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Application of optimal boundary control to reaction-diffusion system with time-varying spatial domain

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Abstract— **This paper considers the optimal boundary control of a parabolic partial differential equation (PDE) with time-varying spatial domain which is coupled to a second order ordinary differential equation (ODE) describing the time-evolution of the domain boundary. The infinite-dimensional state space representation of the PDE yields a linear non-autonomous evolution system with an operator which generates a two-parameter semigroup with analytic expression provided in this work. The nonautonomous evolution system is transformed into an extended system which enables the optimal boundary control problem to be considered. The optimal control law of the extended system is determined and numerical results of the closed-loop feedback system are provided.**

I. INTRODUCTION

Reaction-diffusion systems with time-varying spatial domains arise naturally in many physical processes including the industrial processes of metal casting, Czochralski (CZ) crystal growth and annealing processes where the domain motion is characterized by a change in the material boundary. In annealing processes the boundary motion is determined by a mechanical pulling arm actuator which draws a solid slab from a fluid bath, as depicted in the Fig.1. The purity, component concentration, and metallurgical properties of the material are often dependent on the temperature evolution and the rate at which the slab cools during its processing, see [1], which requires temperature regulation throughout the slab in order to maintain a desired nominal temperature distribution during the process.

The slab temperature dynamics are modeled in general by a reaction-diffusion-convection PDE defined on the time-dependent spatial domain. The boundary motion contributes to a convective transport term in the parabolic PDEs expression associated with a time-dependent system coefficient. Representation of the parabolic PDE on an appropriately defined function space yields a nonautonomous evolution system with solution expressed in terms of a two-parameter semigroup, see [2]. Although the distributed control of parabolic PDE systems with moving boundaries has been considered in several works, see for example [3], [4], [5], the ability to regulate the desired temperature spatial distribution depends on the process setup which may impose limitations on the actuators placement. For example in the CZ crystal growth process, the temperature regulation of the crystal by heaters placed along the pulled crystal, at the initial stage of the process is not feasible in the presence of an ecapsulant

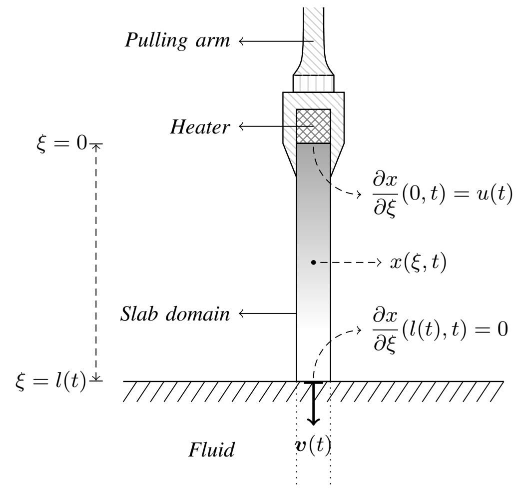

Fig. 1. Annealing of a solid slab where $f(\xi, t)$ represents the temperature. The boundary $\xi = l(t)$ is moving with velocity $\boldsymbol{v}(t)$ and heat input $u(t)$ is applied at the boundary $\xi = 0$.

which embodies the grown crystal and prevents the crystal surface to be exposed to the heaters, [6]. Another example is given by annealing processes in which the placement of heaters along the slab domain becomes economically infeasible if the slab is treated in a non-isolated thermal environment. However, one can consider the boundary control of such processes in which the heat is applied via conduction at the domain boundary and it is usually applied in the process control realizations when distributed actuation can not be applied, see [9], [10].

Motivated by processes with the time-varying spatial domains and boundary control formulation we consider the system setup as depicted in the Fig.1 where the heat input is applied at the stationary boundary. The process model for the temperature dynamics are given by a reaction-diffusion system modelled by a parabolic PDE which is unidirectionally coupled with the mechanical subsystem determining the time-evolution of the spatial domain with dynamics described by an ODE which describes the rigid body dynamics of the moving slab. This paper is organized as follows: In Section 2, the parabolic PDE model of the reaction-diffusion system with time-varying spatial domain is presented and the appropriate functional space setting is provided which enables the representation of the PDE as a nonautonomous linear parabolic evolution system with evolution operator described by a two-parameter semigroup. In Section 3, the evolution system is transformed in order to consider the

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optimal boundary control problem and we explore conditions of spectral assignability of time-varying linear parabolic PDE. Section 4 provides numerical simulation results of the optimal control synthesis, and Section 5 concludes the paper with the summary of results.

II. Preliminaries

This section introduces the notation and the function space setting and the parabolic PDE model on time-varying spatial domain is presented. The nonautonomous evolution system representation of the parabolic PDE is formulated and the analytic expression of the solution operator of the initial value problem is provided in terms of a two-parameter evolution operator.

A. Notation and function space description

The following notation will be used throughout this work: A general Banach space will be denoted as $\mathcal{X}$. If $\mathcal{Y}$ is a Banach space, $\mathcal{L}(\mathcal{X},\mathcal{Y})$ denotes the space of bounded linear operators $\mathcal{T}:\mathcal{X}\to\mathcal{Y}$ and $\mathcal{L}(\mathcal{X})=\mathcal{L}(\mathcal{X},\mathcal{X})$. The time index t is taken in the interval $[0,T]$ for notational convenience. The time-dependent spatial domain Ω_t at some time $t\in[0,T]$ is a bounded open set of $\mathbb{R}^n$ with smooth boundary $\partial\Omega_t$, closure $\bar{\Omega}_t$, and with the initial configuration Ω_0. The largest set will be denoted as $\boldsymbol{\Omega}$ with boundary $\partial\boldsymbol{\Omega}$ such that $\Omega_t\subseteq\boldsymbol{\Omega}$ for all $t\in[0,T]$, and spatial points are denoted by $\xi\in\mathbb{R}^n$. The space of continuous functions on Ω_t is denoted as $C(\Omega_t)$ and $C^k([0,T];\mathcal{X})$ consist of all functions which are k times continuously differentiable, with $k\in\mathbb{N}$, defined in the time interval $[0,T]$ and taking values in $\mathcal{X}$. The space $C^k(\Omega_t)$, $k\in\mathbb{N}$ consists of the functions having all derivatives up to order k continuous on Ω_t and the Hilbert space $L^2(\Omega_t)$ denotes the set of all square integrable functions defined on Ω_t. We will also use the notion of precompact function spaces in describing compact imbedding $L^2(\Omega_t)\subset L^2(\boldsymbol{\Omega})$ which enables the use of a single inner product $\langle\cdot,\cdot\rangle_{L^2(\boldsymbol{\Omega})}=\langle\cdot,\cdot\rangle$, see [11, Theorem 2.21-2.22]. In this way we can handle time-dependent functions defined on Ω_t at each $t\in[0,T]$ by using the $L^2(\boldsymbol{\Omega})$ inner product. The Hilbert spaces $H^{m,p}(\boldsymbol{\Omega})$ with norm $\|\cdot\|_{m,p}$ follow with standard definitions, properties and continuous imbeddings. For simplicity, we denote $H^{1,2}(\boldsymbol{\Omega}):=H^1(\boldsymbol{\Omega})$ and $H^{2,2}(\boldsymbol{\Omega}):=H^2(\boldsymbol{\Omega})$, which are dense in $L^2(\boldsymbol{\Omega})$, see [11], [12], [13].

B. Process model description

The parabolic PDE which describes the temperature dynamics in a spatial domain with time-dependent boundary motion arises from the following, see [14].

Theorem 1 (Transport theorem): Consider a bounded function $x(\xi,t)\in C^1(\Omega_t)$ on Ω_t which is continuous on $\partial\Omega_t$. Let the boundary be moving with finite velocity $\boldsymbol{v}(t)\in C^1([0,T])$. The Transport Theorem describes rate of change of x with respect to time in Ω as:

$$\rho C_p\frac{d}{dt}\int_{\Omega_t}x\,d\xi=\rho C_p\int_{\Omega_t}\left(\frac{\partial x}{\partial t}+\boldsymbol{v}\cdot\nabla x\right)d\xi \tag{1}$$

where ∇ is the gradient operator on $z\in\Omega$.

The positive constants ρ and C_p in the Eq.1 denote the density and specific heat capacity of $\boldsymbol{\Omega}$ which is assumed to be a homogeneous and uniform material. The convective transport term in the Eq.1 is expressed as:

$$\boldsymbol{v}(t)\cdot\nabla x=\frac{d\xi_i}{dt}\frac{\partial x_i}{\partial\xi_j},\qquad i,j=\{1,\dots,n\} \tag{2}$$

and arises from the deformation of the domain. Application of energy balance principles yields the reaction-diffusion-convection PDE which is defined as follows.

Definition 1: The parabolic PDE which describes the temperature dynamics in a region Ω_t is given by:

$$\frac{\partial x}{\partial t}=A(\xi,t)x \tag{3}$$

where there operator $A(\xi,t)$ is defined as:

$$A(\xi,t):=\sum_{i,j=1}^{n}\frac{\partial}{\partial\xi_i}\kappa_{ij}(\xi)\frac{\partial}{\partial\xi_j}-\sum_{k=1}^{n}\boldsymbol{v}_k(t)\frac{\partial}{\partial\xi_k}-g(\xi) \tag{4}$$

with $\kappa(\xi)\in C(\bar{\Omega}_t)$ denoting the thermal diffusivity of the region $\boldsymbol{\Omega}$ which is positive and satisfies $\kappa_{ij}(\xi)=\kappa_{ji}(\xi)$, and $g(\xi)\in C(\bar{\Omega}_t)$ is the linearized reaction term.

Since $\kappa(\xi)$ is symmetric and positive, the principle part of the operator $A(\xi,t)$ satisfies:

$$\sum_{i,j}^{n}\kappa_{ij}(\xi)\eta_i\eta_j\geq\varepsilon|\eta|^2\quad\text{for}\quad\xi\in\Omega_t,\ \eta\in\mathbb{R}^n \tag{5}$$

for constant $\varepsilon>0$, which implies that $A(\xi,t)$ in the Eq.4 is strongly elliptic for each $t\in[0,T]$. The strong ellipticity property in the Eq.5 is important in defining the evolution system representation of the PDE in the Eq.3, together with the continuity of the coefficients $\kappa(\xi)$, $g(\xi)$ and that $\boldsymbol{v}(t)\in C^1([0,T])$ is continuous with $C^1([0,T])\subset C^\alpha([0,T])$, for $\alpha\in[0,1)$, where C^α denotes the space of Hölder continuous functions.

Remark 1: In the case of a time invariant region where the motion of Ω_t is isochronic, i.e. the boundary velocity $\boldsymbol{v}(t)=0$, the contribution by convective transport vanishes which leads to the standard expression of the reaction-diffusion PDE.

C. Evolution system representation

Consider the 1-dimensional form of the parabolic PDE in the Eq.3 on the time dependent spatial domain $\Omega_t=(0,l(t))\subset\mathbb{R}$, with the homogeneous Neumann boundary conditions:

$$\frac{\partial x}{\partial\xi}(0,t)=0,\qquad\frac{\partial x}{\partial\xi}(l(t),t)=0 \tag{6}$$

and define the nonautonomous linear operator:

$$A(t)x=A(\xi,t)x,\quad\text{for}\quad x\in D(A(t)) \tag{7}$$

where $D(A(t))=H^1(\boldsymbol{\Omega})\cap H^2(\boldsymbol{\Omega})$ is the domain of the operator $A(t)$ associated with the family of strongly elliptic operators $A(\xi,t)$. The nonautonomous evolution system representation of the initial and boundary value problem formed by the PDE in the Eq.3 together with the Eq.6 is given by:

$$\frac{dx}{dt} = A(t), \qquad x(0) = x_0 \tag{8}$$

for initial state $x_0 \in L^2(\mathbf{\Omega})$. The properties of the operator $A(\xi,t)$ and the associated nonautonomous operator $A(t)$ yields the solution $x(t)$ of the initial value problem in the Eq.8 in terms of a two-parameter evolution operator $U(t,s)$ such that $x(t)$ is the generalized solution of the initial and boundary value problem, see [15, Theorem 6.1, Chapter 5; Lemma 6.1, Chapter 7] and [16, Chapter 6]. In order to construct the two-parameter semigroup $U(t,s)$, let $\kappa(\xi) = \kappa > 0$ (constant), and consider the eigenvalue problem $A(t)\phi = \lambda\phi$ at each $t \in [0,T]$ which yields the time dependent family of eigenvalues $\{\lambda_n(t)\}_{t\in[0,T]}$, for $n \in \mathbb{N}$, determined as:

$$\lambda_n(t) = -\kappa\left(\frac{n\pi}{l(t)}\right)^2 - \frac{1}{2}\kappa^{-1}\frac{\boldsymbol{v}(t)^2}{2} - g \tag{9}$$

One can note that the eigenvalues in the Eq.9 are real and negative and for each $t \in [0,T]$ the spectrum $\sigma(A(t))$, which is the same as the spectrum of the adjoint operator $A^*(t) = \partial^2/\partial\xi^2 + \boldsymbol{v}(t)\partial/\partial\xi - g$, is discrete and lies in the left half-plane of $\mathbb{C}$, with $\{0\} \notin \sigma(A(t))$ for $\boldsymbol{v}(t) \neq 0$. The growth bound $\omega_0 \in \mathbb{R}$ is given by:

$$\omega_0 = \sup_{n\geq 1, t\in[0,T]} \mathrm{Re}(\lambda_n(t)) < 0 \tag{10}$$

Then the operator $A(t)$ is a sectorial operator for each $t \in [0,T]$, i.e. there exists a sector:

$$S_\omega = \{\lambda \in \mathbb{C} : |\mathrm{arg}\lambda| < \frac{\pi}{2} + \omega\}/\{0\}, \quad \omega \in (0, \pi/2] \tag{11}$$

in the resolvent set $\rho(A(t))$ in which the spectrum of $A(t)$ is contained, i.e. $\sigma(A(t)) \subset \mathbb{C}/S_\omega$, which implies that the operator $A(t) : D(A(t)) \subset L^2(\mathbf{\Omega}) \rightarrow L^2(\mathbf{\Omega})$ is the infinitesimal generator a family of exponentially stable semigroups (see, for example, [13], [15], [16]). This result is demonstrated from the fact that at each $t \in [0,T]$, $A(t)$ is the negative of a Sturm-Liouville operator, see [17].

The eigenspace formed by the corresponding time-dependent family of eigenfunctions $\{\phi(\xi,t)\}_{t\in[0,T]}$ is one dimensional, where the eigenfunctions $\phi(\xi,t) \in C^1([0,T], L^2(\mathbf{\Omega}))$ are determined as:

$$\phi_n(\xi,t) = B_n(t)e^{\frac{1}{2\kappa}\boldsymbol{v}(t)\xi} \left(\cos\left(\frac{n\pi}{l(t)}\xi\right) - \frac{1}{2\kappa}\frac{\boldsymbol{v}(t)}{(n\pi/l(t))}\sin\left(\frac{n\pi}{l(t)}\xi\right)\right) \tag{12}$$

The coefficients

$$B_n(t) = \sqrt{\frac{2}{l(t)}}\left(1 + \left(\frac{\boldsymbol{v}(t)}{2\kappa\,(n\pi/l(t))}\right)^2\right)^{-\frac{1}{2}} \tag{13}$$

orthonormalize $\phi_n(\xi,t)$ with respect to the adjoint eigenvectors $\psi_n \in C^1([0,T], L^2(\mathbf{\Omega}))$ determined as:

$$\psi_n(\xi,t) = e^{-\kappa^{-1}\boldsymbol{v}(t)\xi}\phi(\xi,t) \tag{14}$$

Utilizing the expressions for the eigenvalues and eigenfunctions, the two-parameter semigroup $U(t,s)$ is defined in the following form.

Theorem 2: Denote $\phi_n(t) := \phi_n(\xi,t)$ and $\psi_n(t) := \psi_n(\xi,t)$. Consider the operator $A(t) : D(A(t)) \rightarrow L^2(\Omega)$ defined as:

$$A(t) := \sum_{n=1}^{\infty} \Lambda_n(t)\langle\,\cdot\,,\psi_n\rangle\phi_n \tag{15}$$

with

$$\Lambda_n(t) = \left\{\left(t\frac{d}{dt}\lambda_n + \lambda_n\right)\phi_n + \frac{\partial}{\partial t}\phi_n\right\}\phi_n^{-1} \tag{16}$$

The operator $A(t)$ is the infinitesimal generator of the two-parameter semigroup $U(t,s)$ with the analytic expression:

$$U(t,s)x(s) := \sum_{n=1}^{\infty} e^{\lambda_n(t)t}e^{-\lambda_n(s)s}\langle x(s),\psi_n(s)\rangle\phi_n(t) \tag{17}$$

for $x(s) \in L^2(\Omega)$, $0 \leq s \leq t \leq T$.

The operator $U(t,s)$ in the Eq.15 satisfies the following properties, see [15], [16]:

P1. $U(t,t) = I$, $U(t,s) = U(t,r)U(r,s)$ for $0 \leq s \leq r \leq t \leq T$

P2. For $x(s) \in L^2(\Omega)$

$$A(t)U(t,s) = \frac{\partial U(t,s)}{\partial t}$$

and similarly,

$$-U(t,s)A(s) = \frac{\partial U(t,s)}{\partial s}$$

P3. $\|U(t,s)\| \leq L_1$, $\|A(t)U(t,s)\| \leq L_2(t-s)^{-1}$, and $\|A(t)U(t,s)A(s)^{-1}\| \leq L_3$ for constants $L_i > 0$.

Then the generalized solution to the nonautonomous evolution system in the Eq.8 is expressed in terms of the two-parameter semigroup as:

$$x(t) = U(t,s)x(s), \quad \text{for} \quad 0 \leq s \leq t \leq T \tag{18}$$

which is the generalized solution of the initial and boundary value problem for the 1-dimensional form of the PDE in the Eq.3 together with the prescribed boundary conditions in the Eq.6. The two-parameter semigroup is utilized in the development of the boundary control problem in the Section III.

D. Spatial domain motion

In the context of the annealing process depicted in the Fig.1, whereby the material is pulled from a fluid medium by a mechanical actuator subsystem, the spatial domain motion is determined by the second order ODE for rigid body mechanics:

$$M\frac{d^2\hat{l}(t)}{dt^2} + c\frac{d\hat{l}(t)}{dt} + a\hat{l}(t) = F_m(t) \tag{19}$$

with initial conditions $l(0) = l_0$, $dl(0)/dt = \dot{l}_0$ where M, a and c denote the constant coefficients of mass, elasticity and dampening of the rigid body system. The input to this mechanical subsystem is the force applied by the actuator,

$F_m(t)$ and it is assumed that Eq.19 is in deviation form such that $l(t) = \hat{l}(t) + C > 0$ for all $t \in [0, T]$ for constant $C > 0$. One can see that the mechanical subsystem influences the dynamics of the PDE system through the evolution of the spatial domain length $l(t)$ and the boundary velocity $dl(t)/dt = \boldsymbol{v}(t)$ such that the Eq.3 is unidirectionally coupled with the Eq.19. Moreover, the controller for the mechanical subsystem is considered entirely decoupled from the controller for the temperature dynamics modelled by the PDE in the Eq.3 which is formulated in the following section.

Remark 2: The analysis and representation of solutions of the Eq.8 with time-dependent eigenvalues has also been studied in [18] where it is presumed the nonautonomous operator can be separated into time invariant and time dependent parts, i.e. $A(t) = (A + B(t))$. The corresponding eigenvalues consist of a discrete principal spectrum and time-dependent part, which corresponds to the case considered in this work.

III. Optimal boundary controller synthesis

We consider the boundary control problem for the 1-dimensional form of the PDE in the Eq.3 in which the boundary conditions in the Eq.6 are replaced with:

$$\frac{\partial x}{\partial \xi}(l(t), t) = 0, \qquad \frac{\partial x}{\partial \xi}(0, t) = u(t) \tag{20}$$

The function $u(t)$ is the manipulated input at the domain's boundary. In contrast to the case of distributed control, i.e. control within the spatial domain, the application of control to the boundary requires some additional modifications to the original system.

A. Boundary control system representation

The methodology proposed in [9], [19] enables the transformation of the boundary control problem into a and distributed control problem with linear state space representation. To this end, consider the following linear system on the state space $\mathcal{X} = L^2(\boldsymbol{\Omega})$:

$$\begin{aligned} \frac{dx(t)}{dt} &= \mathscr{A}(t)x(t) \\ \mathscr{B}u(t) &= u(t) \end{aligned} \tag{21}$$

The operators $\mathscr{A}(t)$ and $\mathscr{B}$ are defined as follows. For $\phi \in L^2(\boldsymbol{\Omega})$,

$$\mathscr{A}(t)\phi := \frac{\partial}{\partial \xi}\left(\kappa \frac{\partial \phi}{\partial \xi}\right) - \boldsymbol{v}(t)\frac{\partial \phi}{\partial \xi} \tag{22}$$

with the domain,

$$D(\mathscr{A}(t)) = \left\{ \phi \in L^2(\boldsymbol{\Omega}) : \ \phi, \ \frac{\partial \phi}{\partial \xi} \text{ are a.c.}, \ \mathscr{A}(t)\phi \in L^2(\boldsymbol{\Omega}), \text{ and } \frac{\partial \phi}{\partial \xi}(l(t), t) = 0 \right\}$$

where a.c. means absolutely continuous. Let the boundary operator $\mathscr{B} : \mathcal{X} \to \mathbb{R}$ be a linear operator defined as:

$$\mathscr{B}\phi := \frac{\partial \phi}{\partial \xi}(0, t) \tag{23}$$

with the domain $D(\mathscr{A}(t)) \subseteq D(\mathscr{B})$ defined as:

$$D(\mathscr{B}) := \left\{ \phi \in L^2(\boldsymbol{\Omega}) : \ \phi \text{ is a.c.}, \ \frac{\partial \phi}{\partial \xi} \in L^2(\boldsymbol{\Omega}) \right\}$$

Consider a function $b(\xi, t) := b(t)$ which is assumed to exist such that for all $u(t)$ and $\mathscr{B}u(t) \in D(\mathscr{A}(t))$, the relation $\mathscr{B}b(t)u(t) = u(t)$ is satisfied. The function $b(t) \in D(\mathscr{A}(t))$ is selected as $b(t) = \xi - \frac{1}{2}l(t)\xi^2$ which satisfies the above relation. The transformation $p(t) = x(t) - b(t)u(t)$ is introduced which leads to the system:

$$\begin{aligned} \frac{dp(t)}{dt} &= A(t)p(t) + (\mathscr{A}(t)b(t))u(t) - b(t)\dot{u}(t) \\ p(0) &= p_0 \in D(A) \end{aligned} \tag{24}$$

where the function $\dot{u}(t) = du(t)/dt$ is the time derivative of the input. The associated operator $A(t)$ with domain $D(A(t)) = \{\phi \in D(\mathscr{A}(t)) / \mathscr{B}\phi = 0\}$ is defined on the state space $\mathcal{X}$ such that:

$$A(t)\phi = \mathscr{A}(t)\phi \quad \text{in} \quad D(A(t)) \tag{25}$$

with the domain

$$\begin{aligned} &D(A(t)) := D(\mathscr{A}(t)) \cap \ker(\mathscr{B}) \\ &= \left\{ \phi \in L^2(\boldsymbol{\Omega}) : \ \phi, \ \frac{\partial \phi}{\partial \xi} \text{ are a.c.}, \ \mathscr{A}(t)\phi \in L^2(\boldsymbol{\Omega}), \right. \\ &\qquad \left. \text{and } \frac{\partial \phi}{\partial \xi}(0, t) = 0, \ \frac{\partial \phi}{\partial \xi}(l(t), t) = 0 \right\} \end{aligned}$$

The operator $A(t)$ which is the nonautonomous linear operator in the Section II, with $g = 0$, and is the infinitesimal generator of a family of strongly continuous semigroups for each $t \in [0, T]$, and that the operators $b(\xi, t)$ and $\mathscr{A}(t)b(\xi, t)$ are bounded such that the Eq.24 has the unique solution:

$$\begin{aligned} p(t) = U(t, s)p_0 &- \int_s^t U(t, \tau)b(\tau)\, \dot{u}(\tau)d\tau \\ &+ \int_s^t U(t, \tau)(\mathscr{A}(\tau)b(\tau))u(\tau)d\tau \end{aligned} \tag{26}$$

where for $0 \leq s \leq t \leq T$ the operator $U(t, s)$ is the two parameter evolution operator in the Eq.17. The solution of the system in the Eq.21 takes the form:

$$x(t) = p(t) + b(t)u(t) \tag{27}$$

where $x_0 = p_0 + b(0)u(0)$ is the initial condition of the Eq.21. From the Eq.24, the original boundary control problem is then represented as a distributed control problem by the following system on the extended state space $\mathcal{X}^e = \mathbb{R} \oplus \mathcal{X}$

$$\begin{aligned} \frac{dp^e}{dt} &= \begin{pmatrix} 0 & 0 \\ \mathscr{A}(t)b(t) & A(t) \end{pmatrix} p^e + \begin{pmatrix} 1 \\ -b(t) \end{pmatrix} u^e \\ p^e(0) &= \begin{pmatrix} u(0) \\ p(0) \end{pmatrix} \end{aligned} \tag{28}$$

where the state and the input are given by:

$$p^e = \begin{pmatrix} u(t) \\ p(t) \end{pmatrix} \quad \text{and} \quad u^e = \frac{du(t)}{dt} \tag{29}$$

The Eq.28 is represented as the abstract boundary control system representation:

$$\frac{dp^e(t)}{dt} = A^e(t)p^e(t) + B^e(t)u^e(t) \tag{30}$$

B. Spectrum assignability

In this part, we remark on some aspects regarding the stabilizability of the boundary control problem which has been formulated as the distributed control problem in the form of the Eq.30. One of the primary concerns in designing a feedback regulator is the ability of the controller to stabilize or enhance the stability of the system. Among the controllability criteria is the implied arbitrary spectral assignability by state feedback. The spectral assignability of boundary control feedback systems in the form of the Eq.30 has been considered in several works, see [9], [20], [21]. The results therein included the following conditions which have been modified accordingly to reflect the nonautonomous operator $A(t)$ considered in this present work. The conditions for stabilizability of the boundary control system in the Eq.30 are as follows:

C1. $A(t)$ is an unbounded spectral operator with discrete spectrum $\sigma(A(t))$ and normalized eigenvectors ϕ_n. The eigenvalues are distinct and the eigenspaces are one-dimensional;

C2. $\inf_{\substack{i\neq j\\ t\in[0,T]}} |\lambda_i(t) - \lambda_j(t)| = \varepsilon > 0$

C3. $\sup_{j\leq n<\infty} \sum_{\substack{j=1\\ j\neq n}}^{\infty} \frac{1}{|\lambda_j(t) - \lambda_n(t)|^2} < \infty$

C4. $0 \notin \sigma(A(t)), \quad \inf|\lambda_j(t)| > 0, \quad \sum_{j=1}^{\infty} \frac{1}{|\lambda_j(t)|^2} < \infty$

For unstable systems it is presumed that there is an index J such that $\mathrm{Re}\lambda_j \geq 0$ for $j \in J$ and $\mathrm{Re}\lambda_j < 0$ for $j \notin J$ which means that there exists only a finite number of unstable modes. Under the conditions C1-C4, and an additional completeness assumption, see [9], the system in the Eq.30 is stable under integral feedback a necessary and sufficient condition is that:

$$\sum_{j\in J} \mathrm{Re}\lambda_j(t) < \infty \tag{31}$$

In this case, the spectrum of the closed loop operator can be assigned such that:

$$\Gamma = \sigma(A^e(t) + \langle\cdot, g\rangle B^e) < 0 \tag{32}$$

We note that the operator $A(t)$ given by the Eq.15 satisfies the conditions C1-C3 since $A(t)$ is a sectorial operator having a discrete spectrum with $\{0\} \notin \sigma(A(t))$ for each $t \in [0, T]$, and by the properties P1-P3 the $A(t)$ is the infinitesimal generator of a family of analytic semigroups.

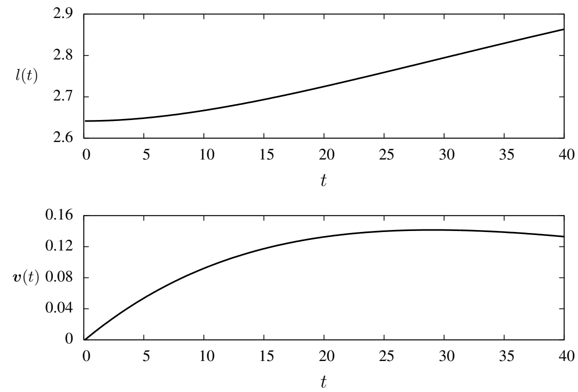

Fig. 2. Domain and boundary velocity evolution with system parameters, $M = 1.95$, $c = 2.5$ and $a = 1$.

C. Linear Quadratic Regulator synthesis

In order to obtain a stabilizing feedback regulator for above boundary control formulation, we consider the following quadratic optimization problem:

$$\min_{u^e} \int_0^T \left(|Qp^e(\tau)|^2 + |Ru^e(\tau)|^2\right) d\tau + \langle Qp^e(T), p^e(T)\rangle$$
$$\text{subject to} \tag{33}$$
$$\frac{dp^e(t)}{dt} = A^e(t)p^e(t) + B^e(t)u^e(t)$$

where $p^e(t)$ and $u^e(t)$ are the input and state defined in the Eq.29, see [22] and $p^e(0) \in \mathbb{R} \oplus \mathcal{X}$. The input is minimized over all possible controls $u^e(t)$ subject to the differential constraint given by the boundary control system. The operator $Q \in \mathcal{L}(\mathbb{R} \oplus \mathcal{X})$ is the self-adjoint and nonnegative, and the operator $R \in \mathcal{L}(\mathbb{R})$ is coercive. Since $A(t)$ generates a C_0-semigroup on $L^2(\mathbf{\Omega})$ for all $t \in [0, T]$ which gives the state evolution in the Eq.26, the optimization problem in the Eq.33 has the continuous and unique minimizing solution $u^e(t)$ given by the feedback formula:

$$u^e_{\mathrm{min}}(t) = -R^{-1}(B^e(t))^T \Pi(t) p^e_{\mathrm{min}}(t) \tag{34}$$

where the operator $\Pi(t) \in \mathcal{L}(\mathbb{R} \oplus \mathcal{X})$ is the strongly continuous, self adjoint, nonegative solution of the differential Riccati equation

$$\begin{aligned} &\frac{d}{dt}\Pi(t) + (A^e(t))^*\Pi(t) + \Pi(t)A^e(t) \\ &\quad - \Pi(t)B^e(t)R^{-1}(B^e(t))^T\Pi(t) + Q = 0 \end{aligned} \tag{35}$$

with final condition $\Pi(T) = Q$ where $(A^e(t))^*$ is the adjoint of $A^e(t)$, see [9], [22].

Remark 3: The structure of the extended state system in the Eq.30 plays a role in the choice of state weights in the control parameter $Q = \begin{pmatrix} Q_{11} & 0 \\ 0 & Q_{nn} \end{pmatrix}$ which is typically taken with only positive entries along its main diagonal. In particular, the first entry $\tilde{Q}_{11}$ influences the first state of the extended state system $\tilde{p}_1 = u(t)$ and therefore acts as the input penalty term to the extended state system.

IV. Simulation and Numerical Results

The boundary control problem for the annealing process discussed in the Section I and depicted in the Fig.1 is

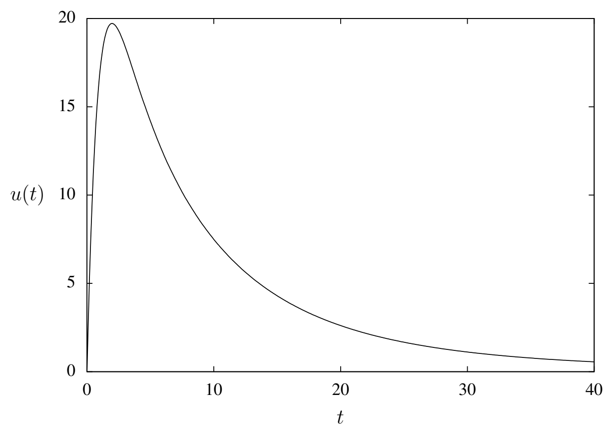

Fig. 3. Optimal boundary input $u(t)$ applied to boundary of the slab at the boundary $\xi = 0$. Control parameters selected as $R = 0.001$, $Q_{11} = 1$, $Q_{nn} = 15$.

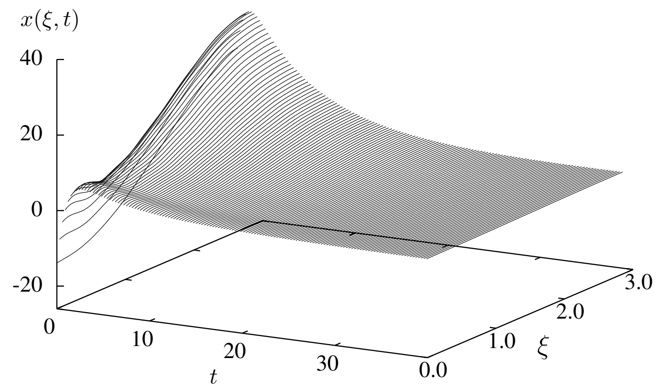

Fig. 4. Closed-loop temperature evolution $x(\xi, t)$ of slab with diffusivity constant $\kappa = 1.5$, $g = 0$, initial condition $x(\xi, 0) = x^2$ with heat input at the boundary, $\xi = 0$, and moving boundary at $\xi = l(t)$.

considered. The 1-dimensional form of the PDE in the Eq.3 on time-dependent spatial domain is utilized to approximate the temperature dynamics in the slab. The spatial domain, as described in the Section II-D with moving boundary at $\xi = l(t)$, has length and boundary velocity evolution depicted the Fig.2. The evolution system is simulated using $n = 10$ modes which were sufficient in capturing the dominant system dynamics such that increasing the number of modes did not significantly change the numerical results. Utilizing the boundary control formulation in the Section III, the operator $\Pi(t)$ is determined by solving the finite dimensional backwards differential Riccati equation analogous to the Eq.35. The the optimal control law in the Eq.34 is determined and the time evolution of the input is shown in the Fig.3. The time evolution of the closed loop temperature distribution of the slab is shown in the Fig.4. Beginning from the initial temperature distribution, the influence of heat input to the boundary at $\xi = 0$ can be seen as the system evolves and settles around the nominal distribution of $x(\xi, t) = 0$. At the simulation time of $t = 40$ the boundary input to the system converges towards zero as the slab temperature is almost completely dissipated.

V. SUMMARY

In this paper, we have considered the optimal boundary control of reaction-diffusion processes with time-varying spatial domain. The domain evolution was modelled using the second order ODE for rigid body mechanics. The functional space framework was developed in connection with nonautonomous evolution system representation of the PDE model of the reaction-diffusion process. The solution of the nonautonomous evolution equation was provided by the two-parameter semi-group $U(t, s)$ and used in the expression of the boundary control problem extended system representation. The optimal control law for the extended system was determined and numerical simulations results demonstrate that the temperature distribution in the time-dependent spatial domain is stabilized by the optimal heat input applied at the boundary.

REFERENCES

[1] J. Verhoeven, *Fundamentals of Physical Metallurgy*. Wiley, 1975.

[2] P. Acquistapace, F. Flandoli, and B.Terreni, "Initial boundary value problems and optimal control for nonautonomous parabolic systems," *SIAM J. Math. Anal.*, vol. 29, pp. 89–118, 1991.

[3] A. Armaou and P. D. Christofides, "Crystal temperature control in the czochralski crystal growth process," *AIChE J.*, vol. **47**, pp. 79–106, 2001.

[4] ——, "Robust control of parabolic PDE systems with time-dependent spatial domains," *Automatica*, vol. 37, pp. 61–69, 1999.

[5] W. B. Dunbar, N. Petit, and P. M. P. Rouchon, "Boundary control for a nonlinear stefan problem," in *Proc. of the 42nd IEEE Conf. on Decision and Control, Maui, HI*, 2003.

[6] R. Brown, "Theory of transport processes in single crystal growth from the melt," *AIChE J*, vol. 34, pp. 881–911, 1988.

[7] P. K. C. Wang, "Stabilization and control of distributed systems with time-dependent spatial domains," *J. Optim. Theor. & Appl.*, vol. 65, pp. 331–362, 1990.

[8] ——, "Feedback control of a heat diffusion system with time-dependent spatial domains," *Optim. Contr. Appl. & Meth.*, vol. 16, pp. 305–320, 1995.

[9] R. F. Curtain, "On stabilizability of linear spectral systems via state boundary feedback," *SIAM J. Control and Optimization*, vol. 23, pp. 144–152, 1985.

[10] R. F. Curtain and H. Zwart, *An introduction to Infinite-Dimensional Linear Systems Theory*. New York: Springer-Verlag, 1995.

[11] R. Adams, *Sobolev Spaces*. New York, U.S.A.: Academic Press, 1978.

[12] L. Evans, *Partial differential equations*. USA: Amer. Mathematical Society, 1998.

[13] R. McOwen, *Partial differential equations: methods and applications, 2nd ed.* USA: Prentice Hall, 2002.

[14] J. Marsden and T. Hughes, *Mathematical Foundations of Elasticity*. New York, U.S.A.: Dover Publications, 1983.

[15] A. Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations*. New York, U.S.A.: Springer-Verlag, 1983.

[16] H. Tanabe, *Functional analytic methods for partial differential equations*. New York: Marcel Dekker Inc., 1997.

[17] D. Cedric, D. Dochain, and J. Winkin, "Sturm-Liouville systems are Riesz-spectral operators," *Int. J. Appl. Math. Comput. Sci.*, vol. 13, pp. 481–484, 2003.

[18] D. Henry, *Geometric theory of semilinear parabolic equations*. New York: Springer-Verlag, 1981.

[19] H. O. Fattorini, "Boundary control systems," *SIAM Journal on Control*, vol. 6, pp. 349–385, 1968.

[20] C. Z. Xu and G. Sallet, "On spectrum and riesz basis assignment of infinite-dimensional linear systems by bounded linear feedbacks," *SIAM J. Control and Optimization*, vol. 34, pp. 521–541, 1996.

[21] S. Sun, "On spectrum distribution of completely controllable linear systems," *Acta Mathematica Sinica*, vol. 21, pp. 193–205, 1981.

[22] A. Bensoussan, G. D. Prato, M. Delfour, and S. Mitter, *Representation and Control of Infinite Dimensional Systems*. Boston: Birkhäuser, 2007.