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Robust stability analysis for feedback interconnections of unstable time-varying systems

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Abstract— **We consider the problem of robust stability analysis for feedback interconnections of causal linear systems that are potentially time-varying with unbounded gain over the space of finite-energy inputs. To this end, a combination of ν-gap metric and integral-quadratic-constraint based analysis is employed. The main results are developed in an abstract setting using generalized Wiener-Hopf and Hankel operators and the Fredholm index. Underlying assumptions are then verified for two important classes of system: multiplication by constantly proper Callier-Desoer transfer functions; and stabilizable and detectable finite-dimensional state-space systems with coefficients of bounded and continuous variation across time.**

Index Terms— **Feedback, robust stability, integral quadratic constraints, ν-gap metric, time-varying linear systems**

I. Introduction

Recent work [16], [6] considers the problem of structured robustness analysis for feedback interconnections of transfer functions, via a combination of ν-gap metric [21] and integral-quadratic-constraints (IQCs) [18] based analysis. Importantly, the use of IQCs to characterize the elements of the interconnection permits exploitation of structure, beyond small gain and passivity. On the other hand, ν-gap metric based analysis facilitates the direct accommodation of potentially unstable transfer functions in the interconnection.

The main objective of this paper is to extend the results of [16], [6] to accommodate feedback interconnections of causal linear time-varying systems, including those without bounded gain over the space of finite-energy input signals. Towards this end, an abstract framework is developed in terms of generalized Wiener-Hopf and Hankel operators which are defined with respect to bounded linear operators over doubly-infinite time. Eventually, a ν-gap measure of distance is established in terms of the Fredholm index of a Wiener-Hopf operator associated with a composition of normalized coprime representations of the system graphs and the compactness of a corresponding Hankel operator. Together, these generalize the well-known winding number condition used for the time-invariant case [21], [6]. In the development, the normalized representations of the graphs are assumed to exist on the basis that this is true for important classes of linear system. These include: time-invariant mappings corresponding to multiplication by constantly proper Callier-Desoer transfer functions (see also [6]); and linear time-varying systems that admit stabilizable and detectable finite-dimensional state-space realizations with coefficients of bounded and continuous variation across time, as recently shown in [19].

As in [6], the main result provides a sufficient condition for the robust stability of a family of potentially time-varying feedback interconnections, parametrised in terms of (i) a path of corresponding open-loop systems that is assumed to be continuous with respect to the ν-gap measure of distance introduced; and (ii) the satisfaction of an IQC along this path. Its development involves consideration of behaviour over the doubly-infinite time axis, as also arises in the purely time-invariant case. In keeping with the time-varying setting of the paper, we take care to not attribute special significance to any particular time dividing the past and the future, while still maintaining a handle on the causality of inverses via a condition on the corresponding instantaneous gains. Indeed, these points differentiate the work presented here from previous studies of general linear feedback interconnections via the gap metric, where behaviour is considered over a fixed positive time index set, and causality is either not treated at all, or only as an after-thought; see e.g. [11], [7].

The paper evolves along the following line. First some preliminaries are developed in Section II, including the generalized Wiener-Hopf and Hankel operator framework. The stability of feedback interconnection of possible unbounded but causal linear systems is then considered in Section III, in terms of causal and causally invertible normalized representations of the system graphs. Section IV provides an initial definition of a ν-gap metric for time-varying systems and the main robust stability result. An alternative, more familiar, formulation of this is then established in Section V, under an assumption that the Hankel operator associated with normalized left representations of the system graphs are compact. In Section VI, all assumptions underpinning the abstract development are then verified for the aforementioned classes of system. Finally, some illustrative examples involving the application of the main result are presented.

We refer to [15] for proofs of the results in this paper.

II. Preliminaries

A. Basic Notation and Operator Theory

The real and complex numbers are denoted $\mathbb{R}$ and $\mathbb{C}$. The transpose of a matrix $M \in \mathbb{R}^{p\times m}$ is denoted M^T, the complex conjugate transpose of an $M \in \mathbb{C}^{p\times m}$ is denoted M^*, and the largest and smallest singular values of M in either case are denoted $\sigma_{\max}(M)$ and $\sigma_{\min}(M)$, respectively.

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Let $\mathcal{H}_1, \mathcal{H}_2$ be Hilbert spaces and $\mathbf{X} : \mathcal{H}_1 \to \mathcal{H}_2$ be a bounded linear operator. The Hilbert adjoint $\mathbf{X}^* : \mathcal{H}_2 \to \mathcal{H}_1$ is uniquely defined by the equation

$$\langle \mathbf{X}w,\, v\rangle_{\mathcal{H}_2} = \langle w,\, \mathbf{X}^* v\rangle_{\mathcal{H}_1} \;\forall\, w \in \mathcal{H}_1 \text{ and } v \in \mathcal{H}_2.$$

It follows that $\mathrm{img}(\mathbf{X})^\perp = \ker(\mathbf{X}^*)$ and $\ker(\mathbf{X})^\perp = \mathrm{cl}\,\mathrm{img}(\mathbf{X}^*)$, where $\perp$ denotes the orthogonal complement and cl the closure of a subspace; see e.g. [17, Theorem 3]. Next we make note of some useful gain relations. Given bounded $\mathbf{X}, \mathbf{X}_1 : \mathcal{H}_1 \to \mathcal{H}_2$ and $\mathbf{X}_2 : \mathcal{H}_2 \to \mathcal{H}_3$, let

$$\overline{\gamma}(\mathbf{X}) := \sup_{\|w\|_{\mathcal{H}_1}=1} \|\mathbf{X}w\|_{\mathcal{H}_2} \text{ and}$$
$$\underline{\gamma}(\mathbf{X}) := \inf_{\|w\|_{\mathcal{H}_1}=1} \|\mathbf{X}w\|_{\mathcal{H}_2},$$

which satisfy the relations $\overline{\gamma}(\mathbf{X}^*) = \overline{\gamma}(\mathbf{X})$, $\overline{\gamma}(\mathbf{X}^*\mathbf{X}) = \overline{\gamma}(\mathbf{X})^2$,

$$\overline{\gamma}(\mathbf{X}_2\mathbf{X}_1) \le \overline{\gamma}(\mathbf{X}_2)\overline{\gamma}(\mathbf{X}_1), \quad \underline{\gamma}(\mathbf{X}_2\mathbf{X}_1) \ge \underline{\gamma}(\mathbf{X}_2)\underline{\gamma}(\mathbf{X}_1)$$

and, if $\underline{\gamma}(\mathbf{X}) > 0$, then $\underline{\gamma}(\mathbf{X}^*) = \underline{\gamma}(\mathbf{X})$. When $\mathbf{X}$ has a bounded inverse we have $\overline{\gamma}(\mathbf{X}^{-1}) = 1/\underline{\gamma}(\mathbf{X})$. Finally, if $\left[\begin{smallmatrix}\mathbf{X}\\ \mathbf{Y}\end{smallmatrix}\right] : \mathcal{H}_1 \to \mathcal{H}_2 \times \mathcal{H}_3$ is such that $\mathbf{X}^*\mathbf{X} + \mathbf{Y}^*\mathbf{Y} = I$ (i.e. it is an isometry) then $\overline{\gamma}(\mathbf{Y})^2 = 1 - \underline{\gamma}(\mathbf{X})^2$; similarly, if $\begin{bmatrix}\mathbf{X} & \mathbf{Y}\end{bmatrix} : \mathcal{H}_1 \times \mathcal{H}_2 \to \mathcal{H}_3$ is such that $\mathbf{X}\mathbf{X}^* + \mathbf{Y}\mathbf{Y}^* = I$ (i.e. it is a co-isometry) then $\overline{\gamma}(\mathbf{Y})^2 = 1 - \underline{\gamma}(\mathbf{X})^2$; see e.g. [5, Lemma 3].

In this paper, Fredholm operators play an important role. Indeed, the associated Fredholm index is eventually used to define a ν-gap measure of distance between systems.

Definition 1: A bounded linear operator $\mathbf{X} : \mathcal{H}_1 \to \mathcal{H}_2$ from a Hilbert space $\mathcal{H}_1$ to a Hilbert space $\mathcal{H}_2$ is of Fredholm type if both $\dim\ker(\mathbf{X})$ and $\mathrm{codim}\,\mathrm{img}(\mathbf{X}) = \dim\mathrm{coker}(\mathbf{X})$ are finite, where $\dim$ denotes the dimension of a subspace and coker denotes the quotient space of the codomain by the image; note $\mathrm{img}(\mathbf{X})$ is necessarily closed. The Fredholm index is defined to be

$$\mathrm{ind}(\mathbf{X}) = \dim\ker(\mathbf{X}) - \mathrm{codim}\,\mathrm{img}(\mathbf{X})$$

Lemma 1: Let $\mathbf{X}, \mathbf{X}_1 : \mathcal{H}_1 \to \mathcal{H}_2$ and $\mathbf{X}_2 : \mathcal{H}_2 \to \mathcal{H}_3$ be Fredholm operators. Then:

- (i) $\mathrm{ind}(\mathbf{X}^*) = -\mathrm{ind}(\mathbf{X})$;
- (ii) $\mathrm{ind}(\mathbf{X}_2\mathbf{X}_1) = \mathrm{ind}(\mathbf{X}_2) + \mathrm{ind}(\mathbf{X}_1)$;
- (iii) if $\mathbf{Y} : \mathcal{H}_1 \to \mathcal{H}_2$ is a bounded linear operator such that $\underline{\gamma}(\mathbf{X}) > \overline{\gamma}(\mathbf{Y})$, we have $\mathrm{ind}(\mathbf{X} + \mathbf{Y}) = \mathrm{ind}(\mathbf{X})$; and
- (iv) if $\mathbf{K} : \mathcal{H}_1 \to \mathcal{H}_2$ is a *compact* linear operator, we have $\mathrm{ind}(\mathbf{X} + \mathbf{K}) = \mathrm{ind}(\mathbf{X})$.

In what follows, operators mappings between time-domain signal spaces are of central concern. For any time interval $\mathbb{X} \subseteq \mathbb{T} := (-\infty, \infty) = \mathbb{R}$, the notation $\mathscr{L}_2^m(\mathbb{X})$ is used for the Hilbert space of square integrable $\mathbb{R}^m$-valued functions with support on $\mathbb{X}$, inner product

$$\langle w,\, v\rangle_{\mathscr{L}_2(\mathbb{X})} = \int_{\mathbb{X}} w(t)^T v(t)dt$$

and norm $\|w\|_{\mathscr{L}_2(\mathbb{X})} = \langle w,\, w\rangle_{\mathscr{L}_2(\mathbb{X})}^{1/2}$. We sometimes suppress the spatial dimension m from the notation.

When studying feedback interconnection of linear mappings that act causally on subspaces of $\mathscr{L}_2(\mathbb{T})$ it is convenient to split the doubly-infinite time axis in to the past $\mathbb{T}_- = (-\infty, t_0)$ and future $\mathbb{T}_+ = [t_0, \infty)$ relative to a $t_0 \in \mathbb{T}$; see Section III. Defining $v(t) = 0$ when $t < t_0$ for $v \in \mathscr{L}_2(\mathbb{T}_+)$, and $v(t) = 0$ when $t \ge t_0$ for $\mathscr{L}_2(\mathbb{T}_-)$, we get the useful subset inclusions $\mathscr{L}_2(\mathbb{T}_-), \mathscr{L}_2(\mathbb{T}_+) \subset \mathscr{L}_2(\mathbb{T})$ and the direct sum decomposition $\mathscr{L}_2(\mathbb{T}_-) \oplus \mathscr{L}_2(\mathbb{T}_+) = \mathscr{L}_2(\mathbb{T})$.

The frequency domain space $\mathscr{L}_2^m(j\mathbb{R})$ comprises the Fourier transforms of signals in $\mathscr{L}_2^m(\mathbb{T})$. The time and frequency domain spaces are isometrically isomorphic; i.e. $\|v\|_{\mathscr{L}_2(\mathbb{T})} = \|\hat{v}\|_{\mathscr{L}_2(j\mathbb{R})}$, where $\hat{v}$ denotes the Fourier transform of v. Finally, we let

$$\mathscr{L}_\infty^{m\times m}(j\mathbb{R}) = \{\Pi : j\mathbb{R} \to \mathbb{C}^{m\times m} : \|\Pi\|_\infty < \infty\},$$

where $\|\Pi\|_\infty = \sup_{\omega\in\mathbb{T}} \sigma_{\max}(\Pi(j\omega))$. In developing the main stability result in Section IV we use quadratic forms defined by a frequency-dependent multiplier $\Pi \in \mathscr{L}_\infty^{m\times m}$ as

$$\langle \hat{v},\, \Pi\hat{v}\rangle_{\mathscr{L}_2(j\mathbb{R})} = \int_{-\infty}^{\infty} \hat{v}(j\omega)^*\Pi(j\omega)\hat{v}(j\omega)d\omega.$$

This satisfies the bound $\langle \hat{v},\, \Pi\hat{v}\rangle_{\mathscr{L}_2(j\mathbb{R})} \le \|\Pi\|_\infty \|\hat{v}\|_{\mathscr{L}_2(j\mathbb{R})}^2$.

Let $\mathbf{M} : \mathscr{L}_2^m(\mathbb{T}) \to \mathscr{L}_2^p(\mathbb{T})$ be a bounded linear operator. Its restrictions to $\mathscr{L}_2^m(\mathbb{T}_+)$ and $\mathscr{L}_2^m(\mathbb{T}_-)$ are denoted by $\mathbf{M}|_{\mathscr{L}_2^m(\mathbb{T}_+)}$ and $\mathbf{M}|_{\mathscr{L}_2^m(\mathbb{T}_-)}$, respectively. $\mathbf{M}$ is said to be *causal* if

$$\mathbf{P}_T\mathbf{M}(I - \mathbf{P}_T) = 0 \quad \forall\, T \in \mathbb{T},$$

where $\mathbf{P}_T$ denotes the truncation operator defined by $(\mathbf{P}_T v)(t) = v(t)$, $t \le T$ and $(\mathbf{P}_T v)(t) = 0$, $t \ge T$ for any $v \in \mathscr{L}_2^m(\mathbb{T})$, which is a projection; in this case, $\mathrm{img}(\mathbf{M}|_{\mathscr{L}_2^m(\mathbb{T}_+)}) \subset \mathscr{L}_2^p(\mathbb{T}_+)$. $\mathbf{M}$ is called *anti-causal* if

$$(I - \mathbf{P}_T)\mathbf{M}\mathbf{P}_T = 0 \quad \forall\, T \in \mathbb{T};$$

in this case, $\mathrm{img}(\mathbf{M}|_{\mathscr{L}_2^m(\mathbb{T}_-)}) \subset \mathscr{L}_2^p(\mathbb{T}_-)$. Note that $\mathbf{M}$ is causal if, and only if, $\mathbf{M}^*$ is anti-causal.

B. Wiener-Hopf and Hankel Operators

In this section we introduce some preliminary results on generalized Wiener-Hopf and Hankel operators. These are used in the proof of the main robust stability result.

Definition 2 (Wiener-Hopf and Hankel operators): Let $\mathbf{M} : \mathscr{L}_2^m(\mathbb{T}) \to \mathscr{L}_2^p(\mathbb{T})$ be a bounded linear operator and given a $t_0 \in \mathbb{T}$, let $\mathbf{P}_+$ denote the projection from $\mathscr{L}_2^p(\mathbb{T})$ to $\mathscr{L}_2^p(\mathbb{T}_+)$ defined by $(\mathbf{P}_+ v)(t) = v(t)$ when $t \in \mathbb{T}_+ = [t_0, \infty)$ and $(\mathbf{P}_+ v)(t) = 0$ when $t \in \mathbb{T}_- = (-\infty, t_0)$. Finally, let $\mathbf{P}_- := I - \mathbf{P}_+$. We associate with $\mathbf{M}$:

1) the Wiener-Hopf operator $\mathbf{T}_\mathbf{M} : \mathscr{L}_2^m(\mathbb{T}_+) \to \mathscr{L}_2^p(\mathbb{T}_+)$ defined as $\mathbf{T}_\mathbf{M} = \mathbf{P}_+\mathbf{M}|_{\mathscr{L}_2(\mathbb{T}_+)}$;
2) the forward Hankel operator $\mathbf{H}_\mathbf{M}^{+-} : \mathscr{L}_2^m(\mathbb{T}_-) \to \mathscr{L}_2^p(\mathbb{T}_+)$ defined as $\mathbf{H}_\mathbf{M}^{+-} = \mathbf{P}_+\mathbf{M}|_{\mathscr{L}_2(\mathbb{T}_-)}$; and
3) the backward Hankel operator $\mathbf{H}_\mathbf{M}^{-+} : \mathscr{L}_2^m(\mathbb{T}_+) \to \mathscr{L}_2^p(\mathbb{T}_-)$ defined as $\mathbf{H}_\mathbf{M}^{-+} = \mathbf{P}_-\mathbf{M}|_{\mathscr{L}_2(\mathbb{T}_+)}$.

Remark 1: Note that the projections and, in general, the Wiener-Hopf and Hankel operators defined above all depend on the choice of $t_0 \in \mathbb{T}$. For notational convenience this dependence is left implicit. Reference to the corresponding t_0 is made where necessary, often implicitly (again) via the intervals $\mathbb{T}_- = (-\infty, t_0]$ and $\mathbb{T}_+ = [t_0, \infty)$.

The following result establishes some basic properties of the generalised Wiener-Hopf and Hankel operators.

Lemma 2: Let $\mathbf{M} : \mathscr{L}_2^m(\mathbb{T}) \to \mathscr{L}_2^p(\mathbb{T})$ be a bounded linear operator. For any $t_0 \in \mathbb{T}$, we have

(i) $\mathbf{T}_{\mathbf{M}}^* = \mathbf{T}_{\mathbf{M}^*}$ and

(ii) $(\mathbf{H}_{\mathbf{M}}^{+-})^* = \mathbf{H}_{\mathbf{M}^*}^{-+}$ and thus, $\mathbf{H}_{\mathbf{M}}^{-+} = (\mathbf{H}_{\mathbf{M}^*}^{+-})^*$.

Lemma 3: For any $t_0 \in \mathbb{T}$, the Wiener-Hopf operator $\mathbf{T}_{\mathbf{M}} : \mathscr{L}_2^n(\mathbb{T}_+) \to \mathscr{L}_2^n(\mathbb{T}_+)$ associated with a bounded linear operator $\mathbf{M} : \mathscr{L}_2^n(\mathbb{T}) \to \mathscr{L}_2^n(\mathbb{T})$ has a bounded inverse if, and only if,

(i) $\underline{\gamma}(\mathbf{T}_{\mathbf{M}}) > 0$ and

(ii) $\text{ind}(\mathbf{T}_{\mathbf{M}}) = 0$.

In the applications of this paper we need to establish conditions under which the inverse of a Wiener-Hopf operator is causal. To this end, the following results prove useful.

Lemma 4: Let $\mathbf{M}, \mathbf{M}_1 : \mathscr{L}_2^m(\mathbb{T}) \to \mathscr{L}_2^p(\mathbb{T})$ and $\mathbf{M}_2 : \mathscr{L}_2^p(\mathbb{T}) \to \mathscr{L}_2^q(\mathbb{T})$ be bounded linear operators. The following properties hold for any $t_0 \in \mathbb{T}$:

(i) If $\mathbf{M}$ is causal then $\mathbf{T}_{\mathbf{M}} = \mathbf{M}|_{\mathscr{L}_2(\mathbb{T}_+)}$ and $\mathbf{T}_{\mathbf{M}}$ is causal in the sense that $\mathbf{P}_T\mathbf{T}_{\mathbf{M}}(I - \mathbf{P}_T) = 0$ for all $T > t_0$;

(ii) $\mathbf{T}_{\mathbf{M}_2\mathbf{M}_1} = \mathbf{T}_{\mathbf{M}_2}\mathbf{T}_{\mathbf{M}_1} + \mathbf{H}_{\mathbf{M}_2}^{+-}\mathbf{H}_{\mathbf{M}_1}^{-+}$;

(a) if $\mathbf{M}_1$ is causal, then $\mathbf{T}_{\mathbf{M}_2\mathbf{M}_1} = \mathbf{T}_{\mathbf{M}_2}\mathbf{T}_{\mathbf{M}_1}$; and

(b) if $\mathbf{M}_1, \mathbf{M}_2$ are both causal, then $\mathbf{T}_{\mathbf{M}_2\mathbf{M}_1} = \mathbf{T}_{\mathbf{M}_2}\mathbf{T}_{\mathbf{M}_1} = \mathbf{M}_2\mathbf{M}_1|_{\mathscr{L}_2(\mathbb{T}_+)}$.

Definition 3: A bounded causal $\mathbf{M} : \mathscr{L}_2^n(\mathbb{T}) \to \mathscr{L}_2^n(\mathbb{T})$ is said to have non-singular instantaneous gain if

$$\rho_I(\mathbf{M}) \stackrel{\text{def}}{=} \inf_{t' \in \mathbb{T}} \inf_{\delta t > 0} \underline{\gamma}((\mathbf{P}_{t'+\delta t} - \mathbf{P}_{t'})\mathbf{M}(\mathbf{P}_{t'+\delta t} - \mathbf{P}_{t'})) > 0.$$

Lemma 5: Suppose a bounded causal operator $\mathbf{M} : \mathscr{L}_2^n(\mathbb{T}) \to \mathscr{L}_2^n(\mathbb{T})$ has non-singular instantaneous gain. Then, for each $t_0 \in \mathbb{T}$, the Wiener-Hopf operator $\mathbf{T}_{\mathbf{M}}$ is injective and the corresponding inverse map $\mathbf{T}_{\mathbf{M}}^{-1} : \text{img}(\mathbf{T}_{\mathbf{M}}) \to \mathscr{L}_2^n(\mathbb{T}_+)$ is causal in the sense that $\mathbf{P}_T\mathbf{T}_{\mathbf{M}}^{-1}\tilde{v} = 0 \ \forall \tilde{v} \in \{v : v \in \text{img}(\mathbf{T}_{\mathbf{M}}); \mathbf{P}_T v = 0\}$ and all $T > t_0$.

The above lemma can be strengthened when the Wiener-Hopf operator has a bounded inverse, as summarised below.

Lemma 6: For a bounded causal operator $\mathbf{M} : \mathscr{L}_2^n(\mathbb{T}) \to \mathscr{L}_2^n(\mathbb{T})$ suppose the Wiener-Hopf operator $\mathbf{T}_{\mathbf{M}} : \mathscr{L}_2^n(\mathbb{T}_+) \to \mathscr{L}_2^n(\mathbb{T}_+)$ has a bounded inverse for all $t_0 \in \mathbb{T}$. Then $\mathbf{T}_{\mathbf{M}}^{-1}$ is causal if, and only if, $\mathbf{M}$ has non-singular instantaneous gain.

Finally, in developing the main stability result we exploit particular relationships between the gains

$$\begin{aligned}
\overline{\gamma}(\mathbf{M}) &= \sup_{\|w\|_{\mathscr{L}_2^n(\mathbb{T})}=1} \|\mathbf{M}w\|_{\mathscr{L}_2^n(\mathbb{T})}, \\
\underline{\gamma}(\mathbf{M}) &= \inf_{\|w\|_{\mathscr{L}_2^n(\mathbb{T})=1}} \|\mathbf{M}w\|_{\mathscr{L}_2^n(\mathbb{T})}, \\
\overline{\gamma}(\mathbf{T}_{\mathbf{M}}) &= \sup_{\|w\|_{\mathscr{L}_2^n(\mathbb{T}_+)}=1} \|\mathbf{T}_{\mathbf{M}}w\|_{\mathscr{L}_2^n(\mathbb{T}_+)} \text{ and} \\
\underline{\gamma}(\mathbf{T}_{\mathbf{M}}) &= \inf_{\|w\|_{\mathscr{L}_2^n(\mathbb{T}_+)=1}} \|\mathbf{T}_{\mathbf{M}}w\|_{\mathscr{L}_2^n(\mathbb{T}_+)}.
\end{aligned}$$

Lemma 7: Suppose $M : \mathscr{L}_2^n(\mathbb{T}) \to \mathscr{L}_2^n(\mathbb{T})$ is a bounded linear operator. Then $\overline{\gamma}(\mathbf{T}_{\mathbf{M}}) \leq \overline{\gamma}(\mathbf{M})$ and if in addition $\mathbf{M}$ is causal we have $\underline{\gamma}(\mathbf{M}) \leq \underline{\gamma}(\mathbf{T}_{\mathbf{M}})$.

III. Main Feedback Stability Criterion

Let linear operators $\mathbf{H} : \text{dom}(\mathbf{H}) \subset \mathscr{L}_2^m(\mathbb{T}) \to \mathscr{L}_2^p(\mathbb{T})$ and $\mathbf{\Delta} : \text{dom}(\mathbf{\Delta}) \subset \mathscr{L}_2^p(\mathbb{T}) \to \mathscr{L}_2^m(\mathbb{T})$ be *causal* in the following generalized sense:

$$\mathbf{P}_T\mathcal{G}_{\mathbf{H}} \subset \mathscr{L}_2^{m+p}(\mathbb{T}) \quad (\text{ resp. } \mathbf{P}_T\mathcal{G}'_{\mathbf{\Delta}} \subset \mathscr{L}_2^{m+p}(\mathbb{T}) \text{ })$$

is a graph (resp. inverse graph) of a *linear* operator for all $T \in \mathbb{T}$,[1] where

$$\mathcal{G}_{\mathbf{H}} = \left\{ \begin{bmatrix} e_1 \\ \mathbf{H}e_1 \end{bmatrix} : e_1 \in \text{dom}(\mathbf{H});\ e_2 = \mathbf{H}e_1 \in \mathscr{L}_2^p(\mathbb{T}) \right\}$$

(resp.

$$\mathcal{G}'_{\mathbf{\Delta}} = \left\{ \begin{bmatrix} \mathbf{\Delta}e_2 \\ e_2 \end{bmatrix} : e_2 \in \text{dom}(\mathbf{\Delta});\ e_1 = \mathbf{\Delta}e_2 \in \mathscr{L}_2^m(\mathbb{T}) \right\})$$

is the graph of $\mathbf{H}$ (resp. inverse graph of $\mathbf{\Delta}$). Correspondingly, $\text{img}(\mathbf{H}|_{\text{dom}(\mathbf{H}) \cap \mathscr{L}_2^m(\mathbb{T}_+)}) \subset \mathscr{L}_2^p(\mathbb{T}_+)$ and $\text{img}(\mathbf{\Delta}|_{\text{dom}(\mathbf{\Delta}) \cap \mathscr{L}_2^p(\mathbb{T}_+)}) \subset \mathscr{L}_2^m(\mathbb{T}_+)$ for all $t_0 \in \mathbb{T}$; recall that $\mathbb{T}_+ := [t_0, \infty)$.

Consider the feedback interconnection $[\mathbf{\Delta}, \mathbf{H}]$ defined by

$$\begin{cases} e_1 = -\mathbf{\Delta}e_2 + r_1 \\ e_2 = -\mathbf{H}e_1 + r_2. \end{cases} \tag{1}$$

Definition 4 (Feedback Stability): We say that the interconnection (1) is *stable* if, for all $t_0 \in \mathbb{T}$,

$$\mathbf{F}_{t_0} := \begin{bmatrix} I & \mathbf{\Delta} \\ \mathbf{H} & I \end{bmatrix}\Bigg|_{(\text{dom}(\mathbf{H}) \times \text{dom}(\mathbf{\Delta})) \cap \mathscr{L}_2^{m+p}(\mathbb{T}_+)} \tag{2}$$

has a bounded and causal inverse.

Towards characterizing the existence of a bounded causal inverse

$$\begin{aligned}
\mathbf{F}_{t_0}^{-1} = \begin{bmatrix} r_1 \\ r_2 \end{bmatrix} &\in \mathscr{L}_2^{m+p}(\mathbb{T}_+) \\
&\mapsto \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} \in \mathscr{L}_2^{m+p}(\mathbb{T}_+) \cap (\text{dom}(\mathbf{H}) \times \text{dom}(\mathbf{\Delta})),
\end{aligned}$$

we now assume that the graphs of $\mathbf{H}$ and $\mathbf{\Delta}$ admit normalized right and left coprime representations; this is known to be the case for various classes of linear systems, including time-invariant systems with constantly proper transfer functions in the Callier-Desoer algebra [6] and time-varying systems with stabilizable and detectable finite-dimensional state-space realisations [19], for example. In particular, we assume the existence of bounded *causal* operators $\mathbf{V}, \mathbf{U}, \widetilde{\mathbf{V}}, \widetilde{\mathbf{U}}, \mathbf{X}_{\mathbf{H}}, \mathbf{Y}_{\mathbf{H}}, \widetilde{\mathbf{X}}_{\mathbf{H}}, \widetilde{\mathbf{Y}}_{\mathbf{H}}$ and $\mathbf{N}, \mathbf{M}, \widetilde{\mathbf{N}}, \widetilde{\mathbf{M}}, \mathbf{X}_{\mathbf{\Delta}}, \mathbf{Y}_{\mathbf{\Delta}}, \widetilde{\mathbf{X}}_{\mathbf{\Delta}}, \widetilde{\mathbf{Y}}_{\mathbf{\Delta}}$, defined on the whole of $\mathscr{L}_2(\mathbb{T})$, such that the following properties hold:

(A1) the double Bezout identity

$$\begin{bmatrix} \mathbf{X}_{\mathbf{H}} & \mathbf{Y}_{\mathbf{H}} \\ -\widetilde{\mathbf{U}} & \widetilde{\mathbf{V}} \end{bmatrix} \begin{bmatrix} \mathbf{V} & -\widetilde{\mathbf{Y}}_{\mathbf{H}} \\ \mathbf{U} & \widetilde{\mathbf{X}}_{\mathbf{H}} \end{bmatrix} = I;$$

[1] A subspace $\mathcal{G}$ is a graph of linear operator if $\binom{0}{w} \in \mathcal{G}$ implies $w = 0$, and an inverse graph if $\binom{v}{0} \in \mathcal{G}$ implies $v = 0$.

(A2) the double Bezout identity

$$\begin{bmatrix}\mathbf{Y_\Delta} & \mathbf{X_\Delta}\\ \widetilde{\mathbf{M}} & -\widetilde{\mathbf{N}}\end{bmatrix}\begin{bmatrix}\mathbf{N} & \widetilde{\mathbf{X}}_\mathbf{\Delta}\\ \mathbf{M} & -\widetilde{\mathbf{Y}}_\mathbf{\Delta}\end{bmatrix} = I;$$

(B1) the graph symbols

$$\mathbf{G_H} = \begin{bmatrix}\mathbf{V}\\ \mathbf{U}\end{bmatrix} \quad \text{and} \quad \widetilde{\mathbf{G}}_\mathbf{H} = \begin{bmatrix}-\widetilde{\mathbf{U}} & \widetilde{\mathbf{V}}\end{bmatrix}$$

are normalized – i.e. $\mathbf{G_H^*G_H} = I$ and $\widetilde{\mathbf{G}}_\mathbf{H}\widetilde{\mathbf{G}}_\mathbf{H}^* = I$;

(B2) the (inverse) graph symbols

$$\mathbf{\Gamma_\Delta} = \begin{bmatrix}\mathbf{N}\\ \mathbf{M}\end{bmatrix} \quad \text{and} \quad \widetilde{\mathbf{\Gamma}}_\mathbf{\Delta} = \begin{bmatrix}-\widetilde{\mathbf{M}} & \widetilde{\mathbf{N}}\end{bmatrix}$$

are normalized – i.e. $\mathbf{\Gamma_\Delta^*\Gamma_\Delta} = I$ and $\widetilde{\mathbf{\Gamma}}_\mathbf{\Delta}\widetilde{\mathbf{\Gamma}}_\mathbf{\Delta}^* = I$;

(C1) for every $t_0 \in \mathbb{T}$ we have

$$\begin{aligned}\mathcal{G}_\mathbf{H}^+ &:= \mathcal{G}_\mathbf{H} \cap \mathscr{L}_2(\mathbb{T}_+)\\ &= \text{img}(\mathbf{G_H}|_{\mathscr{L}_2(\mathbb{T}_+)}) = \ker(\widetilde{\mathbf{G}}_\mathbf{H}|_{\mathscr{L}_2(\mathbb{T}_+)}).\end{aligned}$$

(C2) for every $t_0 \in \mathbb{T}$ we have

$$\begin{aligned}\mathcal{G}_\mathbf{\Delta}'^+ &:= \mathcal{G}_\mathbf{\Delta}' \cap \mathscr{L}_2(\mathbb{T}_+)\\ &= \text{img}(\mathbf{\Gamma_\Delta}|_{\mathscr{L}_2(\mathbb{T}_+)}) = \ker(\widetilde{\mathbf{\Gamma}}_\mathbf{\Delta}|_{\mathscr{L}_2(\mathbb{T}_+)}).\end{aligned}$$

(D1) $\mathbf{V}$ and $\widetilde{\mathbf{V}}$ have non-singular instantaneous gains; and

(D2) $\mathbf{M}$ and $\widetilde{\mathbf{M}}$ have non-singular instantaneous gains.

Note that the graph symbols just defined are *causal* and that by properties (A1), (A2) and Lemma 4, the following additional properties hold: $\mathcal{G}_\mathbf{H}^+ = \text{img}(\mathbf{T}_{\mathbf{G_H}}) = \ker(\mathbf{T}_{\widetilde{G}_\mathbf{H}})$;

$$\mathbf{T}_{\widetilde{\mathbf{G}}_\mathbf{H}\mathbf{G_H}} = \mathbf{T}_{\widetilde{\mathbf{G}}_\mathbf{H}}\mathbf{T}_{\mathbf{G_H}} = 0;$$

$\mathbf{T}_{\mathbf{G_H}}$ has a causal left inverse $\mathbf{T}_{[\,\mathbf{X_H}\ \mathbf{Y_H}\,]}$; $\mathbf{T}_{\widetilde{\mathbf{G}}_\mathbf{H}}$ has a causal right inverse $\mathbf{T}_{[\,-\widetilde{Y}_\mathbf{H}^*\ \widetilde{X}_\mathbf{H}^*\,]^*}$; $\mathcal{G}_\mathbf{\Delta}'^+ = \text{img}(\mathbf{T}_{\mathbf{\Gamma_\Delta}}) = \ker(\mathbf{T}_{\widetilde{\mathbf{\Gamma}}_\mathbf{\Delta}})$;

$$\mathbf{T}_{\widetilde{\mathbf{\Gamma}}_\mathbf{\Delta}\mathbf{\Gamma_\Delta}} = \mathbf{T}_{\widetilde{\mathbf{\Gamma}}_\mathbf{\Delta}}\mathbf{T}_{\mathbf{\Gamma_\Delta}} = 0;$$

$\mathbf{T}_{\mathbf{\Gamma_\Delta}}$ has a causal left inverse $\mathbf{T}_{[\,\mathbf{Y_\Delta}\ \mathbf{X_\Delta}\,]}$; and $\mathbf{T}_{\widetilde{\mathbf{\Gamma}}_\mathbf{\Delta}}$ has a causal right inverse $\mathbf{T}_{[\,\widetilde{X}_\mathbf{\Delta}^*\ -\widetilde{Y}_\mathbf{\Delta}^*\,]^*}$.

We are now in a position to characterize the existence of a bounded inverse for $\mathbf{F}_{t_0}$; causality follows under an additional assumption as described after the following result.

Lemma 8: For any $t_0 \in \mathbb{T}$ the operator $\mathbf{F}_{t_0}$ in (2) has bounded inverse if, and only if, $\mathbf{T}_{\widetilde{\mathbf{G}}_\mathbf{H}\mathbf{\Gamma_\Delta}}$ has a bounded inverse. Moreover, in this case,

$$\mathbf{F}_{t_0}^{-1} = \begin{bmatrix}-I & 0\\ 0 & I\end{bmatrix}\mathbf{T}_{\mathbf{\Gamma_\Delta}}(\mathbf{T}_{\widetilde{\mathbf{G}}_\mathbf{H}\mathbf{\Gamma_\Delta}})^{-1}\mathbf{T}_{\widetilde{\mathbf{G}}_\mathbf{H}} + \begin{bmatrix}I & 0\\ 0 & 0\end{bmatrix}.$$

By the formula for $\mathbf{F}_{t_0}^{-1}$ in Lemma 8 and the properties that $\mathbf{T}_{\mathbf{\Gamma_\Delta}}$ has a causal left inverse and $\mathbf{T}_{\widetilde{\mathbf{G}}_\mathbf{H}}$ has a causal right inverse, it is immediate that $\mathbf{F}_{t_0}^{-1}$ is causal if, and only if, $(\mathbf{T}_{\widetilde{\mathbf{G}}_\mathbf{H}\mathbf{\Gamma_\Delta}})^{-1}$ is causal. This leads to the following feedback stability result.

Lemma 9: Suppose $\widetilde{\mathbf{G}}_\mathbf{H}\mathbf{\Gamma_\Delta}$ has non-singular instantaneous gain. Then the feedback interconnection $[\mathbf{\Delta}, \mathbf{H}]$ is stable if, and only if, for all $t_0 \in \mathbb{T}$,

(i) $\underline{\gamma}(\mathbf{T}_{\widetilde{\mathbf{G}}_\mathbf{H}\mathbf{\Gamma_\Delta}}) > 0$ and

(ii) $\text{ind}(\mathbf{T}_{\widetilde{\mathbf{G}}_\mathbf{H}\mathbf{\Gamma_\Delta}}) = 0$.

To this point the normalization properties (B1) and (B2) have not been used. These lead to the following identities on $\mathscr{L}_2(\mathbb{T})$, which play an important role in establishing the main robust stability result of the next section:

$$\begin{bmatrix}\widetilde{\mathbf{G}}_\mathbf{H}\\ \mathbf{G}_\mathbf{H}^*\end{bmatrix}\begin{bmatrix}\widetilde{\mathbf{G}}_\mathbf{H}^* & \mathbf{G_H}\end{bmatrix} = \begin{bmatrix}I & 0\\ 0 & I\end{bmatrix} \tag{3}$$

and

$$\widetilde{\mathbf{G}}_\mathbf{H}^*\widetilde{\mathbf{G}}_\mathbf{H} + \mathbf{G_H}\mathbf{G}_\mathbf{H}^* = I. \tag{4}$$

The last property follows from (3) by an argument used in the proof of [5, Proposition 4]. Indeed, from (3) we have

$$\text{img}\left(\begin{bmatrix}\widetilde{\mathbf{G}}_\mathbf{H}\\ \mathbf{G}_\mathbf{H}^*\end{bmatrix}\right) = \mathscr{L}_2^{p+m}(\mathbb{T})$$

and by the structure of this that

$$\begin{aligned}\ker\left(\begin{bmatrix}\widetilde{\mathbf{G}}_\mathbf{H}\\ \mathbf{G}_\mathbf{H}^*\end{bmatrix}\right) &= \ker(\widetilde{\mathbf{G}}_\mathbf{H}) \cap \ker(\mathbf{G}_\mathbf{H}^*)\\ &= \ker(\widetilde{\mathbf{G}}_\mathbf{H}) \cap \text{img}(\mathbf{G_H})^\perp = \{0\},\end{aligned}$$

which together imply that $\begin{bmatrix}\widetilde{\mathbf{G}}_\mathbf{H}\\ \mathbf{G}_\mathbf{H}^*\end{bmatrix}$ is boundedly invertible, whereby (4) holds.

IV. Robust Stability Analysis (Main Result)

Consider a family of feedback interconnections of the form (1), parametrized by $\mathbf{H}_\theta$ with $\theta \in [0,1]$. We let $\mathbf{H}_1 = \mathbf{H}$ so that $\theta = 1$ corresponds to the system of interest. Furthermore, as in the preceding section, we assume that each $\mathbf{H}_\theta$, $\theta \in [0,1]$, has normalized graph symbols $\mathbf{G}_{\mathbf{H}_\theta}$ and $\widetilde{\mathbf{G}}_{\mathbf{H}_\theta}$ satisfying the properties described above, so that the following is well defined for $a, b \in [0,1]$:

$$\delta_\nu(\mathbf{H}_a, \mathbf{H}_b) := \begin{cases}\overline{\gamma}(\widetilde{\mathbf{G}}_{\mathbf{H}_a}\mathbf{G}_{\mathbf{H}_b}) & \text{if } \widetilde{\mathbf{G}}_{\mathbf{H}_a}\widetilde{\mathbf{G}}_{\mathbf{H}_b}^* \in \mathcal{N}\\ 1 & \text{otherwise}\end{cases}, \tag{5}$$

where

$$\mathcal{N} = \{\mathbf{M} : \mathscr{L}_2^p(\mathbb{T}) \to \mathscr{L}_2^p(\mathbb{T}) : \mathbf{M} \text{ has bounded inverse; } \text{ind}(\mathbf{T_M}) = 0 \text{ and } \mathbf{H}_\mathbf{M}^{-+} \text{ is compact}\}.\, \forall\, t_0 \in \mathbb{T}$$

The condition in (5) requires $\underline{\gamma}(\widetilde{\mathbf{G}}_{\mathbf{H}_a}\widetilde{\mathbf{G}}_{\mathbf{H}_b}^*) > 0$ and $\mathbf{T}_{\widetilde{\mathbf{G}}_{\mathbf{H}_a}\widetilde{\mathbf{G}}_{\mathbf{H}_b}^*}$ to be Fredholm with $\text{ind}(\mathbf{T}_{\widetilde{\mathbf{G}}_{\mathbf{H}_a}\widetilde{\mathbf{G}}_{\mathbf{H}_b}^*}) = 0$. The nature of $\delta_\nu(\cdot,\cdot)$ is considered further in Section V.

Assumption 1: The path $\theta \in [0,1] \mapsto \mathbf{H}_\theta$ is continuous in the following sense: for any $\eta > 0$, there exists $\delta > 0$ such that $\delta_\nu(\mathbf{H}_a, \mathbf{H}_b) \le \eta$ whenever $|a - b| \le \delta$.

The main result below is formulated in terms of IQCs. Specifically, given $\Pi \in \mathscr{L}_\infty^{(p+m)\times(p+m)}(j\mathbb{R})$, we say $\mathbf{H}_\theta$ satisfies the strict IQC defined by Π (denoted $\mathbf{H} \in \text{SIQC}(\Pi)$) when there exists $\epsilon > 0$ such that

$$\langle \hat{v},\, \Pi\hat{v}\rangle_{\mathscr{L}_2(j\mathbb{R})} \ge \epsilon\|\hat{v}\|_{\mathscr{L}_2(j\mathbb{R})}^2\, \forall\, v \in \text{img}(\mathbf{G}_{\mathbf{H}_\theta}) \subset \mathscr{L}_2(\mathbb{T}),$$

where $\hat{v}$ denotes the Fourier transform of v. Similarly, $\mathbf{\Delta}$ with normalized right graph symbol $\mathbf{\Gamma_\Delta}$ is said to satisfy the complementary IQC (denoted $\mathbf{\Delta} \in \text{IQC}^c(\Pi)$) when

$$\langle \hat{w},\, \Pi\hat{w}\rangle_{\mathscr{L}_2(j\mathbb{R})} \le 0\, \forall\, w \in \text{img}(\mathbf{\Gamma_\Delta}) \subset \mathscr{L}_2(\mathbb{T}).$$

Assumption 2: for each $\theta \in [0,1]$, $\widetilde{\mathbf{G}}_{\mathbf{H}_\theta}\mathbf{\Gamma}_{\mathbf{\Delta}}$ has non-singular instantaneous gain.

Theorem 1: Suppose Assumptions 1 and 2 hold. Then the feedback interconnection $[\mathbf{\Delta}, \mathbf{H}]$ in (1) is stable if

(i) $[\mathbf{\Delta}, \mathbf{H}_0]$ is stable and

(ii) there exists $\Pi \in \mathscr{L}_\infty^{(p+m)\times(p+m)}(j\mathbb{R})$ such that:

- (a) $\mathbf{\Delta} \in \mathrm{IQC}^c(\Pi)$; and
- (b) $\mathbf{H}_\theta \in \mathrm{SIQC}(\Pi)\,\forall\,\theta \in [0,1]$.

V. An Alternative Formulation of the ν-Gap

In this section we consider the nature of the ν-gap defined in (5), including an alternative formulation of it, under the following assumption, which is known to hold for LTI systems with constantly proper transfer functions in the Callier-Desoer algebra and LTV systems with finite-dimensional state-space realisation; see the next section.

Assumption 3: For all $a \in [0,1]$, the Hankel operator $\mathbf{H}^{+-}_{\widetilde{\mathbf{G}}_{\mathbf{H}_a}}$ is compact.

The next result shows that $\delta_\nu(\cdot,\cdot)$ as defined in (5) under the above assumption has simpler and more familiar formulations. The arguments employed in [21] can be used to conclude that $\delta_\nu(\cdot,\cdot)$ is a metric.

Proposition 1: Suppose Assumption 3 holds. Then

$$\delta_\nu(\mathbf{H}_a,\mathbf{H}_b)=\begin{cases}\overline{\gamma}(\widetilde{\mathbf{G}}_{\mathbf{H}_a}\mathbf{G}_{\mathbf{H}_b}) & \text{if } \underline{\gamma}(\widetilde{\mathbf{G}}_{\mathbf{H}_a}\widetilde{\mathbf{G}}^*_{\mathbf{H}_b})>0 \text{ and}\\ & \mathrm{ind}(\mathbf{T}_{\widetilde{\mathbf{G}}_{\mathbf{H}_a}\widetilde{\mathbf{G}}^*_{\mathbf{H}_b}})=0\,\forall t_0\in\mathbb{T},\\ 1 & \text{otherwise}\end{cases} \tag{6}$$

where $\delta_\nu(\cdot,\cdot)$ is defined in (5). Furthermore,

$$\begin{aligned}\underline{\gamma}(\mathbf{G}^*_{\mathbf{H}_a}\mathbf{G}_{\mathbf{H}_b}) &= \underline{\gamma}(\widetilde{\mathbf{G}}_{\mathbf{H}_a}\widetilde{\mathbf{G}}^*_{\mathbf{H}_b}),\\ \overline{\gamma}(\widetilde{\mathbf{G}}_{\mathbf{H}_a}\mathbf{G}_{\mathbf{H}_b})^2 &= 1-\underline{\gamma}(\widetilde{\mathbf{G}}_{\mathbf{H}_a}\widetilde{\mathbf{G}}^*_{\mathbf{H}_b})^2 \text{ and}\\ \mathrm{ind}(\mathbf{T}_{\mathbf{G}^*_{\mathbf{H}_a}\mathbf{G}_{\mathbf{H}_b}})=0 \quad &\Leftrightarrow \quad \mathrm{ind}(\mathbf{T}_{\widetilde{\mathbf{G}}_{\mathbf{H}_a}\widetilde{\mathbf{G}}^*_{\mathbf{H}_b}})=0,\end{aligned} \tag{7}$$

whereby

$$\delta_\nu(\mathbf{H}_a,\mathbf{H}_b)=\begin{cases}\overline{\gamma}(\widetilde{\mathbf{G}}_{\mathbf{H}_a}\mathbf{G}_{\mathbf{H}_b}) & \text{if } \underline{\gamma}(\mathbf{G}^*_{\mathbf{H}_a}\mathbf{G}_{\mathbf{H}_b})>0 \text{ and}\\ & \mathrm{ind}(\mathbf{T}_{\mathbf{G}^*_{\mathbf{H}_a}\mathbf{G}_{\mathbf{H}_b}})=0\,\forall t_0\in\mathbb{T}.\\ 1 & \text{otherwise}\end{cases}$$

VI. System Classes

In this section we illustrate with some examples of systems that satisfy the required properties.

A. The Callier Desoer Class of LTI Systems

We let $\mathcal{A}^{p\times m}(\beta)$ be algebra of transfer functions obtained as the Laplace transforms of the impulse response functions (see [10], [9])

$$h(t) = h_c(t)\theta(t) + \sum_{k=0}^{\infty} h_k\delta(t-\tau_k),$$

where $e^{-\beta t}h_c(t) \in \mathscr{L}_1^{p\times m}[0,\infty) := \{f : [0,\infty) \to \mathbb{R}^{p\times m} : \int_0^\infty |f_{ij}|\,dt < \infty\}$, $h_k \in \mathbb{R}^{p\times m}$, $\tau_0 = 0$, $\tau_k > 0$, $k \geq 1$, $\sum_{k=0}^\infty e^{-\beta\tau_k}|h_k| < \infty$, $\theta(\cdot)$ is the unit step function and $\delta(\cdot)$ is the Dirac distribution. For convenience let $\mathcal{A} := \mathcal{A}(0)$. To each $H \in \mathcal{A}^{p\times m}$ we associate the causal time-domain operator $\mathbf{H} : \mathscr{L}_2^m(\mathbb{T}_+) \to \mathscr{L}_2^p(\mathbb{T}_+)$ defined by

$$(\mathbf{H}v)(t) = \int_{t_0}^t h_c(t-\tau)v(\tau) + \sum_{k=0}^{t} h_k v(t-\tau_k).$$

This is bounded, with induced norm $\|\mathbf{H}\| = \sup_{\omega\in\mathbb{R}}\sigma_{\max}(H(j\omega))$. The extension $\mathbf{H} : \mathscr{L}_2^m(\mathbb{T}) \to \mathscr{L}_2^p(\mathbb{T})$ is defined similarly, except that the lower bound of the integral is now $t_0 = -\infty$. Finally, we let $\mathcal{A}_-^{p\times m} = \{\mathcal{A}^{\times m}(\beta) : \beta < 0\}$ and $\mathcal{A}_{cp,-}^{p\times m}$ be the subclass of constantly proper transfer functions corresponding to $h_k = 0$ for $k \geq 1$, which implies that the transfer functions are constant at infinity.

The Callier-Desoer class $\mathcal{B}^{p\times m}$ consists of transfer functions where each element belongs to the quotient algebra $\mathcal{A}_-[\mathcal{A}_\infty]$ and $\mathcal{A}_\infty = \{G \in \mathcal{A}_- : \lim_{|s|\to\infty}\sigma_{\min}(G(s)) > 0\}$. Each transfer function from $\mathcal{B}^{p\times m}$ has a normalized right and left coprime factorization $H = UV^{-1} = \widetilde{V}^{-1}\widetilde{U}$, where $U, \widetilde{U} \in \mathcal{A}_-^{p\times m}$, $V \in \mathcal{A}_\infty^{m\times m}$, $\widetilde{V} \in \mathcal{A}_\infty^{p\times p}$ are such that

$$\begin{aligned}U(j\omega)^*U(j\omega) + V(j\omega)^*V(j\omega) &= I,\\ \widetilde{U}(j\omega)\widetilde{U}(j\omega)^* + \widetilde{V}(j\omega)\widetilde{V}(j\omega)^* &= I;\end{aligned}$$

see [3], [4]. To $H \in \mathcal{B}^{p\times m}$ we associated a causal time-domain operator $\mathbf{H} : \mathrm{dom}(\mathbf{H}) \subset \mathscr{L}_2^m(\mathbb{T}_+) \to \mathscr{L}_2^p(\mathbb{T}_+)$, with graph defined by

$$\mathcal{G}^+_{\mathbf{H}} = \left\{ v = \begin{bmatrix} v_1\\ v_2\end{bmatrix} : \hat{v}_1 = V\hat{w};\ \hat{v}_2 = U\hat{w};\ \hat{w} \in e^{-j\omega t_0}\mathscr{H}_2^m \right\},$$

where the explicit expression for the domain is

$$\mathrm{dom}(\mathbf{H}) = \{v : \hat{v} = V\hat{w};\ \hat{w} \in e^{-j\omega t_0}\mathscr{H}_2^m\},$$

and the frequency domain space $\mathscr{H}_2^m$ comprises the Fourier transforms of signals in $\mathscr{L}_2^m[0,\infty)$. The operator $\mathbf{H}$ is unbounded outside this domain of definition and therefore regarded as an unstable system.

In order to define the ν-gap we use the subclass of constantly proper transfer functions $\mathcal{B}_{cp}^{p\times m}$, where each matrix element belongs to the quotient algebra $\mathcal{A}_{cp,-}[\mathcal{A}_{cp,\infty}]$ and $\mathcal{A}_{cp,\infty} = \{G \in \mathcal{A}_{cp,-} : \lim_{|s|\to\infty}\sigma_{\min}(G(s)) > 0\}$. Any two $H_1, H_2 \in \mathcal{B}_{cp}^{p\times m}$ have normalized right and left coprime factorizations $H_k = U_kV_k^{-1} = \widetilde{V}_k^{-1}\widetilde{U}_k$, from which we can define so-called right and left graph symbols

$$\begin{aligned}G_{H_k} &= \begin{bmatrix} V_k\\ U_k\end{bmatrix} \in \mathcal{A}_{cp,-}^{(p+m)\times m} \text{ and}\\ \widetilde{G}_{H_k} &= \begin{bmatrix} -\widetilde{U}_k & \widetilde{V}_k\end{bmatrix} \in \mathcal{A}_{cp,-}^{p\times(p\times m)}.\end{aligned}$$

The Hankel operator $\mathbf{H}^{+-}_{\widetilde{G}_{\mathbf{H}_k}}$ is compact by [9, Lemma 8.2.4]. So Assumption 3 holds. Therefore, by Proposition 1 the ν-gap in (5) can be formulated as in (6), which in turn reduces to

$$\delta_\nu(\mathbf{H}_1,\mathbf{H}_2) = \begin{cases}\overline{\gamma}(\widetilde{G}_{H_1}G_{H_2}), & \underline{\gamma}(\widetilde{G}_{H_1}\widetilde{G}^*_{H_2}) > 0 \text{ and}\\ & \mathrm{wno}(\widetilde{G}_{H_1}\widetilde{G}^*_{H_2}) = 0\\ 1, & \text{otherwise}\end{cases}$$

where

$$\overline{\gamma}(\widetilde{G}_{H_1}G_{H_2}) = \sup_{\omega\in\mathbb{T}} \sigma_{\max}(\widetilde{G}_{H_1}G_{H_2})(j\omega)$$
$$\underline{\gamma}(\widetilde{G}_{H_1}\widetilde{G}^*_{H_2}) = \inf_{\omega\in\mathbb{T}} \sigma_{\min}(\widetilde{G}_{H_1}\widetilde{G}^*_{H_2})(j\omega)$$

and where the winding number is defined as

$$\mathrm{wno}(G) = \lim_{\omega\to\infty} \frac{\arg(\det(G(j\omega))) - \arg(\det(G(-j\omega)))}{2\pi}.$$

The fact that the index is equal to the winding number is well known; see e.g. [12, Chapter XII].

B. Finite Dimensional LTV Operators

Consider a time-varying linear system on the form

$$\begin{aligned} \dot{x}(t) &= A(t)x(t) + B(t)u(t) \\ y(t) &= C(t)x(t) + D(t)u(t). \end{aligned} \tag{8}$$

It is assumed that A, B, C, D are continuous and bounded matrix valued functions

$$\begin{aligned} A &: \mathbb{T} \to \mathbb{R}^{n\times n}, \qquad & B &: \mathbb{T} \to \mathbb{R}^{n\times m}, \\ C &: \mathbb{T} \to \mathbb{R}^{p\times n}, \qquad & D &: \mathbb{T} \to \mathbb{R}^{p\times m}. \end{aligned}$$

We let $X(\cdot)$ denote the invertible fundamental matrix defined by the solution of $\frac{dX}{dt}(t) = A(t)X(t)$ with $X(t) = I$ for $t \in \mathbb{T}$, which exists by the assumptions on $A(\cdot)$; see e.g. [2], [13]. It follows that the state transition matrix $\Phi_A(t,s) = X(t)X(s)^{-1}$ satisfies

$$\frac{d}{dt}\Phi_A(t,s) = A(t)\Phi_A(t,s), \qquad \Phi_A(t,t) = I.$$

Definition 5: The continuous and bounded matrix function $A : \mathbb{T} \to \mathbb{R}^{n\times n}$ defines an exponentially dichotomous evolution with associated exponential dichotomy (projection) $P = P^2 \in \mathbb{R}^{n\times n}$ if there exist constants $\rho > 0$ and $\sigma > 0$ such that

$$\|X(t)PX(s)^{-1}\| \le \rho e^{-\sigma(t-s)}\ \forall\, t \ge s \text{ and}$$
$$\|X(t)(I-P)X(s)^{-1}\| \le \rho e^{-\sigma(s-t)}\ \forall\, s \ge t.$$

This is equivalent to the following requirement: for each $t_0 \in \mathbb{T}$, there exist a projection $P_{t_0} (= X(t_0)PX(t_0)^{-1})$ such that

$$\|\Phi_A(t,t_0)P_{t_0}\Phi_A(t_0,s)\| \le \rho e^{-\sigma(t-s)}\ \forall\, t \ge s \text{ and}$$
$$\|\Phi_A(t,t_0)(I-P_{t_0})\Phi_A(t_0,s)\| \le \rho e^{-\sigma(s-t)}\ \forall\, s \ge t,$$

which implies

$$\mathrm{img}(P_{t_0}) = \{x \in \mathbb{R}^n : \Phi_A(\cdot,t_0)x \in \mathscr{L}_2^n(\mathbb{T}_+)\} \text{ and}$$
$$\ker(P_{t_0}) = \{x \in \mathbb{R}^n : \Phi_A(-\cdot,t_0)x \in \mathscr{L}_2^n(\mathbb{T}_-)\}.$$

Note that $\mathrm{rank}P = \mathrm{rank}P_{t_0}$. If $P = I$, then A defines an exponentially stable evolution.

When A defines an exponential dichotomy it follows that the state space system in (8) can be interpreted as a convolution operator $\mathbf{M} : \mathscr{L}_2^m(\mathbb{T}) \to \mathscr{L}_2^p(\mathbb{T})$ defined by the integral equation [13, Theorem 1.2.3]

$$(\mathbf{M}u)(t) = \int_{-\infty}^{\infty} k(t,s)u(s)ds + D(t)u(t), \tag{9}$$

where

$$k(t,s) = \begin{cases} C(t)X(t)PX(s)^{-1}B(s), & t \ge s \\ -C(t)X(t)(I-P)X(s)^{-1}B(s), & s > t \end{cases};$$

the first case corresponds to the causal part of the operator while the second case corresponds to the anti-casual part of the operator. The class of all such operators is an algebra [13, Theorem I.2.4]. The corresponding Wiener-Hopf operator is defined by

$$(\mathbf{T_M}u)(t) = \int_{t_0}^{\infty} k(t,s)u(s)ds + D(t)u(t),$$

where $k(t,s)$ is defined as above. We notice that since the state corresponds to

$$\begin{aligned} x(t) = \Phi_A(t,t_0)P_{t_0}\int_{t_0}^{t} \Phi_A(t_0,s)B(s)u(s)ds \\ - \Phi_A(t,t_0)(I-P_{t_0})\int_{t}^{\infty} \Phi_A(t_0,s)B(s)u(s)ds, \end{aligned}$$

we have $x(t_0) \in \mathrm{Im}(I - P_{t_0})$; i.e. $x(t_0) \in \ker P_{t_0}$.

The operator in (9) is henceforth denoted in terms of the state space realisation as

$$\mathbf{M} = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right].$$

In order to define the Fredholm index we use the inverse dynamics, which has stabilizable and detectable state-space representation

$$\mathbf{M}^{-1} = \left[\begin{array}{c|c} A - BD^{-1}C & BD^{-1} \\ \hline -D^{-1}C & D^{-1} \end{array}\right], \tag{10}$$

where it is assumed that D^{-1} is uniformly bounded. If we further assume the matrix $A^\times := A - BD^{-1}C$ has an exponential dichotomy defined by the projection $P^\times$, then the Fredholm index of $\mathbf{T_M}$ can be computed as

$$\begin{aligned} \mathrm{ind}(\mathbf{M}) &= \mathrm{rank}P - \mathrm{rank}P^\times \\ &= \dim \mathrm{Im}P - \dim \mathrm{Im}P^\times \\ &= \dim \ker P^\times - \dim \ker P; \end{aligned}$$

see [13, Theorem II.5.2].

The verification of upper and lower gain bounds must for this class of operators be performed using optimal control theory. Suppose $\overline{\gamma}(\mathbf{M}) < \bar{\gamma}$. This implies that the cost function

$$J(u) = \left\langle \begin{bmatrix} x \\ u \end{bmatrix}, \begin{bmatrix} -C^TC & -C^TD \\ -D^TC & \bar{\gamma}^2 - D^TD \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} \right\rangle_{\mathscr{L}_2(\mathbb{T})}$$

is strictly positive over all solutions $(x^T, u^T)^T \in \mathscr{L}_2^{n+m}(\mathbb{T})$ of $\dot{x} = Ax + Bu$. Similarly $\underline{\gamma}(\mathbf{M}) > \underline{\gamma}$ if

$$J(u) = \left\langle \begin{bmatrix} x \\ u \end{bmatrix}, \begin{bmatrix} -C^TD^{-T}D^{-1}C & -C^TD^{-T}D \\ -D^TD^{-1}C & \underline{\gamma}^{-2} - D^{-T}D^{-1} \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} \right\rangle_{\mathscr{L}_2(\mathbb{T})}$$

is strictly positive over all solutions $(x^T, u^T)^T \in \mathscr{L}_2^{n+m}(\mathbb{T})$ of $\dot{x} = (A - BD^{-1}C)x + BD^{-1}u$. Riccati theory for verifying these conditions can be found in [14]. An alternative technique to estimate gain bounds is obtained using the integral operator formulation in (9). We may the use the following result from [10].

Lemma 10: The integral operator in (9) satisfies

$$\overline{\gamma}(\mathbf{M}) \leq \sqrt{\gamma_1 \gamma_\infty},$$

where

$$\gamma_1 = \sup_{s\in\mathbb{R}} \int_{-\infty}^{\infty} |k(t,s)|dt,$$
$$\gamma_\infty = \sup_{t\in\mathbb{R}} \int_{-\infty}^{\infty} |k(t,s)|ds.$$

Remark 2: Notice that both γ_1 and γ_∞ are bounded since A defines an exponential dichotomy.

Now, without assuming that A defines an exponentially dichotomous evolution, if the pairs (A,B) and (A,C) are stabilizable and detectable, respectively, then from results in [19] there exists normalized coprime representations of the graph satisfying the properties identified in Section III; see also [1], [20] where the same result is obtained under stronger assumptions;

$$\mathbf{G_H} = \begin{bmatrix}\mathbf{V}\\ \mathbf{U}\end{bmatrix} = \left[\begin{array}{c|c} A+BF & BR^{-1/2} \\ \hline C+DF & DR^{-1/2} \\ F & R^{-1/2}\end{array}\right] \text{ and}$$

$$\widetilde{\mathbf{G}}_{\mathbf{H}} = \begin{bmatrix}-\widetilde{\mathbf{U}} & \widetilde{\mathbf{V}}\end{bmatrix} = \left[\begin{array}{c|cc} A+LC & -L & B+LD \\ \hline \widetilde{R}^{-1/2}C & -\widetilde{R}^{-1/2} & \widetilde{R}^{-1/2}D\end{array}\right],$$

where

$$R = I + D^TD, \qquad F = -R^{-1}(D^TC + B^TX),$$
$$\widetilde{R} = I + DD^T, \qquad L = -(BD^T + YC^T)\widetilde{R}^{-1},$$

and $X = X^T$, respectively $Y = Y^T$, are the exponentially stabilizing solutions to the differential Riccati equations

$$\begin{aligned}\dot{X} =& X(A - BR^{-1}D^TC) + ((A - BR^{-1}D^TC)^TX \\ &- XBR^{-1}B^TX + C^T\tilde{R}^{-1}C, \text{ respectively} \\ \dot{Y} =& (A - BD^T\widetilde{R}^{-1}C)Y + Y(A - BD^T\widetilde{R}^{-1}C)^T \\ &- YC^T\widetilde{R}^{-1}CY + BR^{-1}B^T.\end{aligned}$$

Indeed, any such time-varying system $\left[\begin{array}{c|c} A & B \\ \hline C & D\end{array}\right]$ may be associted with a causal operator $\mathbf{H} : \mathrm{dom}(\mathbf{H}) \subset \mathscr{L}_2^m(\mathbb{T}_+) \to \mathscr{L}_2^p(\mathbb{T}_+)$, with graph defined by

$$\mathcal{G}_{\mathbf{H}}^+ = \left\{ v = \begin{bmatrix}v_1\\v_2\end{bmatrix} \,:\, v_1 = \mathbf{V}w;\, v_2 = \mathbf{U}w;\, w \in \mathscr{L}_2(\mathbb{T}_+)\right\},$$

where the explicit expression for the domain is

$$\mathrm{dom}(\mathbf{H}) = \{v : v = \mathbf{V}w; w \in \mathscr{L}_2^m(\mathbb{T}_+)\},$$

and $\mathbf{U}$ and $\mathbf{V}$ are defined as in the normalized right coprime realization above.

The Hankel operator can be factorized as $\mathbf{H}^{+-}_{\widetilde{G}_{\mathbf{H}}} = \mathbf{L}_C\mathbf{L}_B$, where $\mathbf{L}_C : \mathbb{R}^n \to \mathscr{L}_2^p(\mathbb{T}_+)$ and $\mathbf{L}_B : \mathscr{L}_2^m(\mathbb{T}_-) \to \mathbb{R}^n$ are defined by

$$(\mathbf{L}_C x_0)(t) = C(t)\Phi(t,t_0)x_0 \text{ and}$$
$$(\mathbf{L}_B v)(t) = \int_{-\infty}^{t_0} \Phi(t_0,s)v(s)ds.$$

Since both $\mathbf{L}_C$ and $\mathbf{L}_B$ have finite rank it follows that $\mathbf{H}^{+-}_{\widetilde{G}_{\mathbf{H}}}$ is compact. Hence, Assumption 3 holds and therefore, by Proposition 1 the ν-gap in (5) can be formulated as in (6), which here reduces to

$$\delta_\nu(\mathbf{H}_1,\mathbf{H}_2) = \begin{cases}\overline{\gamma}(\widetilde{\mathbf{G}}_{\mathbf{H}_1}\mathbf{G}_{\mathbf{H}_2}), & \underline{\gamma}(\widetilde{\mathbf{G}}_{\mathbf{H}_1}\widetilde{\mathbf{G}}^*_{\mathbf{H}_2}) > 0 \text{ and} \\ & \mathrm{rank}P_{\widetilde{\mathbf{G}}_1\widetilde{\mathbf{G}}_2^*} - \mathrm{rank}P^\times_{\widetilde{\mathbf{G}}_1\widetilde{\mathbf{G}}_2^*} = 0 \\ 1, & \text{otherwise,}\end{cases}$$

where

$$\widetilde{\mathbf{G}}_{\mathbf{H}_1}\mathbf{G}_{\mathbf{H}_2} = \left[\begin{array}{c|c} A_{\widetilde{\mathbf{G}}_1\mathbf{G}_2} & B_{\widetilde{\mathbf{G}}_1\mathbf{G}_2} \\ \hline C_{\widetilde{\mathbf{G}}_1\mathbf{G}_2} & D_{\widetilde{\mathbf{G}}_1\mathbf{G}_2}\end{array}\right],$$
$$\widetilde{\mathbf{G}}_{\mathbf{H}_1}\widetilde{\mathbf{G}}^*_{\mathbf{H}_2} = \left[\begin{array}{c|c} A_{\widetilde{\mathbf{G}}_1\widetilde{\mathbf{G}}_2^*} & B_{\widetilde{\mathbf{G}}_1\widetilde{\mathbf{G}}_2^*} \\ \hline C_{\widetilde{\mathbf{G}}_1\widetilde{\mathbf{G}}_2^*} & D_{\widetilde{\mathbf{G}}_1\widetilde{\mathbf{G}}_2^*}\end{array}\right],$$

and where $P_{\widetilde{\mathbf{G}}_1\widetilde{\mathbf{G}}_2^*}$ and $P^\times_{\widetilde{\mathbf{G}}_1\widetilde{\mathbf{G}}_2^*}$ are the projections defining the exponential dichotomies of $A_{\widetilde{\mathbf{G}}_1\mathbf{G}_2}$ and $A^\times_{\widetilde{\mathbf{G}}_1\widetilde{\mathbf{G}_2}^*} = A_{\widetilde{\mathbf{G}}_1\widetilde{\mathbf{G}}_2^*} - B_{\widetilde{\mathbf{G}}_1\widetilde{\mathbf{G}}_2^*}D^{-1}_{\widetilde{\mathbf{G}}_1\widetilde{\mathbf{G}}_2^*}C_{\widetilde{\mathbf{G}}_1\widetilde{\mathbf{G}}_2^*}$, respectively. Bounds on $\overline{\gamma}(\widetilde{\mathbf{G}}_{\mathbf{H}_1}\mathbf{G}_{\mathbf{H}_2})$ and $\underline{\gamma}(\widetilde{\mathbf{G}}_{\mathbf{H}_1}\widetilde{\mathbf{G}}^*_{\mathbf{H}_2})$ can be verified using optimal control techniques, as discussed above.

C. Interconnections of LTV and LTI Systems

The theory in the previous sections can be applied to interconnections where an unstable system from the Callier Desoer class is stabilized by a finite dimensional linear time-varying system. We will here show that the assumption on nonsingular instantaneous gain necessarily will be satisfied if the interconnection is well-posed. The remaining conditions can be verified using the results discussed in the previous subsections.

Let $\mathbf{H}$ be defined by a transfer function from $\mathcal{B}^{p\times m}_{cl}$ and let $\mathbf{\Delta}$ be a finite dimensional LTV system. To establish the assumption on nonsingular instantaneous gain we use the left graph representation

$$(\widetilde{\mathbf{G}}_{\mathbf{H}}w)(t) = D_{\widetilde{\mathbf{G}}_{\mathbf{H}}}(t)w(t) + \int_{-\infty}^{t} h_{\widetilde{\mathbf{G}}_{\mathbf{H}}}(t-s)w(s)ds$$

and the right graph representation

$$(\mathbf{\Gamma_\Delta}v)(t) = D_{\mathbf{\Gamma_\Delta}}(t)v(t) + \int_{-\infty}^{t} k_{\mathbf{\Gamma_\Delta}}(t,s)v(s)ds,$$

where

$$k_{\mathbf{\Gamma_\Delta}}(t,s) = C_{\mathbf{\Gamma_\Delta}}(t)\Phi_{A_{\mathbf{\Gamma_\Delta}}}(t,s)B_{\mathbf{\Gamma_\Delta}}(s).$$

Note that both representations are causal with exponentially decaying integral kernels. This implies that there exists $\alpha > 0$ such that

$$e^{\alpha t}h_{\widetilde{\mathbf{G}}_{\mathbf{H}}}(t) \in \mathscr{L}_1[0,\infty) \tag{11}$$

and $|\Phi_{A_{\mathbf{\Gamma_\Delta}}}(t,s)| \leq ce^{-\alpha(t-s)}$ which in turn implies

$$|k_{\mathbf{\Gamma_\Delta}}(t,s)| \leq \rho e^{-\alpha(t-s)} \tag{12}$$

for some $\rho > 0$.

The next result shows that the instantaneous gain is equal to the product of the direct terms. This implies that the assumption on nonsingular instantaneous gain necessarily will be valid in a well-posed interconnection.

Proposition 2:

$$\rho_I(\widetilde{\mathbf{G}}_{\mathbf{H}}\mathbf{\Gamma}_{\mathbf{\Delta}}) \geq \inf_{t\in\mathbb{T}} \sigma_{\min}(D_{\widetilde{\mathbf{G}}_{\mathbf{H}}} D_{\mathbf{\Gamma}_{\mathbf{\Delta}}}(t)).$$

VII. Example

Consider an interconnection $[\mathbf{\Delta}, \mathbf{H}]$, where $\mathbf{H} \in \mathcal{B}_{cp}^{n\times n}$ and $\mathbf{\Delta}$ is an exponentially stable LTV system with stabilizable and detectable state space realization as in (8). We assume the *circle constraint*

$$\overline{\gamma}(\mathbf{\Delta} - cI) \leq r$$

for some center position c and radius $r > 0$. By Lemma 10 this holds if

$$\sup_{t\in\mathbb{R}} |D(t) - cI| + \sqrt{\gamma_1\gamma_\infty} \leq r$$

where γ_1 and γ_∞ are computed as in the lemma. This implies that $\mathbf{\Delta} \in \mathrm{IQC}^c(\Pi)$, where $\Pi = \begin{bmatrix} I & -cI \\ -cI & (c^2 - r^2)I \end{bmatrix}$.

By Theorem 1 it follows that $[\mathbf{\Delta}, \mathbf{H}]$ is stable if

(i) there exists a ν-gap continuous parametrization $\mathbf{H}_\theta$ with $\mathbf{H}_1 = \mathbf{H}$ and $[\mathbf{\Delta}, \mathbf{H}_0]$ stable.

(ii) $\mathbf{H}_\theta \in \mathrm{SIQC}(\Pi)$, $\forall\theta \in [0, 1]$.

Let us for simplicity specialize to the single-input single-output case. Let $c = -2$, $r = 3/4$ and $H_0(s) = \frac{1}{s-1}$. Then the interconnection $[\mathbf{\Delta}, \mathbf{H}_0]$ is stable, which is straightforward to verify since $r \cdot \|\frac{1}{1-cH_0}\|_\infty = \frac{3}{4} < 1$ and the stability follows from the small gain theorem. The IQC condition in (ii) above simplifies to the circle condition

$$\left|H_\theta(j\omega) - \frac{c}{c^2 - r^2}\right|^2 > \frac{r^2}{(c^2 - r^2)^2}, \quad \forall\omega \in \mathbb{R} \cup \{\infty\}. \tag{13}$$

Consider the following parametrizations

$$H_\theta = \frac{1}{s - 1 + 2\theta}, \quad \text{and} \quad H_\theta = \frac{e^{-s\theta\bar{h}}}{s-1}$$

for $\theta \in [0, 1]$. It is shown in [8] that these parametrizations are ν-gap continuous and the Nyquist plots in Figure 1 shows that the circle condition in (13) is satisfied in both cases. We conclude that all systems are stable.

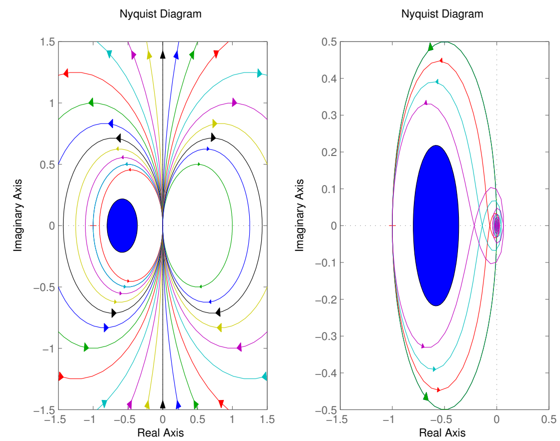

Fig. 1. The ν-gap continuous paths $H_\theta(s) = \frac{1}{s-1+2\theta}$ and $H_\theta = \frac{e^{-s\theta\bar{h}}}{s-1}$ in the case when $\bar{h} = 0.3$.

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