of this paper, however, we shall assume that a perfect model of the plant is available (i.e. no plant/model mismatch is present).

Remark 1: The architecture depicted in Figure 1 is a special case of the main architecture used in [20] when the plant $\mathbf{G}$ is stable. In this paper we prefer to use the architecture depicted in Figure 1 because it enables parallels to be drawn with the fault detection literature, and also because it enables a more transparent manipulation of the governing equations. □

The plant $\mathbf{G}$ has the following state-space realisation

$$\mathbf{G} \sim \left[\begin{array}{c|c} A_p & B_p \\ \hline C_p & D_p \end{array}\right] \in \mathbb{R}^{(n+p)\times(n+m)} \tag{3}$$

Consequently, an n^{th} order left coprime factorisation [24] has the following state-space realisation

$$[\mathbf{N}_0 \quad \mathbf{M}_0] \sim \left[\begin{array}{c|cc} A_p + LC_p & B_p & L \\ \hline C_p & 0 & I \end{array}\right] \tag{4}$$

where L is chosen such that $A_p + LC_p$ is Hurwitz. The n^{th} order Bezout complements have the following state-space realisations

$$\begin{bmatrix}\mathbf{Y}_l \\ \mathbf{X}_l\end{bmatrix} \sim \left[\begin{array}{c|c} A_p + B_pF & -L \\ \hline -F & 0 \\ C_p & I \end{array}\right] \tag{5}$$

It can be seen that the filters which generate the residual y_{aw} are selected by altering the coprime factorisation of the plant, and in light of equation (4), achieve an appropriate observer design. At the moment however, design of the anti-windup compensator Θ has not been addressed.

B. An equivalent block diagram

In [22], [23] the anti-windup problem for systems with input saturation was clarified substantially by re-drawing the standard AW block diagram in a mathematically equivalent, but more illuminating, form. This new "decoupled" structure brings to the fore many useful features and has been the basis for several subsequent developments (e.g. [19], [21]); this new representation essentially captures many of the features

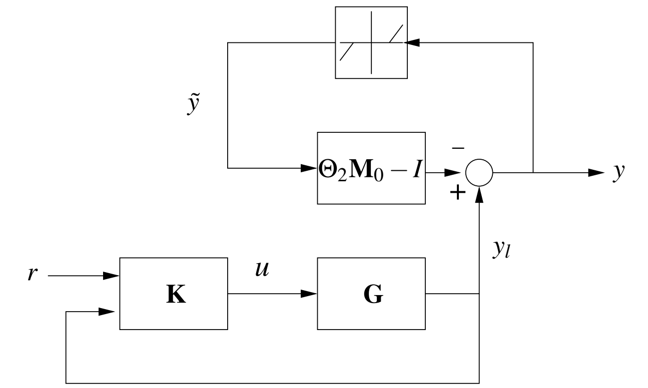

Fig. 2. Equivalent representation of system with sensor saturation and anti-windup

of the scheme proposed in [17]. It is thus natural to apply similar logic to the sensor saturation problem. From Figure 1 it follows that

$$\begin{aligned} y_{aw} &= \mathbf{N}_0 u - \mathbf{M}_0 y_m \\ &= \mathbf{N}_0 u - \mathbf{M}_0 (y - \tilde{y}) \\ &= \mathbf{N}_0 u - \mathbf{M}_0 (\mathbf{G}u - \tilde{y}) \\ &= \mathbf{M}_0 \tilde{y} \end{aligned} \tag{6}$$

Note also that

$$\begin{aligned} y_l &= y_m + \Theta_2 y_{aw} \\ &= y - \tilde{y} + \Theta_2 \mathbf{M}_0 \tilde{y} \\ &= \mathbf{G}u - (I - \Theta_2 \mathbf{M}_0)\tilde{y} \\ &= \mathbf{G}(u_l + \Theta_1 \mathbf{M}_0 \tilde{y}) - (I - \Theta_2 \mathbf{M}_0)\tilde{y} \\ &= \mathbf{G}u_l + \mathbf{M}_0^{-1}(\mathbf{N}_0\Theta_1 + \mathbf{M}_0\Theta_2 - I)\mathbf{M}_0\tilde{y} \end{aligned} \tag{7}$$

Therefore it follows that choosing Θ_1 and Θ_2 as the Bezout complements of $\mathbf{N}_0$ and $\mathbf{M}_0$ (i.e. $\Theta_1 = \mathbf{Y}_l$ and $\Theta_2 = \mathbf{X}_l$) makes

$$\mathbf{N}_0\Theta_1 + \mathbf{M}_0\Theta_2 = I \tag{8}$$

and hence $y_l = Gu_l$

Next, from the block diagram, $y_m = y_l - \Theta_2 y_{aw}$ so as $y_m = y - \tilde{y}$ this becomes

$$\begin{aligned} y - \tilde{y} &= y_l - \Theta_2 \mathbf{M}_0 \tilde{y} \\ y &= y_l - (\Theta_2 \mathbf{M}_0 - I)\tilde{y} \end{aligned} \tag{9}$$

Thus, recalling that $\tilde{y} = \mathrm{Dz}(y)$ and using the fact that

$$u_l = \mathbf{K}\begin{bmatrix} r \\ y_l \end{bmatrix} \tag{10}$$

we can combine equations (10), (7) and (9) to yield the decoupled block diagram given in Figure 2. It is important to mention that this is achieved, provided that Θ_1 and Θ_2 are chosen as the Bezout complements of $\mathbf{N}_0$ and $\mathbf{M}_0$.

Figure 2 consists of a *nominal linear loop* which is not affected by saturation, and a *nonlinear loop* which governs the performance of the system during saturation and, moreover, the stability of the overall nonlinear closed-loop system. Hence, when studying the stability of the system, attention can be focused on the nonlinear loop alone.

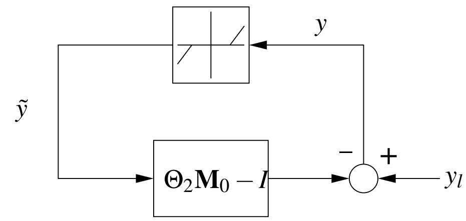

Fig. 3. Nonlinear loop

III. STABILITY ANALYSIS

A. The general case

The foregoing section noted that, providing Θ_1 and Θ_2 were chosen as the Bezout compliments of N_0, M_0, then Figure 1 is mathematically equivalent to Figure 2. Furthermore provided that the nominal linear systems is stable, stability of the system in Figure 2 (and therefore Figure 1) is purely dependent on the stability of the nonlinear loop (this is re-drawn in Figure 3 for convenience). Notice that the nonlinear loop depicted in Figure 3, is a feedback interconnection of a linear system and a static (memoryless) nonlinearity, and thus, falls within the set of standard absolute stability problems. For simplicity, and at the expense of some conservatism, the standard multivariable Circle Criterion will be used in this paper to provide stability guarantees.

Proposition 1: Consider the feedback system in Figure 3 with $\mathbf{M}_0$ having state-space realisation as given by equation (4) and $\Theta_2 = \mathbf{X}_l$ having state-space realisation given by equation (5). Then, the origin of the system is locally exponentially stable with $\mathscr{E}(P) = \{x \in \mathbb{R}^{2n} : x'\mathrm{diag}(P_1, P_2)x \leq 1\}$ contained within the basin of attraction if there exist a scalar $\varepsilon \in (0,1)$, positive definite matrices $P_1 > 0$ and $P_2 > 0$, a diagonal positive definite matrix $W > 0$ and two further matrices L and F, such that the following matrix inequalities are satisfied

$$\mathrm{He}\begin{bmatrix} P_1(A_p + B_pF) & -P_1LC_p & -P_1L \\ 0 & P_2(A_p + LC_p) & P_2L \\ -\varepsilon WC_p & -\varepsilon WC_p & -W \end{bmatrix} < 0 \tag{11}$$

$$\frac{1}{1-\varepsilon}\bar{y}_i^2\begin{bmatrix} P_1 & 0 \\ \star & P_2 \end{bmatrix} - \begin{bmatrix} C'_{p,i}C_{p,i} & C'_{p,i}C_{p,i} \\ C'_{p,i}C_{p,i} & C'_{p,i}C_{p,i} \end{bmatrix} \geq 0 \quad \forall i \in \{1,\dots,p\} \tag{12}$$

Proof: Note that from the realisations given in equations (4)-(5), a state-space realisation of the compensator is derived

$$\Theta_2\mathbf{M}_0 - I \sim \left[\begin{array}{cc|c} A_p + B_pF & -LC_p & -L \\ 0 & A_p + LC_p & L \\ \hline C_p & C_p & 0 \end{array}\right] \tag{13}$$

Next note that $\mathrm{Dz}(.)$ is sector bounded by $\mathrm{Sector}[0, \varepsilon I]$, and that this sector condition holds globally for $\varepsilon = 1$ and *locally* if $\varepsilon \in (0,1)$; the latter signifies a reduced sector bound. Note further ([7]) that $\mathrm{Dz}(.) \in \mathrm{Sector}[0, \varepsilon I]$ for all $y \in \mathscr{Y}$ where

$$\mathscr{Y} : \{y \in \mathbb{R}^p : \quad |y_i| \leq (1-\varepsilon)^{-1}\bar{y}_i\} \quad \forall\{i \in 1, 2, \dots, p\}$$

Assume that we consider an ellipsoidal region of the state-space of the system $\Theta_2\mathbf{M}_0 - I$, denoted by

$$\mathscr{E}(P) := \{x \in \mathbb{R}^{2n} : x'Px \leq 1\}$$

Then it follows, from the results in [7], [14], that $\mathscr{E}(P) \subset \mathscr{Y}$ if the matrix inequalities in (12) are satisfied. Thus, assume that $x \in \mathscr{E}(P)$, then it follows that $y \in \mathscr{Y}$ and hence $\mathrm{Dz}(.) \in \text{Sector}[0, \varepsilon I]$. Thus by standard application of the Circle Criterion (see for example [10], [13]) the origin of the system will be locally asymptotically stable if $\varepsilon(\Theta_2 \mathbf{M}_0 - I) + I$ is passive, as the deadzone nonlinearity belongs to the reduced sector $\text{Sector}[0, \varepsilon I]$. Passivity of $\varepsilon(\Theta_2 \mathbf{M}_0 - I) + I$ can be examined, using the KYP Lemma ([15]). Hence, the system is locally asymptotically stable, with $\mathscr{E}(P)$ in the region of attraction if there exist a scalar $\varepsilon \in (0,1)$, positive definite matrices $P > 0$ and diagonal $W > 0$ such that

$$\mathrm{He}\begin{bmatrix} P\begin{bmatrix} A_p + B_pF & -LC_p \\ 0 & A_p + LC_p \end{bmatrix} & P\begin{bmatrix} -L \\ L \end{bmatrix} \\ \varepsilon W\,[-C_p \quad -C_p] & -W \end{bmatrix} < 0 \tag{14}$$

and inequality (12) holds. Setting $P = \mathrm{diag}(P_1, P_2)$ then yields the inequalities in the proposition. □□

The matrix inequalities (11)-(12)are, unfortunately, nonlinear. Despite this, the inequalities are useful because they can be used to obtain *existence* conditions for matrices F and L which stabilise the nonlinear loop. This will in turn ensure the existence of an anti-windup compensator which guarantees stability of the system proposed in Figure 2, and hence, that in Figure 1. This prompts the following result.

Proposition 2: There exist positive definite matrices $P_1 > 0$ and $P_2 > 0$, a diagonal positive definite matrix $W > 0$, two matrices L and F, and a scalar $\varepsilon \in (0,1)$ such that Proposition 1 is satisfied for any (arbitrarily large, but bounded) $\mathscr{E}(P)$ if (A_p, B_p) is controllable and (C_p, A_p) is detectable.

Before proving this result the following fact will be useful.

Fact 1: Consider $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$. Then there exist matrices F and $X > 0$ such that, for any scalar α

$$(A+BF)'X + X(A+BF) = \alpha M_1(X) < 0 \tag{15}$$

$$X^{-1}(A+BF)' + (A+BF)X^{-1} = \alpha M_2(X^{-1}) < 0 \tag{16}$$

where X is independent of α, and $M_1(X)$ and $M_2(X^{-1})$ are linear functions of X and X^{-1} respectively, if and only if (A,B) is controllable.

Proof: See appendix for details. □□

Proof of Proposition 2

First note that inequality (11) can be written as

$$\underbrace{\begin{bmatrix} P_1(A_p + B_pF) + (A_p + B_pF)'P_1 & 0 & -\varepsilon C_p' W \\ \star & P_2A_p + A_p'P_2 & -\varepsilon C_p' W \\ \star & \star & -2W \end{bmatrix}}_{\Psi}$$

$$+ \underbrace{\begin{bmatrix} -P_1 \\ P_2 \\ 0 \end{bmatrix}}_{G} L\,[0 \quad C_p \quad I] + \underbrace{\begin{bmatrix} 0 \\ C_{p'} \\ I \end{bmatrix}}_{H} L'\,[-P_1 \quad P_2 \quad 0] < 0 \tag{17}$$

Application of the projection lemma ([3]) yields existence conditions, for an L satisfying inequality (17), as

$$W_G' \Psi W_G < 0 \qquad W_H' \Psi W_H < 0 \tag{18}$$

where W_G and W_H are matrices whose columns span, respectively, the null spaces of G' and H'. Noting that $G \in \mathbb{R}^{(2n+p)\times n}$ and $\mathrm{rank}(G) = n$ and $H \in \mathbb{R}^{(2n+p)\times p}$ and $\mathrm{rank}(H) = p$, suitable choices for W_G' and W_H' are

$$W_G' = \begin{bmatrix} I & I & 0 \\ 0 & 0 & I \end{bmatrix} \mathrm{diag}(P_1^{-1}, P_2^{-1}, I) \qquad W_H' = \begin{bmatrix} I & 0 & 0 \\ 0 & I & -C_p' \end{bmatrix} \tag{19}$$

Thus $W_G' \Psi W_G < 0$ is equivalent to

$$\begin{bmatrix} Q_1(A_p{+}B_pF)'{+}(A_p{+}B_pF)Q_1{+}Q_2A_p'{+}A_pQ_2 & -\varepsilon(Q_1{+}Q_2)C_p'W \\ \star & -2W \end{bmatrix} < 0 \tag{20}$$

where $Q_1 := P_1^{-1} > 0$ and $Q_2 := P_2^{-1} > 0$. Similarly $W_H' \Psi W_H < 0$ can be simplified as

$$\begin{bmatrix} (A_p{+}B_pF)'P_1{+}P_1(A_p{+}B_pF) & \varepsilon C_p' W C_p \\ \star & P_2A_p + A_p'P_2 - 2(1-\varepsilon)C_p'WC_p \end{bmatrix} < 0 \tag{21}$$

As (A_p, B_p) is controllable, Fact 1 implies there exists a matrix F such that $Q_1(A_p + B_pF)' + (A_p + B_pF)Q_1$ and $(A_p + B_pF)'P_1 + P_1(A_p + B_pF)$ can be replaced by two negative definite linear functions $\alpha M_1(P_1)$ and $\alpha M_2(Q_1)$. Furthermore P_1 and Q_1 are independent of α, so α can be chosen arbitrarily. Hence, inequalities (20) and (21) become

$$\begin{bmatrix} \alpha M_2(Q_1) + Q_2A_p' + A_pQ_2 & -\varepsilon(Q_1 + Q_2)C_p'W \\ \star & -2W \end{bmatrix} < 0 \tag{22}$$

$$\begin{bmatrix} \alpha M_1(P_1) & \varepsilon C_p'WC_p \\ \star & P_2A_p + A_p'P_2 - 2(1-\varepsilon)C_p'WC_p \end{bmatrix} < 0 \tag{23}$$

Inequalities (22) and (23) can now be enforced (as $M_2(Q_1) < 0$ and $-2W < 0$) by choosing $\alpha > 0$ large enough, providing there exists a P_2, W and $\varepsilon \in (0,1)$ such that

$$P_2A_p + A_p'P_2 - 2(1-\varepsilon)C_p'WC_p < 0 \tag{24}$$

Thus, there will exist an L such that $\mathscr{E}(P)$ is contained within the region of attraction of the system if inequalities (24) and (12) are simultaneously satisfied. To see that $\mathscr{E}(P)$ can be made arbitrarily large, it is useful to write $P_1 = \eta \tilde{P}_1$ and $P_2 = \eta \tilde{P}_2$ where $\eta > 0$. Thus our expression for $\mathscr{E}(P)$ becomes

$$\mathscr{E}(P) = \left\{ x \in \mathbb{R}^{2n} : \quad x' \begin{bmatrix} \tilde{P}_1 & 0 \\ 0 & \tilde{P}_2 \end{bmatrix} x \le \frac{1}{\eta} \right\}$$

So for fixed $\tilde{P}_1, \tilde{P}_2$, $\mathscr{E}(P)$ can be made arbitrarily large by choosing η arbitrarily small. With this notation, we can then write equation (24) as

$$\eta(\tilde{P}_2A_p + A_p'\tilde{P}_2 - 2(1-\varepsilon)\eta^{-1}C_p'\tilde{W}C_p) < 0 \tag{25}$$

Now, let $W = (1-\varepsilon)^{-1}\eta\delta I$ for some $\delta > 0$ and rearrange equation (12) such that

$$\eta(\tilde{P}_2A_p + A_p'\tilde{P}_2 - 2\delta C_p'C_p) < 0 \tag{26}$$

$$\frac{\eta}{1-\varepsilon}\bar{y}_i^2 \begin{bmatrix} \tilde{P}_1 & 0 \\ \star & \tilde{P}_2 \end{bmatrix} - \begin{bmatrix} C_{p,i}'C_{p,i} & C_{p,i}'C_{p,i} \\ C_{p,i}'C_{p,i} & C_{p,i}'C_{p,i} \end{bmatrix} \ge 0 \;\forall i \in \{1,\dots,p\} \ge 0 \tag{27}$$

Note that inequalities (26) and (27) are our new feasibility conditions, and that for sufficiently large δ, there always exists a $\tilde{P}_2 > 0$ (independent of η) such that (26) holds, provided (C_p, A_p) is detectable. Furthermore, for any $\eta > 0$, $\tilde{P}_1 > 0$ and $\tilde{P}_2 > 0$, there always exists an $\varepsilon \in (0,1)$ sufficiently close to unity such that inequality (27) holds. Thus, $\eta > 0$ can be made arbitrarily small and hence $\mathscr{E}(P)$ arbitrarily large. This completes the proof. □□

Proposition 2 implies that if a linear plant is completely controllable and detectable, then there exists an anti-windup compensator of the form depicted in Figure 1 which provides semi-global stability, *regardless* of the location of the plant's poles: this result is roughly in agreement with [6], although the control strategy is different. We highlight that the compensation scheme proposed here requires an existing baseline linear controller, which is later augmented with coprime factor based anti-windup filtering; in [6] a sample-and-hold nonlinear control law accompanied by a deadbeat observer is proposed.

B. The case of stable plants

The results of Proposition 2 are not *necessary* and, hence, there may be plants which are not completely controllable for which inequality (11) admits a solution. In fact, for the class of stable plants $\mathbf{G} \in \mathscr{RH}_\infty$ full controllability is not needed, and global stability may be achieved.

If $\mathbf{G} \in \mathscr{RH}_\infty$, we can choose $L = 0$ and consider the global sector bound, that is $\varepsilon = 1$. Hence inequality (11) becomes

$$\mathrm{He}\begin{bmatrix} (A_p + B_pF)'P_1 & 0 & 0 \\ 0 & A_p'P_2 & 0 \\ -C_pW & -C_pW & -W \end{bmatrix} < 0 \qquad (28)$$

This inequality can be thought of as linear (a change of variables linearises it) and it is always possible to satisfy with appropriate choice of P_1, P_2 and W. To see this, note that if A_p is Hurwitz, there always exist a positive definite matrix P_2 such that the inequality $A_p'P_2 + P_2A_p < 0$ is guaranteed. Thus with $P_1 > 0$ chosen arbitrarily large, (28) is seen to hold unconditionally.

Interestingly, choosing $L = 0$ means that $\Theta_2 = I$ and $\mathbf{M}_0 = I$, which implies that $\Theta_2\mathbf{M}_0 - I = 0$. Hence the nonlinear loop "disappears" and stability conditions are unconditionally satisfied. Furthermore, for this special choice of L, we also have $\mathbf{N}_0 = \mathbf{G}$ and $\Theta_1 = 0$. Thus, our anti-windup scheme effectively reduces to an "internal model" type control scheme, as depicted in Figure 4. It is possible to observe the striking duality that exists between internal model control (IMC) anti-windup for actuator and sensor saturation. In the actuator saturation case, it is easy to see [18] that the IMC anti-windup scheme forces the "collapse" of the nonlinear loop and unconditional global stability (for stable linear plants) is ensured; it appears that IMC AW fulfils the same role in the sensor saturation case.

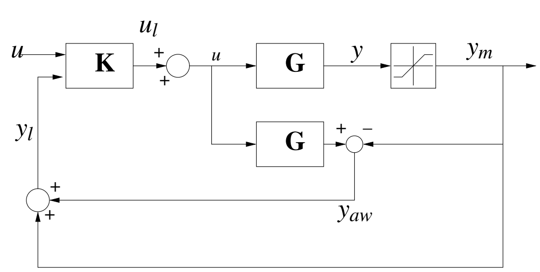

Fig. 4. IMC anti-windup for sensor saturation

Remark 2: In practice, choosing $L = 0$ is not a good idea as this would essentially correspond to disconnecting plant measurements (i.e. y_m) from the controller. However, it is always possible to find matrices F and L (not necessarily $L = 0$) such that the closed-loop sensor saturated system is globally asymptotically stable; note that inequalities (22) and (23) can always be satisfied if the matrix A_p is Hurwitz. □

IV. COMPENSATOR DESIGN AND EXAMPLE

A. Compensator Construction

The previous section demonstrates that there always *exists* choices of $\mathbf{M}_0, \mathbf{N}_0, \mathbf{X}_l$ and $\mathbf{Y}_l$ (and hence an AW compensator), such that the closed-loop will be locally asymptotically stable. However, the actual design of such compensators is problematic since inequality (11), and the derived projection lemma conditions (20) and (21), are all nonlinear. This can be overcome, to some extent, by performing a two step design. Recall that necessary conditions for inequalities (11) and (12) to be satisfied are given by the following inequalities

$$\begin{bmatrix} P_2(A_p + LC_p) + (A_p + LC_p)'P_2 & P_2L - \varepsilon C_p'W \\ \star & -2W \end{bmatrix} < 0 \qquad (29)$$

$$(1-\varepsilon)^{-1}\bar{y}_i^2 P_2 - C_{p,i}'C_{p,i} \geq 0 \quad \forall i \in \{1, \ldots, p\} \qquad (30)$$

Defining the $Y = P_2L$ and for fixed $\varepsilon \in (0,1)$, these can be written as the LMI's

$$\begin{bmatrix} P_2A_p + YC_p + A_p'P_2 + C_p'Y' & Y - \varepsilon C_p'W \\ \star & -2W \end{bmatrix} < 0 \qquad (31)$$

$$\bar{y}_i^2 P_2 - (1-\varepsilon)C_{p,i}'C_{p,i} \geq 0 \quad \forall i \{1, \ldots, p\} \qquad (32)$$

which, can be solved for $Y, P_2 > 0$ and the diagonal $W > 0$.

To make $\mathscr{E}(P)$ as large as possible, the matrix P must be small, hence in the first step we might attempt to

$$\min \mathrm{trace}(P_2)$$

subject to inequalities (31)-(32). Then $L = YP_2^{-1}$, P_2 and W (and our previous choice of ε) can be used in inequalities (11) and (12). Applying a congruence transformation to (11) and letting $Z = FQ_1$, it becomes

$$\mathrm{He}\begin{bmatrix} A_pQ_1 + B_pZ & -LC_p & -L \\ 0 & P_2(A_p + LC_p) & P_2L \\ -\varepsilon WC_pQ_1 & -\varepsilon WC_p & -2W \end{bmatrix} < 0 \qquad (33)$$

which is now linear in the variables $Q_1 = P_1^{-1}$ and Z (as ε L, P_2 and W are now constant), where F can be computed using the relationship $F = ZQ^{-1}$. Similarly, applying a congruence transformation to inequality (12) and applying the Schur complement yields

$$\begin{bmatrix} \bar{y}_i^2 Q_1 & 0 & (1-\varepsilon)QC_{p,i}' \\ \star & \bar{y}_i^2 P_2 & (1-\varepsilon)C_{p,i}' \\ \star & \star & (1-\varepsilon)I \end{bmatrix} \geq 0 \quad \forall i \in \{1, \ldots, p\} \qquad (34)$$

Thus to make $\mathscr{E}(P)$ as large as possible (i.e. P is small as possible in some sense) we might choose to

$$\min \mathrm{trace}(P_1)$$

subject to inequalities (33)-(34).

To summarise, the anti-windup compensator design is performed in three steps:

i) ε is fixed; the closer to unity it is, the larger the region of attraction tends to be
ii) L is determined (hence the coprime factors $\mathbf{M}_0$, $\mathbf{N}_0$)
iii) F is determined (hence the Bezout complements Θ_1, Θ_2).

Although this two-stage solution of nonlinear matrix inequalities is normally not attractive due to concerns over existence of solutions and sub-optimality, in our case it appears acceptable as we are always guaranteed that a solution will exist.

B. Example

Consider the unstable SISO system given by

$$G = \frac{s+1}{s^2-3s+5} \tag{35}$$

A simple proportional control gain, $\mathbf{K} = [11 \quad -11]'$, provides closed-loop stability and some degree of reference tracking. However, when sensor saturation is present, the system may become unstable. This instability can be addressed by including anti-windup compensation, designed using the algorithm given in the previous section. With $\bar{y} = 80$, Table I shows trace(P) and the fastest poles of the AW compensator. Notice that as ε gets closer to unity, the region of attraction of the closed-loop system gets larger. Unfortunately, a by-product of the algorithm is that the fastest poles of the resulting AW compensator are very fast and thus extremely difficult to use for simulation or implementation purposes.

ε	trace(P)	Fastest pole of AW compensator
0.3	6.4625×10^{-4}	-1.5103×10^{16}
0.9	1.1354×10^{-4}	-3.2378×10^{15}
0.99	4.0402×10^{-5}	-1.4161×10^{16}
0.999	7.4722×10^{-6}	-2.6709×10^{15}
0.9999	1.2916×10^{-6}	-3.9055×10^{14}

TABLE I
REGION OF ATTRACTION AND FASTEST POLE OF COMPENSATOR

V. CONCLUSIONS

This paper has presented a novel method for the design of AW compensators which preserve stability in the presence of sensor saturation. It has been shown that, provided the plant is *controllable* and detectable, it is always possible to find an AW compensator (with structure as presented in section II) that guarantees semi-global asymptotic stability of the nonlinear closed-loop system. Thus, in principle, the requirements for semi-global stability of systems with sensor saturation, are much weaker than the corresponding requirements for actuator saturation: it can be achieved *regardless* of the location of the plant's poles or zeros.

The downside of the AW compensators returned by the algorithm proposed, as in the work of [6], is that the compensators are unsuitable for implementation because the algorithm tends to return compensators with extremely fast poles. One must also be aware that the compensators may produce large correction (control) signals during periods of sensor saturation, so any practical anti-windup technique must also account for this during the design stage [5].

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APPENDIX

Proof of Fact 1

Sufficiency: If (A,B) is completely controllable, it follows that there exists a matrix F such that

$$A+BF = \alpha TDT^{-1} \tag{36}$$

where α is any positive scalar and D is any diagonal Hurwitz matrix and T is the matrix of eigenvectors of $(A+BF)$. Then it follows that

$$(A+BF)'X + X(A+BF) = \alpha \underbrace{[(TDT^{-1})'X + X(TDT^{-1})]}_{M_1(X)}$$

As D is Hurwitz there always exists a matrix $X > 0$ such that $M_1(X) < 0$, and furthermore, this X is obviously independent of α. A simple congruence transformation then shows that this implies that (16) also holds.

Necessity If (A,B) is not completely controllable then there does not exists an F such that equation (36) holds for *any* α and *any* diagonal matrix D. In particular for the j uncontrollable modes, we must have that αd_{jj} is fixed by the uncontrollable eigenvalues of (A,B), and thus cannot be arbitrary.