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OPTIMAL CONTROLLED VARIABLE SELECTION FOR INDIVIDUAL PROCESS UNITS IN SELF OPTIMIZING CONTROL WITH MIQP FORMULATIONS

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Abstract— **In order to facilitate optimal operation of process plants in the presence of disturbances, optimal control structure selection is important. In this paper we review the controlled variable selection, $\mathbf{c} = \mathbf{H}\mathbf{y}$, where $\mathbf{y}$ includes all the measurements. The objective is to find the matrix $\mathbf{H}$ such that steady-state operation is optimized when there are disturbances and inputs are adjusted to keep $\mathbf{c}$ constant. Several cases are studied such as optimal individual measurements, optimal combinations of fewer/all measurements and combinations of disjoint measurement subsets of fewer/all measurements. The proposed methods are evaluated on a distillation column case study with 41 trays.**

I. INTRODUCTION

Operating process plants close to the optimal even in the presence of disturbances, aid in improved productivity and profitability. Optimal control structure selection is vital for optimal operation. The decision on which variables to be controlled, which variables to be measured, which inputs to be manipulated and which links should be made between them is called control structure selection. Generally, the decisions of control structure selection are based on heuristic methods or on the intuition of process engineers. The scope of this paper is to select controlled variables (CVs) associated with the unconstrained degrees of freedom. We assume that the CVs ($\mathbf{c}'s$) are selected as individual measurements or combinations of fewer/all available measurements $\mathbf{y}$. This can be written as

$\mathbf{c} = \mathbf{H}\mathbf{y}$ where $n_y \geq n_c$;

n_y: number of measurements; n_c:number of CVs = number of unconstrained MVs = n_u ; where the objective is to find a good choice for the matrix $\mathbf{H}$. In general, we also include the inputs (MVs) in the available measurements set $\mathbf{y}$.

Skogestad and coworkers [11], [9] have proposed to use the steady state process model to find "self-optimizing" variables with an assumption that plant economics are governed by the pseudo/steady state behavior. The idea of "self-optimizing control" can be defined as suitable selection of $\mathbf{c}'s$ and by keeping these CVs ($\mathbf{c}'s$) at constant set points, the operation gives acceptable steady state loss from the optimal operation even in the presence of disturbances. The theory for self-optimizing control (SOC) is well developed for quadratic optimization problems with linear models. This may seem restrictive, but any unconstrained optimization problem may locally be approximated suitably by this method. Alternatively based on dynamic economics, a Mixed Integer Nonlinear Programming formulation is presented to select the controlled variables for the manipulated variables [3]. In this work, we concentrate only on "self-optimizing control". The "exact local method" [2] for SOC accounts for both disturbances and implementation errors. Here after we call "exact local method" as "minimum loss method". The problem of finding CVs as optimal variable combinations ($\mathbf{c} = \mathbf{H}\mathbf{y}$, where $\mathbf{H}$ is a full matrix) was originally believed to be nonconvex and thus difficult to solve numerically [2], but later it has been shown that this problem may be reformulated as a quadratic optimization problem with linear constraints [1]. The problem of selecting individual measurements, selecting combinations of fewer measurements as controlled variables are more difficult because of the combinatorial nature of the problem. As the alternatives increase rapidly with the process dimensions, exhaustive search is computationally intractable. Kariwala and Cao [6] have developed methods that use monotonicity of the objective function, but these cannot be used in the presence of structural constraints. This motivates the need to develop simple and efficient methods that can both handle structural constraints and find $\mathbf{c}'s$ as optimal individual measurements/combinations of fewer measurements.

Structural constraints are needed to improve dynamic controllability (i.e. fast response, control loop localization), to reduce the time delay between the MVs to CVs. In this paper, we consider the case where the $\mathbf{c}'s$ are obtained as combinations of disjoint measurement sets, where each $\mathbf{c}$ is obtained by combining the measurements from the process unit or section associated with an input $\mathbf{u}$ and the cases with structural constraints. Unfortunately for these cases we do not have a convex problem formulation, but we derive upper bounds to SOC problems with structural constraints using 3 approaches that result in convex quadratic programming (QP) problems at each node in MIQP formulation.

In summary, we consider two interesting problems related to finding $\mathbf{H}$ with structural constraints:

1) Selection of CVs as combination of disjoint measurement subsets using all the measurements
2) Selection of CVs as both combination of disjoint measurement subsets using only n measurements and that

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meet few additional structural constraints.

We try to address the above two problems using one of the proposed approaches, when applied to the minimum loss method formulation of [2]. Heldt [4] has reported an iterative method, but it is still non-convex and does not guarantee global optimum. In this study the proposed methods also cannot solve problem 1 and 2 to give a globally optimal $\mathbf{H}$ with specified structures, but the bounds obtained are of significant value from a practical point of view. The developed methods are evaluated on a distillation column case study with 41 measurements, where $\mathbf{c}'s$ are combinations of measurements with specified structures. The developed MIQP methods for SOC are generic and can easily be evaluated for any process plant.

II. MINIMUM LOSS METHOD

We here review the "minimum loss method" formulation from [2] and its optimal solution from [1] and present some new results (Theorems 4,5). We then provide some new ideas for dealing with the nonconvex case with structural constraints on $\mathbf{H}$. We denote measurements, inputs or manipulated variables, disturbances by $\mathbf{y}$,$\mathbf{u}$ and $\mathbf{d}$ respectively. The economic cost function for the steady state operation is denoted by $\mathbf{J}(\mathbf{u},\mathbf{d})$. In order to keep the operation optimal in the presence of varying disturbances the inputs $\mathbf{u}$ are updated according to $\mathbf{d}$ using online optimization (real-time optimization). We denote the optimal cost as $\mathbf{J}_{opt}(\mathbf{u}_{opt}(\mathbf{d}),\mathbf{d})$.

A simple and effective alternative is to update $\mathbf{u}$ using a feedback controller, which manipulates $\mathbf{u}$ to keep the CVs $\mathbf{c}$ at their specified set points $\mathbf{c}_s$.

$$\mathbf{c} = \mathbf{H}\mathbf{y} \tag{1}$$

where $\mathbf{c}_s = \mathbf{H}\mathbf{y}_{opt}(\mathbf{d}^*)$, $\mathbf{H}$ is the combination matrix and $\mathbf{y}$ are measurements.

Note that feedback introduces implementation error (noise) n^c. In the presence of integral action in feedback control the implementation error $n^c = \mathbf{H}n^y$. The difference between the cost functions of these two strategies is defined as the loss [12].

$$\mathbf{L} = \mathbf{J}(\mathbf{u},\mathbf{d}) - \mathbf{J}_{opt}(\mathbf{u}_{opt}(\mathbf{d}),\mathbf{d}) \tag{2}$$

Here "Self optimizing control" can be viewed as the selection of optimal $\mathbf{H}$ in $\mathbf{c} = \mathbf{H}\mathbf{y}$ and by keeping these $\mathbf{c}$ at constant set point $\mathbf{c}_s$ results in the minimal loss or that gives acceptable loss from the optimal operation. The set point $\mathbf{c}_s$ are obtained from the optimal solution for the nominal disturbance $\mathbf{d}$.

In order to express the loss ($\mathbf{L}$) as a function of disturbances, implementation errors locally, the loss is approximated using a second order Taylors series expansion around the "moving" optimal $\mathbf{u}_{opt}(d)$. We assume that the set of active constraints for the process does not change with $\mathbf{d}$ and n^c. The linearized (local) model in terms of the deviation variables is written as

$$\Delta\mathbf{y} = \mathbf{G}^y\Delta\mathbf{u} + \mathbf{G}_d^y\Delta\mathbf{d} \tag{3}$$

$$\Delta\mathbf{c} = \mathbf{G}\Delta\mathbf{u} + \mathbf{G}_d\Delta\mathbf{d} \tag{4}$$

where $\mathbf{G} = \mathbf{H}\mathbf{G}^y$and $\mathbf{G}_d = \mathbf{H}\mathbf{G}_d^y$. For a constant set point policy ($\mathbf{c}_s = 0$) [2].

It is assumed that the number of $\mathbf{c}'s$ is the same as the number of unconstrained degrees of freedom $\mathbf{u}$ and that $\mathbf{G} = \mathbf{H}\mathbf{G}^y$ is invertible. This assumption is needed to guarantee that the CVs are controlled at the specified set points using a controller with integral action.

Theorem 1: [1], [2], [8] Minimum loss method : To minimize the average and worst case loss for expected noise and disturbances, $\left\|\left[\begin{array}{c}\mathbf{d}'\\ n^{y'}\end{array}\right]\right\|_2 \leq 1$, find the $\mathbf{H}$ that solves the problem

$$\min_{\mathbf{H}} \left\|\mathbf{J}_{uu}^{1/2}(\mathbf{H}\mathbf{G}^y)^{-1}\mathbf{H}\mathbf{Y}\right\|_{2,F} \tag{5}$$

where $\mathbf{Y} = [\mathbf{F}\mathbf{W}_d\mathbf{W}_n)]$; $\mathbf{F} = \frac{\partial\mathbf{y}_{opt}}{\partial\mathbf{d}} = \mathbf{G}^y\mathbf{J}_{uu}^{-1}\mathbf{J}_{ud} - \mathbf{G}_d^y$; the 2-norm (maximum singular value) is for worst case loss, frobenius norm (F) is for average loss.

In many cases it is easier to find the optimal disturbance sensitivity matrix $\mathbf{F}$ numerically by reoptimizing for various disturbances. Kariwala et al. [8] prove that the combination matrix $\mathbf{H}$ that minimizes the average loss in (5) is super optimal and in the sense that the same $\mathbf{H}$ minimizes the worst case loss in (5). Hence, only optimization problem (5) involving the frobenius norm (F) is considered in the rest of the paper.

A. Finding full $\mathbf{H}$ without structural constraints

Theorem 2 (Reformulation as a convex problem): The problem in equation (5) may seem non-convex [1], but for the standard case where $\mathbf{H}$ is a full matrix (with no structural constraints), it can be reformulated as a constrained quadratic programming problem [1]

$$\begin{aligned}&\min_{\mathbf{H}} ||\mathbf{H}\mathbf{Y}||_F\\ &\text{s.t. } \mathbf{H}\mathbf{G}^y = \mathbf{J}_{uu}^{1/2}\end{aligned} \tag{6}$$

Proof: From the original problem in equation (5) the optimal solution $\mathbf{H}$ is non-unique. If $\mathbf{H}$ is a solution then $\mathbf{H}_1 = \mathbf{D}\mathbf{H}$ is also a solution as $(\mathbf{J}_{uu}^{1/2}(\mathbf{H}_1\mathbf{G}^y)^{-1}\mathbf{H}_1\mathbf{F}) = (\mathbf{J}_{uu}^{1/2}(\mathbf{H}\mathbf{G}^y)^{-1}\mathbf{H}\mathbf{F})$ for any non-singular matrix $\mathbf{D}$ of $n_u \times n_u$ size. This means the objective function is unaffected by the choice of $\mathbf{D}$. One implication is that $\mathbf{H}\mathbf{G}^y$ can be chosen freely. We can thus make $\mathbf{H}$ unique by adding a constraint, for example $\mathbf{H}\mathbf{G}^y = \mathbf{J}_{uu}^{1/2}$. More importantly this simplifies the optimization problem in equation (5) to optimization problem shown in equation (6). *End Proof*

Theorem 3 ([1]): An analytical solution to (5) in Theorem 1 using Theorem 2 is $\mathbf{H}^T = \left(\mathbf{Y}\mathbf{Y}^T\right)^{-1}\mathbf{G}^y\left(\mathbf{G}^{y^T}\left(\mathbf{Y}\mathbf{Y}^T\right)^{-1}\mathbf{G}^y\right)^{-1}\mathbf{J}_{uu}^{1/2}$.

Theorem 4 (Simplified analytical solution): Another analytical solution for the problem in (5) is

$$\mathbf{H}^T = \left(\mathbf{Y}\mathbf{Y}^T\right)^{-1}\mathbf{G}^y\mathbf{Q} \tag{7}$$

where $\mathbf{Q}$ is any non-singular matrix of $n_c \times n_c$.

Proof. This follows trivially from Theorem 3, since if $\mathbf{H}^T$ is a solution then so is $\mathbf{H}_1^T = \mathbf{H}^T\mathbf{D}^T$ and we simply select $\mathbf{D}^T = (\mathbf{Q}^{-1}(\mathbf{G}^{y^T}(\mathbf{Y}\mathbf{Y}^T)^{-1}\mathbf{G}^y)^{-1}\mathbf{J}_{uu}^{1/2})^{-1} =$

$\mathbf{J}_{uu}^{-1/2}\mathbf{G}^{y^T}(\mathbf{Y}\mathbf{Y}^T)^{-1}\mathbf{G}^y\mathbf{Q}$ which is a $n_c \times n_c$ matrix.*End proof.*

Corollary 1 (Important insight): Theorem 4 gives the very important insight that $\mathbf{J}_{uu}$ is not needed for finding the optimal $\mathbf{H}$, provided we have the standard case where $\mathbf{H}$ can be any $n_c \times n_y$ matrix.

This means that in (5) we can replace $\mathbf{J}_{uu}^{1/2}$ by any non-singular matrix, and still get an optimal $\mathbf{H}$. This can greatly simplify practical calculations, because $\mathbf{J}_{uu}$ may be difficult to obtain numerically because it involves the second derivative. On the other hand, we have that $\mathbf{F}$, which enters in $\mathbf{Y}$, is relatively straightforward to obtain numerically. Although $\mathbf{J}_{uu}$ is not needed for finding the optimal $\mathbf{H}$, it would be required for finding a numerical value for the loss.

Theorem 5 (Generalized convex formulation): An optimal $\mathbf{H}$ for the problem (5) can be written as in (8) using Theorem 4, where $\mathbf{Q}$ is any non-singular matrix of $n_c \times n_c$.

$$\begin{aligned} &\min_{\mathbf{H}} ||\mathbf{H}\mathbf{Y}||_F \\ &\text{s.t. } \mathbf{H}\mathbf{G}^y = \mathbf{Q} \end{aligned} \tag{8}$$

Proof. The result follows from Corollary 1, but can more generally be derived as follows. The problem in (6) is to minimize $\left\| \underbrace{(\mathbf{J}_{uu}^{1/2}(\mathbf{H}\mathbf{G}^y)^{-1}}_{\mathbf{X}}\mathbf{H}\mathbf{Y}) \right\|_F$. The reason why we can omit the $n_c \times n_c$ matrix $\mathbf{X}$, is that if $\mathbf{H}$ is an optimal solution then so is $\mathbf{H}_1 = \mathbf{D}\mathbf{H}$ where $\mathbf{D}$ is any nonsingular $n_c \times n_c$ (see proof of Theorem 2). However, note that the matrix $\mathbf{X}$, or equivalently the matrix $\mathbf{Q}$, must be fixed during the optimization, so it needs to be added as a constraint. *End proof.*

B. Dealing with structural constraints on $\mathbf{H}$

For practical reasons, it may be interesting to obtain the $\mathbf{c}'s$ as combinations of measurements with a specified structure.

$$\begin{aligned} &\min_{\mathbf{H}} \left\| \mathbf{J}_{uu}^{1/2}(\mathbf{H}\mathbf{G}^y)^{-1}\mathbf{H}\mathbf{Y} \right\|_F^2 \\ &\text{s.t.}\mathbf{H} = [\text{specified structure}] \end{aligned} \tag{9}$$

By specified structure we mean that certain elements in $\mathbf{H}$ are fixed to zero. We will consider the following special cases:

Case 1. Selecting subset of measurements (some columns in $\mathbf{H}$ are zero)

(a) **Fixed subset**. For example, $\mathbf{H} = \begin{bmatrix} 0 & \mathbf{h}_{12} & 0 & \mathbf{h}_{14} & \mathbf{h}_{15} \\ 0 & \mathbf{h}_{22} & 0 & \mathbf{h}_{24} & \mathbf{h}_{25} \end{bmatrix}$. In such cases, both Theorem 2 and 5 hold. This implies $\mathbf{J}_{uu}$ is *not* needed. This is quite obvious since it corresponds to deleting some measurements.

(b) **Optimal subset**. where the objective is to select measurements (e.g. 3 out of 5). In this case, only Theorem 2 hold and we need $\mathbf{J}_{uu}$. This is because in Theorem 2, $\mathbf{H}\mathbf{G}^y = \mathbf{J}_{uu}^{1/2}$ and the ordering of the loss in (5) and $||\mathbf{H}\mathbf{F}||_F$ is the same for all possible subsets.

Case 2. Specified structure (specified elements are zero in addition to some columns in $\mathbf{H}$ are zero)

(I) **Decentralized structure**. For example, If a process has 2 inputs and 5 measurements with 2 disjoint measurement sets $\{1,2,3\}$,$\{4,5\}$; then the structure is $\mathbf{H}_I = \begin{bmatrix} \mathbf{h}_{11} & \mathbf{h}_{12} & \mathbf{h}_{13} & 0 & 0 \\ 0 & 0 & 0 & \mathbf{h}_{24} & \mathbf{h}_{25} \end{bmatrix}$

(II) **Triangular structure**. For example, If a process has 2 inputs and 5 measurements with partially disjoint measurement sets as $\{1,2,3,4,5\}$ for one CV and $\{4,5\}$ for another CV, then the structure is $\mathbf{H}_{II} = \begin{bmatrix} \mathbf{h}_{11} & \mathbf{h}_{12} & \mathbf{h}_{13} & \mathbf{h}_{14} & \mathbf{h}_{15} \\ 0 & 0 & 0 & \mathbf{h}_{34} & \mathbf{h}_{35} \end{bmatrix}$;

Theorem 2 do *not* hold in case 2. The reason is that to have same structure as $\mathbf{H}$ in $\mathbf{H}_1 = \mathbf{D}\mathbf{H}$, $\mathbf{D}$ must have a structure $\mathbf{D}_I = \begin{bmatrix} \mathbf{d}_{11} & 0 \\ 0 & \mathbf{d}_{22} \end{bmatrix}$, $\mathbf{D}_{II} = \begin{bmatrix} \mathbf{d}_{11} & \mathbf{d}_{12} \\ 0 & \mathbf{d}_{22} \end{bmatrix}$ respectively so $\mathbf{D}$ is not a full matrix as assumed when deriving Theorem 2.

Case 3. Selecting the best individual measurements for decentralized control, for example, $\mathbf{H} = \begin{bmatrix} \mathbf{h}_{11} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathbf{h}_{24} & 0 \end{bmatrix}$. This is a special case of case 2 (I), but Theorem 2 holds as it can also be viewed as case 1(b) as the selection of the best n_u measurements. Then the non-zero part of $\mathbf{H}$ is a square matrix and later we can choose $\mathbf{D}$ as inverse of this square full matrix to arrive at a decentralized diagonal $\mathbf{H}$.

C. Dealing with specified disjoint structure

Consider using controlled variables $\mathbf{c}'s$ as combinations of disjoint measurement sets, this can be viewed as separate control of individual process units (process sub parts). This is case 2 (I) for $\mathbf{H}$,

$$\mathbf{H} = \begin{bmatrix} \mathbf{H}_1 & 0 & \cdots & 0 \\ 0 & \mathbf{H}_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathbf{H}_{n_{iu}} \end{bmatrix} \tag{10}$$

for which Theorem 2 does not apply. Here each $\mathbf{H}_i$ corresponds to measurements and inputs of process unit i and n_{iu} is the number of individual process units in the plant. Note that, as opposed to cases 1 (a) and 3, $\mathbf{J}_{uu}$ is needed to find the optimal solution for case 2, this may seem a bit surprising. For case 2, we do not have a convex problem formulation, that is, we need to solve the nonconvex problem in (9). This is not surprising as decentralized control is generally a nonconvex problem. Nevertheless, using the ideas from Theorems 2 and 5, with additional constraints on the structure of $\mathbf{H}$, give convex optimization problems that provide *upper bounds* on the optimal $\mathbf{H}$ for case 2. In particular, in Theorem 5 we may make use of the extra degree of freedom provided by the matrix $\mathbf{Q}$ [13].

The idea is to exclude the matrix $\mathbf{J}_{uu}^{1/2}(\mathbf{H}\mathbf{G}^y)^{-1}$ in front of $\mathbf{H}\mathbf{F}$ in (9). However, when $\mathbf{H}$ has a specified structure, we do not generally have enough degrees of freedom to make

$\mathbf{J}_{uu}^{1/2}(\mathbf{HG}^y)^{-1} = I$. To proceed, we have considered the following 3 options :

1) Use the non-zero (n_{nz}) elements in $\mathbf{D}$ to match any n_{nz} number of elements in $\mathbf{HG}^y$ to $\mathbf{J}_{uu}^{1/2}$. This results in multiple choices to select n_{nz} elements in $\mathbf{HG}^y$, so an MIQP formulation is presented to find the optimal $\mathbf{H}$ with specified structure.
2) Introduce a constraint $\mathbf{HG}^y \leq \mathbf{Q}$ [14], this provides extra freedom to choose optimal structured $\mathbf{H}$. $\mathbf{Q}$ must be chosen to have negative elements in each row to obviate the trivial solution.
3) Use a constraint to let $\mathbf{J}_{uu}^{1/2}(\mathbf{HG}^y)^{-1}$ have a structure similar to the $\mathbf{D}$ that preserves the structure in $\mathbf{H}, \mathbf{DH}$ and the remaining problem is to minimize $||\mathbf{HF}||_F$.

Numerical evidence shows that option 1, option 2 and option 3 provide good upper bounds to the problem in (9). We present details of option 1 only in this paper.

The optimal solution to equation (9) is non-unique, so if $\mathbf{H}$ is a solution then $\mathbf{H}_1 = \mathbf{DH}$ is also a solution as for any non-singular matrix $\mathbf{D}$ of $n_u \times n_u$ size that preserves the structure in constraint of equation (10). The number of non-zero elements in matrix $\mathbf{D}$ is n_{nz}.So we can select any n_{nz} number of elements in $\mathbf{Q} = \mathbf{HG}^y$ freely. There are multiple choices of selecting the n_{nz} elements in $\mathbf{Q}$, and all of these can be explored by formulating it as a mixed integer formulation.

In this formulation, we let $n_u n_u$ numbers of binary variables (z_j) denote which elements in $\mathbf{Q}$ should be matched to $\mathbf{HG}^y$. Only n_{nz} non-zero elements in $\mathbf{D}$ that preserve the structure) of these $n_u n_u$ binary variables should be 1 and the rest should be 0. The non-singular matrix $\mathbf{Q}$ can be chosen to solve the resulting MIQP faster. In our case, we simply selected $\mathbf{Q} = 0.01\mathbf{J}_{uu}^{1/2}$ and resulting MIQP formulation is

$$\begin{aligned} &\min_{\mathbf{H}} ||\mathbf{HF}||_F \\ \text{s.t.}\ & -\mathrm{m}(1-\mathrm{z_j}) \leq [\mathbf{HG}^y - \mathbf{Q}]_{\text{element}} \leq m(1-z_j) \\ & z_i \in \{0,1\} \\ & \sum_{j=1}^{n_u n_u} z_j = n_{nz};\ \text{set of eqns} \sum_{l=n_u(k-1)+1}^{n_u n_{u_k}} z_l = n_{u_k} \\ & \forall k = 1,2,\ldots,\ \text{number of blocks} \\ & \mathbf{H} = [\text{specified structure}] \end{aligned} \tag{11}$$

where n_{u_k} is the number of inputs in block k. The constraints in the reformulated problem are: (i) Among the $n_u n_u$ number of elements in $\mathbf{HG}^y$, n_{nz} elements match the elements of $\mathbf{Q}$, (ii) Equality constraint : Sum of binary variables associated to elements in $\mathbf{HG}^y$ to n_{nz}, (iii) The structure imposed on $\mathbf{H}$.

Only certain $\mathbf{D}$ matrices preserve the structure imposed on $\mathbf{H}$ in equation (10), which means that only certain n_{nz} elements of $\mathbf{HG}^y$ can be matched to elements of $\mathbf{Q}$ matrix. For example, if we match an element $1,2$ in $\mathbf{HG}^y$ to $\mathbf{Q}_{12}$ then the associated binary variable z_2 is 1; if we do not match the element $2,1$ in $\mathbf{HG}^y$ to $\mathbf{Q}_{21}$ then the associated binary variable z_{n_u+1} is 0. Similar to the big-M constraints in MIQPs, the scalar m is used to bound the unmatched elements of $\mathbf{HG}^y$ in the range $-m$ to m. The problem in the decision matrix $\mathbf{H}$ in equation (11) is vectorized as described in [15]. Solving equation (11) results in controlled variables $\mathbf{c}'s$ as combinations of disjoint measurements sets of all measurements. This provides the upper bound for problem in equation (9) to find $\mathbf{c}'s$ as the combinations of disjoint measurement subsets.

III. MIQP BASED FORMULATIONS

A. MIQP Formulation for best subset selection

The best measurement subset selection problem is to find $\mathbf{c}'s$ as best combinations of measurement subsets. Some solution approaches are (i) partial branch and bound methods [7] (ii) generalized singular value decomposition methods [4] (iii) MIQP based formulations [15]. We discuss only the MIQP formulations here.

$$\begin{aligned} \min_{\mathbf{x}_{aug}}\quad & \mathbf{x}_{aug}^T \mathbf{F}_{aug} \mathbf{x}_{aug} \\ \text{s.t.}\quad & \mathbf{G}_{new}^{y^T} \mathbf{x}_{aug} = \mathbf{Q}_\delta \\ & \mathbf{P}\mathbf{x}_{aug} = n \\ & \sigma_i \in \{0,1\} \quad \forall i = 1,2,\cdots,n_y \end{aligned} \tag{12}$$

$$\begin{bmatrix} -\mathbf{M} & 0 & \cdots \\ 0 & -\mathbf{M} & \cdots \\ \vdots & \vdots & \ddots \\ 0 & \cdots & -\mathbf{M} \end{bmatrix} \sigma_i \leq \begin{bmatrix} \mathbf{x}_i \\ \mathbf{x}_{n_y+i} \\ \vdots \\ \mathbf{x}_{(n_u-1)n_y+i} \end{bmatrix} \leq \begin{bmatrix} \mathbf{M} & 0 & \cdots \\ 0 & \mathbf{M} & \cdots \\ \vdots & \vdots & \ddots \\ 0 & \cdots & \mathbf{M} \end{bmatrix} \sigma_i$$

where $\mathbf{X}_{aug}^T = \begin{bmatrix} \mathbf{x}_\delta^T & \sigma_1 & \sigma_2 & \ldots & \sigma_{n_y} \end{bmatrix}$; $\mathbf{F}_{aug} = \begin{bmatrix} \mathbf{Y}_\delta \mathbf{Y}_\delta^T & 0 \end{bmatrix}$; $\mathbf{G}_{new}^{y^T} = \begin{bmatrix} \mathbf{G}_\delta^T & 0 \end{bmatrix}$; $\mathbf{P} = [0 \quad 1]$ and **n** is the measurement subset size. where $\mathbf{X}_{aug}, \mathbf{F}_{aug}, \mathbf{G}_{new}^{y^T}, \mathbf{P}$ are of size $(n_u n_y + n_y) \times 1, (n_u n_y + n_y) \times (n_u n_y + n_y), n_u n_u \times (n_u n_y + n_y), 1 \times (n_u n_y + n_y)$ respectively.

Starting from (6), the best measurement subset selection problem can be formulated with the "big M" parameter as in (12) [13]. Note that $\mathbf{Q}$ can be chosen as $\mathbf{Q} = r\mathbf{J}_{uu}^{1/2}$ (where r is positive scalar) to preserve the loss ordering of different measurement sets in the MIQP formulation (12). However r should be selected suitably as very small values can interfere with the MIQP solver tolerances.

B. CVs as combinations of disjoint measurement sets of all measurements

For practical reasons, it may be interesting to obtain the $\mathbf{c}'s$ as combinations of disjoint measurement sets; meaning that it has a specified structure. The problem in equation (9) is non-convex, and, unfortunately, Theorem 2 cannot be used to get a convex QP because of the structural constraints in $\mathbf{H}$. But we can use the ideas from Theorem 2 to derive a convex QP that provides a good upper bound as described in II-C.

C. CVs as combinations of disjoint measurement sets of fewer measurements

It is easy to extend the problem formulation in (9) to find CVs as best combinations of fewer measurements in disjoint sets with some more additional constraints by introducing n_y

new binary variables $(\sigma_1, \sigma_2, \ldots, \sigma_n)$. The MIQP problem becomes

$$
\begin{aligned}
\min_{\mathbf{x}_N} \quad & \mathbf{x}_N^T \mathbf{F}_N \mathbf{x}_N \\
\text{s.t.} \quad & \mathbf{G}_N^{y^T} \mathbf{x}_N = \mathbf{Q}_\delta \qquad (13) \\
\mathbf{P}_N \mathbf{x}_N = n; & \ \text{big} - \text{M constraint in (12)} \\
& \sigma_i \in \{0,1\} \quad \forall i = 1, 2, \cdots, n_y \\
& z_j \in \{0,1\} \quad \forall j = 1, 2, \cdots, n_u n_u \\
& \sum_{i=1}^{n_y} \sigma_i + \sum_{j=1}^{n_u n_u} z_j = n_y + n_{nz} \\
\text{set of eqns} & \sum_{\mathrm{l=n_u(k-1)+1}}^{\mathrm{n_u n_{u_k}}} \mathrm{z_l} = \mathrm{n_{u_k}}; \sum_{m=1}^{n_{y_k}} \sigma_{(n_{y_{(k-1)}}(k-1)+m)} = n_k \\
& \forall k = 1, 2, \ldots, \text{number of blocks} \\
& \text{set of eqns } \mathbf{x}_\mathrm{N}(\mathrm{ind}) = 0 \\
& \text{ind associated to 0 in } \mathbf{H}
\end{aligned}
$$

where $\mathbf{X}_N^T = \begin{bmatrix} \mathbf{x}_{aug}^T & z_1 & z_2 \ldots & z_{n_u n_u} \end{bmatrix}$; $\mathbf{F}_N = [\mathbf{F}_{aug} \quad 0]$; $\mathbf{G}_N^{y^T} = \begin{bmatrix}\mathbf{G}_{new}^T & 0\end{bmatrix}$; $\mathbf{P}_N = [\mathbf{P} \quad 0]$ and n is the measurement subset size, n_{nz} is the number of non-zeros in $\mathbf{D}$, n_{u_k}, n_{y_k} and n_k are the numbers of inputs, measurements and measurements to be selected in disjoint set k. Where $\mathbf{X}_N, \mathbf{F}_N, \mathbf{G}_N^{y^T}, \mathbf{P}_N$ are of size $(n_u n_y + n_y + n_u n_u) \times 1, (n_u n_y + n_y + n_u n_u) \times (n_u n_y + n_y + n_u n_u), n_u n_u \times (n_u n_y + n_y + n_u n_u), 1 \times (n_u n_y + n_y + n_u n_u)$ respectively.

IV. DISTILLATION CASE STUDY

The MIQP formulations for obtaining CVs as combinations of disjoint measurement are evaluated on binary distillation column case study [10], where reflux $\mathbf{L}$ and boil up $\mathbf{V}$ are the remaining steady-state degrees of freedom ($\mathbf{u}$). The 41 stage temperatures are taken as candidate measurements. Note that we do not include the inputs in the candidate measurements for this case study. The economic objective $\mathbf{J}$ for the indirect composition control problem is

$$
\mathbf{J} = \left(\frac{x_{top}^H - x_{top,s}^H}{x_{top,s}^H} \right)^2 + \left(\frac{x_{btm}^L - x_{btm,s}^L}{x_{btm,s}^L} \right)^2 \qquad (14)
$$

where $\mathbf{J}$ is the relative steady state composition deviation. $\mathbf{x}_{top}^H, \mathbf{x}_{btm}^L, \mathbf{x}_{top,s}^H, \mathbf{x}_{btm,s}^L, \mathbf{L}$ and $\mathbf{H}$ denote the heavy component composition in top tray, light component composition in bottom tray, specification of heavy component composition in top tray, specification of light component composition in bottom tray, light and heavy key components respectively. The MIQP formulation described in section III-A is implemented for the distillation column with 41 trays to find the 2 CVs as the combinations of 41 tray temperatures. An MIQP is set up for this distillation column with the choice M=1 for the big-M constraints in equation (12). We solved the MIQP to find the CVs as the combinations of best measurement subset size from 2 to 41. The CPLX solver in IBM ILOG Optimizer was used to solve the MIQP problem [5].

We study the case with two disjoint measurements subsets; one for the top section and one for the bottom section of the distillation column. This structure is desirable, mainly for dynamic reasons, to select one combined measurement $\mathbf{c}_1$ from the top section (trays 21 to 41) and one combined measurement $\mathbf{c}_2$ from the bottom section (trays 1 to 20).

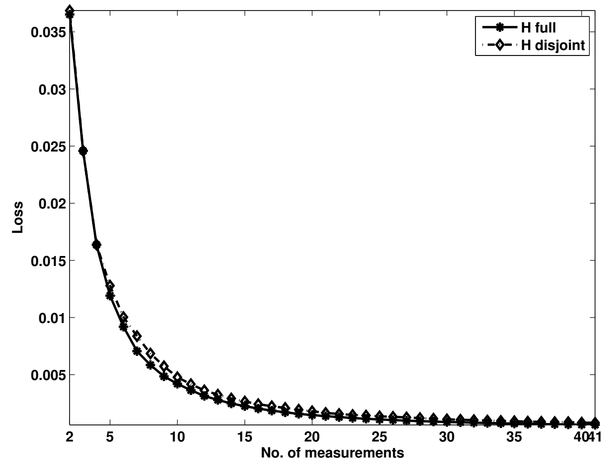

Fig. 1. The loss vs the number of included measurements where the $\mathbf{c}'s$ are combinations of (i) all measurements (solid), (ii) disjoint measurements sets (top and bottom of column)(dash dot)

In addition to the structures mentioned in (10), the following structural constraints are also incorporated. To select n number of measurements, $\lfloor n/2 \rfloor$ number of measurements should be selected from top trays $\lfloor ny/2 + 1 : ny \rfloor$, and rest of the measurements should be selected from $\{1 : \lfloor ny/2 \rfloor\}$;

1) to select 2 measurements, $\lfloor n/2 \rfloor = 1$ measurement should be selected from bottom trays 1 to 20 temperature measurements and other 1 measurement from top trays 21 to 41 temperature measurements
2) to select 9 measurements, $\lfloor ny/2 \rfloor = 4$ measurements should be selected from bottom trays 1 to 20 temperature measurements and rest from top trays 21 to 41 temperature measurements.

Meas	$\mathbf{c}'s$ as combinations of measurements	$Loss_{all}$	$\mathbf{c}'s$ as combinations of measurement	$Loss_{dis}$
	$\mathbf{c}'s$		$\mathbf{c}'s$ with disjoint structure	
2	$\mathbf{c}_1 = T_{12}$ $\mathbf{c}_2 = T_{30}$	0.03657	$\mathbf{c}_1 = T_{12}$ $\mathbf{c}_2 = T_{29}$	0.036868*
3	$\mathbf{c}_1 = T_{12} + 0.0446T_{31}$ $\mathbf{c}_2 = T_{30} + 1.0216T_{31}$	0.024583	$\mathbf{c}_1 = T_{12}$ $\mathbf{c}_2 = T_{30} + 0.9898T_{31}$	0.024593**
4	$\mathbf{c}_1 = T_{11} + 11.2295T_{30} + 11.5251T_{31}$ $\mathbf{c}_2 = T_{12} + 11.5844T_{30} - 11.79T_{31}$	0.016365	$\mathbf{c}_1 = 1.0088T_{11} + T_{12}$ $\mathbf{c}_2 = T_{30} + 0.9898T_{31}$	0.016385**

†disjoint $\mathbf{H}$,(Case 2 for H); $*$ clearly not optimal because this is Case 3 for $\mathbf{H}$ and all structures must give same solution; $**$ is just an upper bound

TABLE I
THE SELF OPTIMIZING VARIABLES $\mathbf{c}'s$ AND ASSOCIATED LOSSES, WHERE $\mathbf{c}'s$ ARE (I) COMBINATIONS OF MEASUREMENTS, (II) COMBINATIONS OF DISJOINT MEASUREMENT SUBSETS

The loss associated to (10), and these structural constraints is also shown in Fig. 1. Fig. 1 show that the loss in terms of the relative composition deviation (14), decreases as the number of included measurements increases from 2 to 41. For the included number of measurements, the actual measurements set, combination weights are determined as part of the MIQP solution. From Fig. 1, we see that the loss with $\mathbf{c}'s$ as combinations of disjoint measurement sets is very close to the loss with $\mathbf{c}'s$ as combinations of all the

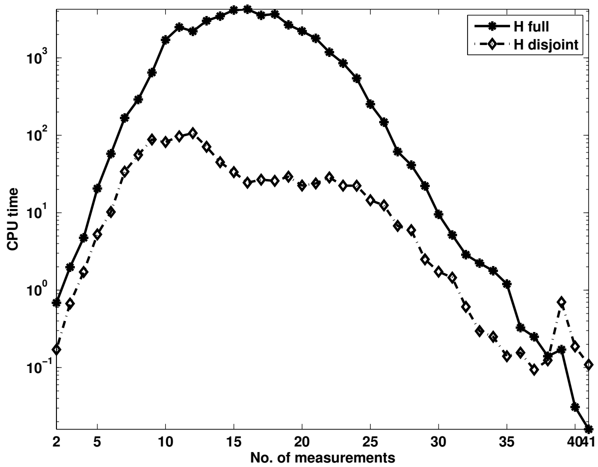

Fig. 2. CPU time requirement for computations in Fig.1

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included measurements; the loss for the worst case increases only by a factor of 1.32. However, the computation time is much shorter; from Fig. 2, we see that obtaining the $\mathbf{c}'s$ as combinations of disjoint measurement sets is approximately 2 orders of magnitude faster than without the structural constraints.

The actual optimal controlled variables (measurement combination $\mathbf{H}$) for the cases with 2, 3 and 4 measurements are shown in Table I. For the case with 2 measurements, we just give the measurement, and not the combination, because we can always choose the $\mathbf{D}$ matrix to make $\mathbf{H} = I$ (identity). For the case with 3 and 4 measurements, we selected $\mathbf{D}$ to make selected elements in $\mathbf{H}$ equal to 1. For the case with 2 measurements, the optimal measurement set is $\{T_{12}, T_{29}\}$. However, the proposed method in (13) only gives an upper bound (because it matches only two elements in $\mathbf{HG}^y$), and this is why it gives a non-optimal set $\{T_{12}, T_{30}\}$ and the loss is increased slightly from 0.0365 to 0.0369. Interestingly, the optimal measurements for the disjoint and full $\mathbf{H}$ case are same for 3 and 4 (Table I) measurement sets. However, since we are restricted in how we can combine measurements in the disjoint case, there is a small difference in the associated losses. Thus, although the method (11) developed for obtaining $\mathbf{c}'s$ as combinations of disjoint measurements sets are not exact, it serves as a tight upper bound for the true optimal solution for the problem in (9).

V. CONCLUSIONS

The minimum loss method of self optimizing control for optimal control structure selection with economic cost function as criterion is addressed. The MIQP based formulations to find controlled variables as best individual measurements, as best combinations of fewer/all measurements are reviewed. As controlled variables are combinations of fewer/all measurements of process plant, there is a possibility of poor controllability between $\mathbf{u}'s$ and $\mathbf{c}'s$. To overcome this disadvantage, the controlled variables are only allowed to be combinations of measurements of disjoint measurement sets, where each measurement subset is associated to a particular unit in a process plant. The method proposed provided a very close upper bound to the exact solution of $\mathbf{c}'s$ as combinations of disjoint measurement sets. Even though the proposed method is not exact the upper bound for the solution is of significant value from a practical point of view. For the distillation column case study the loss increases by a factor of 1.32 with the disjoint measurement set structural constraint.

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