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Necessary and sufficient conditions for quasiconvexity of a class of mixed-integer quadratic programs with applications in hybrid MPC

Stefan Almér, Sébastien Mariéthoz and Manfred Morari

Abstract— **The paper derives necessary and sufficient conditions for quasiconvexity of piecewise quadratic functions. The conditions are stated in terms of linear inequalities which can be verified efficiently. To show the relevance of the result, the paper considers a class of hybrid MPC problems where the system model is piecewise affine and the control input is subject to constraints. Minimizing a quadratic cost results in a mixed integer quadratic program where the objective function is piecewise quadratic. Quasiconvexity can be determined using the result of the paper. The results of the present paper has potential to increase the applicability of hybrid model predictive control in high-speed control applications. In high-speed applications, the only option has been to solve the mixed integer program explicitly and this quickly becomes intractable because of growing complexity. However, if the problem can be shown to be quasiconvex it opens up the possibility to use an efficient on-line approach. A hybrid MPC example is considered which is shown to be quasiconvex for a subset of the initial conditions.**

I. INTRODUCTION

The present paper considers a class of piecewise quadratic continuous functions and derives necessary and sufficient conditions for quasiconvexity of these functions. The functions are defined on an orthogonal partition of the domain of definition. The "pieces" of the piecewise quadratic function are strictly convex quadratic functions defined on rectangular polytopes. The analysis can be extended to functions defined on arbitrary polytopes. However, in the present paper we only consider the case of rectangular polytopes for clarity of presentation.

The quasiconvexity conditions are derived by investigating the change of slope of the function when passing from one region of the partition to its neighbor. The conditions are formulated as linear inequalities which state that the derivative of the function, when restricted to a line orthogonal to the hyperplane separating the regions, must not decrease when crossing the separating hyperplane. In other words, the function is quasiconvex if and only if it is convex in the direction of the normal of the separating hyperplanes.

To show the relevance of the result, the paper presents a class of hybrid model predictive control (MPC) [2] problems where the system model is piecewise affine (PWA) [11] and where we want to minimize a quadratic cost subject to constraints on the control input. The resulting mixed-integer quadratic optimization problem (MIQP) consists in minimizing a piecewise quadratic continuous function where the results of the present paper can be used to verify whether the problem is quasiconvex or not.

In high-speed applications where the control frequency is high, the only way to implement the hybrid MPC solution on-line has been to rely on explicit solutions [12], [4] where the optimal solution is computed off-line and stored in a look-up table (typically in a binary search tree). However, to represent the solution of an MIQP explicitly, one needs to enumerate all combinations of binary variables and solve all resulting QPs explicitly. In the on-line application, all explicit QP solutions must then be evaluated and compared in order to apply the optimal control input. This approach quickly becomes intractable since complexity grows rapidly with the number of subsystems of the PWA model and the prediction horizon. If the MIQP can be shown to be quasiconvex however, the problem could be solved efficiently on-line. Thus, the results of the present paper has potential to increase the applicability of hybrid MPC in high-speed control applications.

In general, a given MPC problem may be quasiconvex for some, but not all relevant initial conditions. Nevertheless, the results of the paper can be used to partition the domain of the initial condition into smaller pieces where the objective function can be shown to be quasiconvex. Such a procedure could be used to reduce the complexity of the optimization problem and thus reduce memory requirements and computation time in on-line applications.

The paper considers an application example from the field of power electronics [10]. A switched mode step-down (buck) DC-DC converter is modeled using PWA approximations [6] which gives a more accurate description of the rippling converter state compared to the conventional averaging approach [9]. Minimizing a quadratic cost subject to the PWA dynamic constraints and input constraints corresponds to minimizing a piecewise quadratic function which is indeed shown to be quasiconvex for a certain set of initial conditions.

The paper is organized as follows. In Section II we introduce the class of piecewise quadratic functions considered. Section III reviews the concept of quasiconvexity. In Section IV we consider the special case of one-variable functions and derive conditions for quasiconvexity. The result is then used in Section V to derive conditions for the general n-dimensional case. Section VI introduces a class of hybrid MPC problems which correspond to minimizing piecewise quadratic functions and in Section VII we consider a practical example from the problem class which is indeed shown to be quasiconvex for a certain set of initial conditions. Finally, Section VIII concludes the paper.

S. Almér, S. Mariéthoz and M. Morari are with the Automatic Control Laboratory, ETH Zürich, Physikstrasse 3, 8092 Zürich, Switzerland [almers,mariethoz,morari]@control.ee.ethz.ch

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II. Piecewise quadratic functions

Consider a rectangular polytope Ω in $\mathbb{R}^n$ defined as

$$\Omega := \{x \in \mathbb{R}^n \mid a_i \leq x_i \leq b_i,\ i = 1, \ldots, n\}$$

where $a_i < b_i$, $i = 1, \ldots, n$. Let Ω be partitioned into smaller rectangular regions by partitioning each interval $[a_i, b_i]$ into N_i pieces according to

$$a_i = p_{i,0} < p_{i,1} < \ \ldots \ < p_{i,N_i} = b_i.$$

Then, the partition of Ω becomes

$$\Omega = \bigcup_{j=1}^{N_{\text{tot}}} \Omega_j$$

where $N_{\text{tot}} = \Pi_{i=1}^{n} N_i$ is the number of regions in the partition and where

$$\Omega_j := \{x \in \mathbb{R}^n \mid p_{i,s_{j_i}-1} \leq x_i \leq p_{i,s_{j_i}},\ i = 1, \ldots, n\}$$

where s_{j_i} is the i^{th} component of the index vector s_j taking value in the set $\{1, \ldots, N_1\} \times \cdots \times \{1, \ldots, N_n\}$.

Let $V : \Omega \to \mathbb{R}$ be a continuous piecewise quadratic function on Ω defined as

$$V(x) = V_j(x) \quad \text{if } x \in \Omega_j \tag{1}$$

where

$$V_j(x) := \frac{1}{2}x' H_j x + f_j' x + g_j, \quad H_j = H_j' > 0 \tag{2}$$

are assumed to satisfy

$$V_j(x) = V_k(x) \quad \forall x \in S_{jk} \tag{3}$$

where

$$S_{jk} := \Omega_j \cap \Omega_k \tag{4}$$

are the intersections of neighboring regions. We note that the assumption $H_j > 0$ implies that each function V_j is strictly convex and that (3) implies that the composite function V is continuous. The function V may be convex, quasiconvex or non-convex. In the sequel we derive necessary and sufficient conditions for V being quasiconvex.

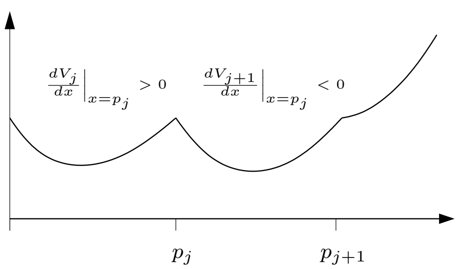

Fig. 1. Non-quasiconvex piecewise quadratic function.

III. Quasiconvexity

Definition 3.1: A function $f : \Omega \to \mathbb{R}$ is quasiconvex iff its domain Ω is convex and for all points $x, y \in \Omega$ and all scalars $\lambda \in [0, 1]$ it holds

$$f((1-\lambda)x + \lambda y) \leq \max\{f(x), f(y)\}. \tag{5}$$

Lemma 3.1: A function $f : \Omega \to \mathbb{R}$ is quasiconvex iff its domain Ω and all sublevel sets

$$S_\alpha := \{x \in \Omega \mid f(x) \leq \alpha\}$$

are convex.

IV. The one-dimensional case

Consider the one-dimensional case where $\Omega = [a, b] \subset \mathbb{R}$, $a < b$. Let $[a, b]$ be partitioned into $N \geq 2$ pieces according to

$$a = p_0 < p_1 < \ \ldots \ < p_N = b$$

and let $V : [a, b] \to \mathbb{R}$ be a piecewise quadratic function defined according to (1)-(3). We note that the continuity conditions (3) reduce to pointwise equalities

$$V_j(p_j) = V_{j+1}(p_j), \quad j = 1, \ldots, N-1$$

where p_j are the points of the partition of $[a, b]$.

Lemma 4.1: V is quasiconvex iff *none* of the points p_j, $j = 1, \ldots, N-1$ of the partition satisfy

$$\left.\frac{dV_j}{dx}\right|_{x=p_j} > 0, \qquad \left.\frac{dV_{j+1}}{dx}\right|_{x=p_j} < 0. \tag{6}$$

Proof: We first prove that if V is quasiconvex, then none of the points p_j of the partition satisfy (6): Assume for the sake of contradiction that V is quasiconvex and there is a point p_j, $j \in \{1, \ldots, N-1\}$ satisfying (6). Since dV_j/dx are continuous functions, there exists $\epsilon > 0$ such that

$$\begin{aligned} V_j(x) &< V_j(p_j) \quad \forall x \in (p_j - \epsilon,\ p_j) \\ V_{j+1}(x) &< V_j(p_j) \quad \forall x \in (p_j,\ p_j + \epsilon). \end{aligned} \tag{7}$$

Take points $x \in (p_j - \epsilon,\ p_j)$, $y \in (p_j,\ p_j + \epsilon)$ and take $\lambda \in [0, 1]$ such that $(1-\lambda)x + \lambda y = p_j$ (such λ exists since $x < p_j < y$). It holds

$$V((1-\lambda)x + \lambda y) = V_j(p_j) > \max\{V(x), V(y)\}$$

where in the last inequality we have used (7). This contradicts that V is quasiconvex.

We now prove that if none of the points p_j, $j = 1, \ldots, N-1$ of the partition satisfy (6), then V is quasiconvex: Assume for the sake of contradiction that no point p_j, $j = 1, \ldots, N-1$ satisfies (6) and V is not quasiconvex. If V is not quasiconvex, by Lemma 3.1 there exists a level set $S_\alpha = \{x \in [a, b] \mid V(x) \leq \alpha\}$ which is not convex. In other words, there exists a level set which can be written as a union of disjoint intervals;

$$S_\alpha = \bigcup_k [\alpha_k, \beta_k]$$

where $\alpha_k \leq \beta_k$ and $\beta_k < \alpha_{k+1}$. Since V_j are convex, the points β_k and α_{k+1} cannot both be in the same interval

$[p_j, p_{j+1}]$ of the partition. This means that there are one or more points p_j between β_k and α_{k+1}, *i.e.*,

$$\beta_k \leq p_j < p_{j+1} < \ \dots \ < p_{j+m} \leq \alpha_{k+1}.$$

Now consider the maximum of V restricted to the interval $[\beta_k, \alpha_{k+1}]$. Because the maximum of a convex function over a polytope is obtained at a vertex of the polytope [3], the maximum will be obtained at a (possibly non-unique) point p_k, $k \in \{j, \dots, j+m\}$.

The assumption $H_j > 0$ implies that V_j are non-constant and thus, there exists $\epsilon > 0$ such that

$$\begin{aligned} V_k(x) < V(p_k) \quad \forall x \in (p_k - \epsilon,\ p_k) \\ V_{k+1}(x) < V(p_k) \quad \forall x \in (p_k,\ p_k + \epsilon) \end{aligned}$$

which implies that

$$\left.\frac{dV_k}{dx}\right|_{x=p_k} > 0, \quad \left.\frac{dV_{k+1}}{dx}\right|_{x=p_k} < 0$$

which contradicts that there is no point p_j satisfying (6). This concludes the proof. ■

V. The N-dimensional case

Consider now the case $\Omega \subset \mathbb{R}^n$, $n \geq 1$. To derive necessary and sufficient conditions for quasiconvexity of the function $V : \Omega \to \mathbb{R}$ we use the definition of quasiconvexity and consider V along all lines in Ω. When restricted to a line, V reduces to a one-variable function and we apply the result of Lemma 4.1 in Section IV to verify inequality (5). The result is stated formally as follows.

Theorem 5.1: Let $V : \Omega \to \mathbb{R}$ be defined according to (1)-(3) where $\Omega \subset \mathbb{R}^n$, $n \geq 1$ and let S_{ij} be defined by (4). Let e_{ij} be the normal of the surface S_{ij} pointing from Ω_i towards Ω_j and let Z_{ij} be a matrix of columns spanning the orthogonal complement of e_{ij}. V is quasiconvex iff for all surfaces S_{ij} of dimension $n-1$ the following conditions hold.

1) For all $x \in S_{ij}$ such that $Z'_{ij} \nabla_x V_i(x) = 0$, it does *not* hold

$$e'_{ij} \nabla V_i(x) > 0, \quad e'_{ij} \nabla V_j(x) < 0. \tag{8}$$

2) For all $x \in S_{ij}$ such that $Z'_{ij} \nabla_x V_i(x) \neq 0$ it holds

$$e'_{ij} \nabla_x V_i(x) \leq e'_{ij} \nabla_x V_j(x). \tag{9}$$

Remark 5.1: In the one-dimensional case ($n = 1$), the surfaces S_{ij} are points and the normals e_{ij} are scalar and thus, $Z_{ij} = 0$. This implies that the second set of conditions in Theorem 5.1 disappears and the remaining inequalities are indeed equivalent to the inequalities stated in Lemma 4.1 which considers the one-dimensional case.

Remark 5.2: We note that the conditions above holds for the index pair (ij) if and only if it holds for the pair (ji). We also note that the conditions of the theorem are linear inequalities which should hold for all points in a polytope. Such conditions can be formulated equivalently in terms of a set of linear inequalities using the S-procedure [5], [7].

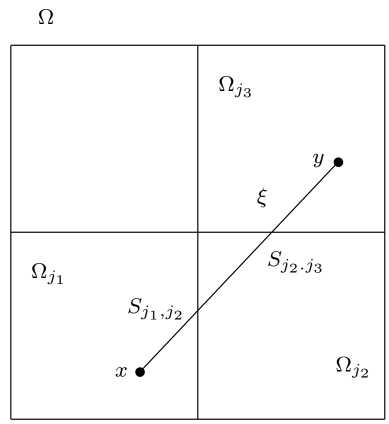

Fig. 2. Partition of Ω into regions Ω_j. The line ξ is contained in regions Ω_{j_1}, Ω_{j_2}, Ω_{j_3} and crosses surfaces S_{j_1,j_2}, $S_{j_2 \cdot j_3}$.

Proof: According to the definition, V is quasiconvex iff

$$V((1-\lambda)x + \lambda y) \leq \max\{V(x), V(y)\} \quad \forall \lambda \in [0,1] \tag{10}$$

holds for all $x, y \in \Omega$. Because of continuity and the structure of V, we need not show (10) for all $x, y \in \Omega$. Firstly, if x and y are both in the same set Ω_j, then (10) follows directly from convexity of V_j. Secondly, because of continuity we may omit points x and y which lie in any set S_{ij} and we may also omit lines $(1-\lambda)x + \lambda y$ which intersect sets S_{ij} of dimension less than $n-1$. Thus, V is quasiconvex iff (10) holds $\forall\ x \in \text{int}(\Omega_i)$, $y \in \text{int}(\Omega_j)$, $i \neq j$ such that the line $(1-\lambda)x + \lambda y$ does not intersect a set S_{ij} of dimension less than $n-1$.

Consider $x \in \text{int}(\Omega_i)$, $y \in \text{int}(\Omega_j)$, $i \neq j$ and let

$$\xi(\lambda) = (1-\lambda)x + \lambda y = x + (y-x)\lambda = x + r\lambda$$

be the line between x and y parameterized by $\lambda \in [0,1]$ having direction

$$r := y - x.$$

The function V restricted to the line ξ is a continuous piecewise quadratic function in λ. By assumption, the line ξ is contained in $m \geq 2$ regions Ω_{j_k}, $k = 1, \dots, m$ and intersects $m-1$ surfaces

$$S_{j_1,j_2},\ S_{j_2,j_3},\ \dots,\ S_{j_{m-2},j_{m-1}}$$

all of dimension $n-1$. Thus we have

$$V_\xi(\lambda) := V(\xi(\lambda)) = V_{\xi,j_k}(\lambda) \text{ if } p_{k-1} \leq \lambda \leq p_k$$

where

$$0 = p_0 < p_1 < \ \dots \ < p_m = 1$$

is a partition of the interval $[0,1]$ where p_k, $k = 1, \dots, m-1$ are the points where $V_\xi(\lambda)$ intersects $S_{j_k,j_{k+1}}$;

$$V_\xi(p_k) \in S_{j_k,j_{k+1}}, \quad k = 1, \dots, m-1.$$

The functions $V_{\xi,j}$ are defined as

$$\begin{aligned} V_{\xi,j}(\lambda) &= \frac{1}{2}\overline{H}_j\lambda^2 + \bar{f}_j\lambda + \bar{g}_j \\ \overline{H}_j &= \frac{1}{2}r'H_jr \\ \bar{f}_j &= (H_jx + f_j)'r \\ \bar{g}_j &= \frac{1}{2}x'H_jx + f_j'x + g_j. \end{aligned}$$

From Lemma 4.1 it follows that (10) holds iff none of the points p_k satisfy

$$\left.\frac{dV_{\xi,j_k}}{d\lambda}\right|_{\lambda=p_k} > 0, \quad \left.\frac{dV_{\xi,j_{k+1}}}{d\lambda}\right|_{\lambda=p_k} < 0. \tag{11}$$

We now reformulate (11) in terms of the gradients of V_j. Consider a scalar $\bar{\lambda} \in [0,1]$ and let $\bar{x}$ be the corresponding point on the line ξ so that $\bar{x} = x + r\bar{\lambda}$. It holds

$$\left.\frac{dV_{\xi,j}}{d\lambda}\right|_{\lambda=\bar{\lambda}} = r'\nabla_x V_j\big|_{x=\bar{x}}. \tag{12}$$

Substituting the expression above in (11) it follows that (10) holds iff none of the points p_k satisfy

$$r'\nabla_x V_{j_k}(x + rp_k) > 0, \quad r'\nabla_x V_{j_{k+1}}(x + rp_k) < 0. \tag{13}$$

The inequality (13) holds for all lines $(1-\lambda)x + \lambda y$ in Ω iff for all surfaces S_{ij}, there is *no* point $x \in S_{ij}$ and direction r pointing from Ω_i to Ω_j satisfying

$$r'\nabla_x V_i(x) > 0, \quad r'\nabla_x V_j(x) < 0, \quad e_{ij}'r > 0. \tag{14}$$

where e_{ij} is the normal of the surface S_{ij} pointing from Ω_i to Ω_j and where the inequality $e_{ij}'r > 0$ restricts r to be directed from Ω_i toward Ω_j.

To show quasiconvexity of V we need to show that for all S_{ij}, there is *no* point $x \in S_{ij}$ and direction r satisfying (14). The inequality (14) is quadratic in x and r and is therefore non-trivial to verify. In order to derive a simpler (equivalent) statement of (14) we note the following.

- The continuity condition implies that V_i and V_j are identical on S_{ij}. This implies that the partial derivatives along directions parallel to and inside S_{ij} are also identical for the two functions and we can therefore write
$$\begin{aligned} \nabla V_i(x) &= \nabla V_\perp(x) + \overline{\nabla V}_i(x) \\ \nabla V_j(x) &= \nabla V_\perp(x) + \overline{\nabla V}_j(x) \end{aligned} \tag{15}$$
where $\nabla V_\perp$ is the projection of V_i (V_j) onto S_{ij} and where $\overline{\nabla V}_i$, $\overline{\nabla V}_j$ are parallel with the normal e_{ij}.
- The surface S_{ij} can be split into two subsets;
$$\begin{aligned} S_{ij}^1 &:= \{x \in S_{ij} \mid V_\perp(x) = 0\} \\ &= \{x \in S_{ij} \mid Z_{ij}'V_i(x) = 0\} \\ S_{ij}^2 &:= \{x \in S_{ij} \mid V_\perp(x) \neq 0\} \\ &= \{x \in S_{ij} \mid Z_{ij}'V_i(x) \neq 0\}. \end{aligned}$$

We now proceed with the reformulation of (14). For each surface S_{ij} we consider the two cases that $x \in S_{ij}^1$ and $x \in S_{ij}^2$.

Case 1: Assume $x \in S_{ij}^1$ so that $\nabla V_\perp(x) = 0$. In this case ∇V_i and ∇V_j are parallel to e_{ij} and we can without loss of generality choose $r = c \cdot e_{ij}$, $c > 0$ in (14). Doing so, one can show that there is *no* direction r satisfying (14) iff it does *not* hold

$$e_{ij}'\nabla V_i(x) > 0, \quad e_{ij}'\nabla V_j(x) < 0.$$

This is guaranteed by the first set of inequalities in Theorem 5.1.

Case 2: Assume $x \in S_{ij}^2$ so that $\nabla V_\perp(x) \neq 0$. By Gordan's Theorem [8] there is *no* solution r to (14) iff there *is* a solution $y \geq 0$, $y \neq 0$ to

$$y_1(-\nabla V_i(x)) + y_2\nabla V_j(x) = y_3 e_{ij}. \tag{16}$$

Using the decomposition (15) in (16) we get

$$(y_2 - y_1)\nabla V_\perp - y_1\overline{\nabla V}_i + y_2\overline{\nabla V}_j = y_3 e_{ij}.$$

By assumption, $\nabla V_\perp$ is non-zero. Furthermore, $\nabla V_\perp$ is orthogonal to e_{ij} and the coefficient multiplying $\nabla V_\perp$ must therefore be zero in order for the equation to have a solution. Thus it must hold $y_2 = y_1$ and we get

$$y_1(\overline{\nabla V}_j - \overline{\nabla V}_i) = y_3 e_{ij}$$

which has a solution $y \geq 0$, $y \neq 0$ iff

$$e_{ij}'\nabla V_j \geq e_{ij}'\nabla V_i.$$

This is guaranteed by the second set of inequalities in Theorem 5.1. This concludes the proof. ∎

Corollary 5.1: Consider the case where V has dimension strictly greater than one so that $\Omega \subset \mathbb{R}^n$, $n \geq 2$. Then V is quasiconvex iff for all surfaces S_{ij} it holds

$$e_{ij}'\nabla_x V_i(x) \leq e_{ij}'\nabla_x V_j(x) \quad \forall x \in S_{ij}.$$

Proof: Since each function V_j is strictly convex, the function V_j restricted to the surface S_{ij} has a unique minimum and thus, the equality $Z_{ij}'\nabla_x V_i(x) = 0$ can hold for at most one point $x \in S_{ij}$. Assume that $Z_{ij}'\nabla_x V_i(x) = 0$ holds for a point $x^* \in S_{ij}$. By Theorem 5.1 we then have to consider inequality (8) for the point x^* and the inequality (9) for all points $x \in S_{ij} \setminus x^*$. However, by continuity, (9) holds for all $x \in S_{ij} \setminus x^*$ iff (9) holds for all $x \in S_{ij}$. Furthermore, if (9) holds for all $x \in S_{ij}$, then this implies that (8) holds and thus, the first set of inequalities in the theorem are redundant. Thus, in the case when the variable dimension is $n \geq 2$ it is sufficient to verify (9) for all $x \in S_{ij}$ to verify quasiconvexity of V. ∎

Corollary 5.2: Consider the case where V has dimension strictly greater than one so that $\Omega \subset \mathbb{R}^n$, $n \geq 2$. Then V is convex if it is quasiconvex.

Proof: Assume for the sake of counterargument that V is quasiconvex, but not convex. Then there exists neighboring sets Ω_i, Ω_j, points $x \in \text{int}(\Omega_i)$, $y \in \text{int}(\Omega_j)$ and a line $\xi(\lambda) = x + r\lambda$, $\lambda \in [0,1]$ where $r := y - x$ is in the direction from Ω_i to Ω_j such that

$$\left.\frac{dV_{\xi,i}}{d\lambda}\right|_{\lambda=\bar{\lambda}} > \left.\frac{dV_{\xi,j}}{d\lambda}\right|_{\lambda=\bar{\lambda}}$$

where $\bar{\lambda} \in [0,1]$ is such that $\bar{x} := \xi(\bar{\lambda}) \in S_{ij}$. Using (12) we conclude that r and $\bar{x}$ satisfy

$$r' \nabla_x V_i(\bar{x}) > r' \nabla_x V_j(\bar{x}).$$

Using the decomposition (15) and decomposing r as $r = r_\perp + r_\parallel$ where $r_\perp$ is orthogonal to the normal e_{ij} of S_{ij} and $r_\parallel$ is parallel to e_{ij}, this is equivalent to

$$\begin{aligned} r_\parallel' \overline{\nabla V}_i(\bar{x}) > r_\parallel' \overline{\nabla V}_j(\bar{x}) \Leftrightarrow \\ e_{ij}' \overline{\nabla V}_i(\bar{x}) > e_{ij}' \overline{\nabla V}_j(\bar{x}) \end{aligned}$$

where the equivalence follows because $r_\parallel = c \cdot e_{ij}$, $c > 0$. By Theorem 5.1, the last inequality above contradicts the assumption that V is quasiconvex and thus we have a contradiction. This concludes the proof. ∎

Remark 5.3: The fact that in dimensions higher than one, the function V is convex if it is quasiconvex is a consequence of the fact that the "pieces" V_j are assumed to be strictly convex.

VI. Application in hybrid MPC

The present section introduces a class of hybrid MPC problems where the result in Theorem 5.1 can be used to determine whether or not the problem is quasiconvex.

Consider a PWA system defined over a partition of the domain of the control input;

$$\begin{aligned} x_{k+1} = A x_k + B_i u_k + f_i \\ \text{if } p_{i-1} \le u_k \le p_i, \ i = 1, \ldots, \nu \end{aligned} \tag{17}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}$, A, B_i, f_i are constant matrices of matching dimensions and where

$$a = p_0 < p_1 < \ \ldots \ < p_\nu = b$$

is the partition of the domain of the control input into ν pieces. The PWA vector field is assumed to be continuous and thus satisfies

$$B_i p_i + f_i = B_{i+1} p_i + f_{i+1}, \quad i = 1, \ldots, \nu - 1.$$

We consider the problem of minimizing a quadratic cost criterion over a finite prediction horizon, *i.e.*, the control objective is to minimize

$$J = \sum_{l=k}^{k+N} (x_l - x_{\text{ref}})' Q (x_k - x_{\text{ref}}) \tag{18}$$

where $Q = Q' > 0$, N is the prediction horizon and x_{ref} is the reference. We minimize (18) subject to the dynamic constraints (17) and initial condition $x_k = x_0$. This problem can be stated as minimizing $V(U) : \Omega \to \mathbb{R}$, $\Omega \subset \mathbb{R}^N$ which is a function of the vector of control inputs $U := [u_k, \ldots, u_{k+N-1}]'$ defined according to (1)-(2) where

$$\begin{aligned} H_j &:= \mathcal{B}_j' Q \mathcal{B}_j \\ f_j &:= f_{1j} + f_{2j} x_0 \\ f_{1j} &:= \mathcal{B}_j' Q (\mathcal{F}_j - \mathcal{R}) \\ f_{2j} &:= \mathcal{B}_j' Q \mathcal{A} \\ g_j &:= \frac{1}{2} (\mathcal{A} x_0 + \mathcal{F} - \mathcal{R})' Q (\mathcal{A} x_0 + \mathcal{F} - \mathcal{R}) \end{aligned}$$

where

$$\mathcal{A} := \begin{bmatrix} A \\ A^2 \\ \vdots \\ A^N \end{bmatrix}, \ \mathcal{B}_j := \begin{bmatrix} B_{s_{j_1}} & 0 & \ldots & 0 \\ A_{s_{j_1}} & B_{s_{j_2}} & \ldots & 0 \\ \vdots & & & \\ A^{N-1} B_{s_{j_1}} & & \ldots & B_{s_{j_{N-1}}} \end{bmatrix}$$

$$\mathcal{R} := \begin{bmatrix} r_{\text{ref}} \\ x_{\text{ref}} \\ \vdots \\ x_{\text{ref}} \end{bmatrix}, \ \mathcal{F}_j := \begin{bmatrix} f_{s_{j_1}} \\ A_{s_{j_1}} + f_{s_{j_2}} \\ \vdots \\ A^{N-1} f_{s_{j_1}} + \cdots + f_{s_{j_{N-1}}} \end{bmatrix}$$

where the index vector $s_{j_i} \in \{1, \ldots, \nu\}^N$ describes the sequence of active regions of the PWA system. We note that the continuity condition (3) is satisfied by continuity of the PWA model (17).

The cost function described above depends on the initial condition x_0. Clearly it is of interest to verify quasiconvexity of the problem for a set of initial points. Thus, we want to know for which initial conditions x_0 inequality (9) of Theorem 5.1 holds. In other words, we want to investigate for which x_0 it holds

$$\begin{aligned} & e_{ij}' (H_i x + f_{1i} + f_{2i} x_0) \le e_{ij}' (H_j x + f_{1j} + f_{2j} x_0) \\ & \forall x \in S_{ij} \end{aligned} \tag{19}$$

for all surfaces S_{ij} of dimension $n - 1$. Using the S-procedure [5], [7] the condition can be equivalently stated as a set of linear inequalities.

VII. Example

As an example of the problem class described in Section VI above we consider the switched mode step-down converter depicted in Fig. 3. The switched dynamics of the converter are described by

$$\dot{x}(t) = A x(t) + s(t) B_0 v_s + B_1 i_o(t)$$

where $x = [v_c, \ i_\ell]'$ is the state (where v_c is the capacitor voltage and i_ℓ is the inductor current), v_s is the source voltage, i_o is the load current and the system matrices are

$$A = \begin{bmatrix} 0 & \frac{1}{x_c} \\ -\frac{1}{x_\ell} & 0 \end{bmatrix}, \quad B_0 = \begin{bmatrix} 0 \\ \frac{1}{x_\ell} \end{bmatrix}, \quad B_1 = \begin{bmatrix} -\frac{1}{x_c} \\ 0 \end{bmatrix}.$$

The switch function $s(t) \in \{0, 1\}$ represents the position of the switch. The switch function is controlled using fixed-frequency switching and a so-called duty cycle $d_k \in [0, 1]$ according to

$$s(t) = \begin{cases} 1 & t \in [kT_s, (k + d_k) T_s) \\ 0 & t \in [(k + d_k) T_s, (k+1) T_s) \end{cases}$$

where $T_s > 0$ is the switch period. The parameter values of the circuit are taken from [1].

To obtain a tractable control model of the switched dynamics we apply the PWA modeling approach introduced in [6]. Compared to the standard averaged model [9], the PWA model in [6] gives a more accurate description of the system state. Unlike the averaged model, the PWA model represents the switching ripple which is inherent in the state

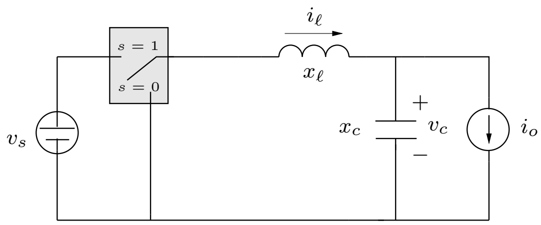

Fig. 3. Synchronous step-down converter.

of a switched mode power converter and thus has potential for improved performance.

To derive the PWA model, the domain $[0, 1]$ of the duty cycle is partitioned into $\nu = 2$ pieces and we thus obtain a PWA model on the form (17) defined on the partition

$$0 = p_0 < p_1 = 1/2 < p_2 = 1$$

with system matrices

$$A = \begin{bmatrix} 0.9075 & 0.0870 \\ -2.0294 & 0.9075 \end{bmatrix}$$
$$B_1 = \begin{bmatrix} 0.1383 \\ 1.9807 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0.0468 \\ 2.0780 \end{bmatrix}$$
$$f_1 = \begin{bmatrix} -0.0870 \\ 0.0925 \end{bmatrix}, \quad f_2 = \begin{bmatrix} -0.0413 \\ 0.0439 \end{bmatrix}.$$

We consider the control objective (18) with $Q = I$, $N = 2$ and $x_{\text{ref}} = [1,\ 1]'$. The resulting optimization problem corresponds to minimizing a piecewise quadratic function on the form (1)-(2) defined over a four-piece partition of $\mathbb{R}^2$.

To see for which initial conditions x_0 the problem is convex we consider the inequality (19) for the four surfaces S_{ij} defining the partition. For each surface the inequality becomes a single hyperplane and the set of initial conditions that yield a convex problem is thus defined by the intersection of four halfspaces. The hyperplanes are illustrated in Fig. 4. The set of initial conditions which lie above the hyperplanes (away from the dashed lines) yield a convex problem.

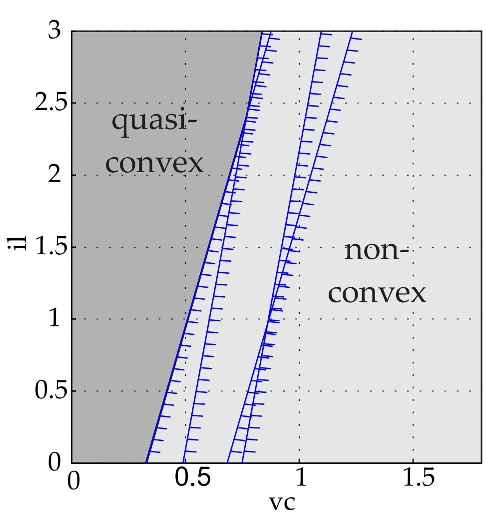

Fig. 4. Subset of the initial conditions for which the MIQP is convex.

VIII. Conclusion

The paper derived necessary and sufficient conditions for quasiconvexity of continuous piecewise quadratic functions. The conditions were first derived for the one-dimensional case and this result was then used to derive conditions for the general n-dimensional case. The conditions are stated in terms of linear inequalities which can be verified efficiently. It is shown that in dimensions higher than one, the function is convex if it is quasiconvex. Thus, there are no functions of the considered class which are quasiconvex, but not convex if the dimension is higher than one. This result is a consequence of the assumption that the "pieces" of the piecewise quadratic function are strictly convex.

A class of hybrid MPC problems were considered where solving the resulting mixed integer quadratic optimization problem consists of minimizing continuous piecewise quadratic functions. The quasiconvexity can be verified using the derived conditions. This allows to determine for which initial points a gradient search could be used to find the optimal system input efficiently. A practical example from the hybrid MPC problem class was considered and proved to be quasiconvex for a certain set of initial conditions.

In general it is unlikely that a problem from the class considered is quasiconvex for all initial conditions of interest. Future work could therefore include using the result of the paper to develop an algorithm for partitioning a problem into subregions where quasiconvexity/concavity can be shown.

References

[1] S. Almér, H. Fujioka, U. Jönsson, C.-Y. Kao, D. Patino, P. Riedinger, T. Geyer, A. Beccuti, G. Papafotiou, M. Morari, A. Wernrud, and A. Rantzer. Hybrid Control Techniques for Synchronous DC-DC Converters, Part I: The Step-Down Topology. In *Proc. American Control Conference*, pages 5450–5457, 2007.

[2] A. Bemporad and M. Morari. Control of systems integrating logic, dynamics and constraints. *Automatica*, 35(3):407–427, 1999.

[3] D. P. Bertsekas. *Nonlinear Programming*. Athena Scientific, second edition, 1999.

[4] F. Borrelli. *Constrained Optimal Control of Linear and Hybrid Systems*. Springer, 2003.

[5] S. Boyd, L. El Ghaoui, E. Feron, and V. Balakrishman. *Linear Matrix Inequalities in System and Control Theory*. SIAM, Philadelphiia, Penn., 1994.

[6] T. Geyer, G. Papafotiou, and M. Morari. Hybrid Model Predictive control of the Step-Down DC-DC Converter. *IEEE Transactions on Control Systems Technology*, 16(6):1112–1124, 2008.

[7] U. T. Jönsson. *A Lecture on the S-Procedure*. Royal Institute of Technology, 2001.

[8] O. L. Mangasarian. *Nonlinear Programming*. McGraw-Hill, 1969.

[9] R. D. Middlebrook and S. Ćuk. A general unified approach to modeling switching-converter power stages. In *Proc. IEEE Power Electronics Specialists Conference*, pages 18–34, 1976.

[10] Ned Mohan, Tore M. Undeland, and William P. Robbins. *Power Electronics. Converters Applications and Design*. John Wiley and Sons Inc, New York, USA, 3rd edition, 2003.

[11] E.D. Sontag. Nonlinear regulation: The piecewise linear approach. *IEEE Transactions on Automatic Control*, 26(2):346–358, April 1981.

[12] P. Tondel, T. A. Johansen, and A. Bemporad. An algorithm for multi-parametric quadratic programming and explicit mpc solutions. *Automatica*, 39(3):489–497, 2003.