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Positive Realness of a Set of Matrix-Valued Time-Varying Uncertainties

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Abstract— **The manuscript is concerned with characterizations of a set of matrix-valued, time-varying, uncertain multiplication operators. The set of uncertain multiplication operators arises from robust stability analysis of systems with aperiodic sampling-and-hold devices, where the sampling periods are assumed to be bounded within a given range. Conditions under which the uncertain operator is positive real are identified. Based on the positive real property, a number of quadratic constraints are derived for the uncertain operator. Such quadratic constraints are proven to be useful in analyzing robustness of aperiodic sampled-data systems against variation of sampling intervals.**

I. Introduction

Consider the state space system

$$\dot{x}(t) = Ax(t) + Bu(t)$$

where x and u respectively denote the state and the control input taking values in $\mathbb{R}^n$ and $\mathbb{R}^m$. A and B are two real-valued matrices of comparable dimensions. We assume the system is operated under the following scenario:

- Measurements of states are taken at time instances $t = \tau[k]$ $(k = 0, 1, \cdots)$ where $\{\tau[k]\}$ is a sequence of unknown real numbers satisfying

  $$\tau[0] = 0, \quad 0 < \underline{\theta} \le \tau[k+1] - \tau[k] \le \bar{\theta} < \infty,$$

  where $\underline{\theta}$ and $\bar{\theta}$ are two given constants.
- The control input u is determined by the feedback rule:

  $$u(t) = Fx(\tau[k]), \quad \forall\, t \in [\tau[k], \tau[k+1]).$$

Under the scenario stated above, the resulting feedback system has the form

$$\dot{x}(t) = Ax(t) + BFx(\tau[k]), \quad \forall\, t \in [\tau[k], \tau[k+1]). \quad (1)$$

Systems of such form are called "aperiodic sampled-data control systems". Applications of such system model can be found in networked and/or embedded control systems (see [1], [2] and references therein), where resources for measurement and control are limited. In view of the widespread of use of networked and embedded control systems, robust stability analysis of system model (1) against variation of sampling intervals is both theoretically and practically important. Various approaches for analyzing such robustness property have been proposed in the literature; see, for examples, [3], [4], [5], [6], [7], [8], [9], [10], [11], [12], [13], [14]. Most existing results are based on the classical Lyapunov theorem, by either explicitly or implicitly constructing various kinds of Lyapunov functions. In contrast, operator-theoretic approaches receive relatively less attention; some previous results are found in [7], [12], [13], [14]. In [14], system (1) is transformed to a feedback interconnection of a linear-time-invariant discrete-time system and a time-varying uncertain multiplication operator, and a stability condition based on the small gain theorem is proposed.

In this manuscript, we aim at refining the stability condition proposed in [14]. The key idea is to identify properties of the uncertain multiplication operator useful to robustness analysis of the aperiodic sampled-data system. In particular, we are interested in the so-called "positive real" property of the uncertain multiplication operator. Conditions under which the uncertain operator is positive real will be derived. Based on the positive real property, a number of quadratic constraints are derived for the uncertain operator. Such quadratic constraints are proven to be useful in analyzing stability of systems with aperiodic sampling-and-hold devices.

The remaining manuscript is organized as follows: the rest of this section is devoted to introducing notations and terminologies that will be used throughout the manuscript. The exact problem formulation is given in Section II and important characterizations of the spectrum of the uncertain operator are presented in Section III. Integral quadratic constraints which characterizes the positive real property of the uncertain operator are derived in Section IV. Finally, we give examples which demonstrate how the positive real property is helpful in stability analysis in Section V and draw a few concluding remarks in Section VI.

Notations and Terminologies

Symbols $\mathbb{R}$, $\mathbb{R}_+$, $\mathbb{R}^{n\times m}$, $\mathbb{C}$, $\mathbb{C}_+$, $\mathbb{C}^{n\times m}$, and $\mathbb{Z}_+$ are used to denote the sets of real numbers, nonnegative real numbers, $n \times m$ real matrices, complex numbers, complex numbers with nonnegative real part, $n\times m$ complex matrices, and nonnegative integers, respectively. Given a matrix M, the transposition and the conjugate transposition of M are denoted by M' and M^*, respectively. The spectrum of M is denoted as $\mathrm{eig}(M)$. The notation $M > 0$ ("$\ge$") is used to denote positive definiteness (positive semi-definiteness).

Symbol l_2^m denotes the space of $\mathbb{C}^m$-valued, square summable functions defined on time interval $[0, \infty)$. In this manuscript we consider the usual norm and inner product functions for l_2 space, which are denoted as $\|\cdot\|_{l_2}$ and $\langle\cdot,\cdot\rangle_{l_2}$, respectively. Likewise, given an operator $\mathcal{H}$ on l_2 space, the l_2-induced norm of $\mathcal{H}$ is denoted by $\|\mathcal{H}\|_{l_2}$ A bounded linear-time-invariant (LTI) self-adjoint operator Π on the l_2 space

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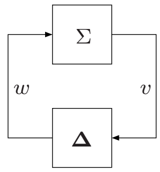

Fig. 1. Feedback interconnection of Σ and $\boldsymbol{\Delta}(A, \theta)$.

defines a quadratic form on l_2:

$$\begin{aligned}\sigma_\Pi(v, w) &:= \left\langle \begin{bmatrix} v \\ w \end{bmatrix}, \Pi \begin{bmatrix} v \\ w \end{bmatrix} \right\rangle_{l_2} \\ &= \sum_{k=0}^{\infty} \begin{bmatrix} v[k] \\ w[k] \end{bmatrix}' \left(\Pi \begin{bmatrix} v \\ w \end{bmatrix} \right) [k] \\ &= \int_{-\pi}^{\pi} \begin{bmatrix} \hat{v}(j\omega) \\ \hat{w}(j\omega) \end{bmatrix}^* \Pi(\omega) \begin{bmatrix} \hat{v}(j\omega) \\ \hat{w}(j\omega) \end{bmatrix} d\omega\end{aligned}$$

where $\hat{v}$ and $\hat{w}$ are Fourier transforms of v and w, respectively. The operator Π is referred to as the multiplier of the quadratic form σ_Π. The multiplier Π is often block partitioned into the form

$$\begin{bmatrix} \Pi_{11} & \Pi_{12} \\ \Pi_{12}^* & \Pi_{22} \end{bmatrix}$$

where the dimensions of Π_{ij} are consistent with those of v and w.

Given an operator $\mathcal{H}$ and a quadratic form $\sigma_\Pi(v, w)$ defined on l_2 space, we said that $\mathcal{H}$ satisfies the integral quadratic constraint defined by σ_Π, or more often "$\mathcal{H}$ satisfies the integral quadratic constraint (IQC) defined by Π" to emphasize the multiplier involved, if $\sigma_\Pi(v, \mathcal{H}(v)) \geq 0$ for all $v \in l_2$. $\mathcal{H}$ is called "positive real" if it satisfies the integral quadratic constraint defined by $\sigma_p(v, \mathcal{H}(v)) := \langle v, \mathcal{H}(v) \rangle_{l_2}$.

II. Problem Formulation

Consider the aperiodic sampled-data system (1). Let $x[k] := x(\tau[k])$. We have

$$x[k+1] = \Phi(\tau[k+1] - \tau[k])x[k], \tag{2}$$

where

$$\Phi(\tau) = \mathrm{e}^{A\tau} + \left(\int_0^\tau \mathrm{e}^{A(\tau-\eta)} \mathrm{d}\eta \right) BF. \tag{3}$$

Let $\tau[k+1] - \tau[k] = h_0 + \theta[k]$, where $h_0 \in [\underline{\theta}, \bar{\theta}]$. In some applications it is natural to assume $\underline{\theta} = 0$. In the sequel we will occasionally make this assumption.

One can verify that (2) can also be expressed as

$$x[k+1] = (\Phi(h_0) + \Delta(\theta[k])\Psi(h_0))\, x[k], \tag{4}$$

where

$$\Psi(h) = A\Phi(h) + BF, \quad \Delta(\theta) := \int_0^\theta \mathrm{e}^{A\eta} \mathrm{d}\eta. \tag{5}$$

System (4) can be equivalently represented as a feedback interconnection of an LTI discrete-time system Σ:

$$\Sigma[z] := \Psi(h_0)\, (zI - \Phi(h_0))^{-1} \tag{6}$$

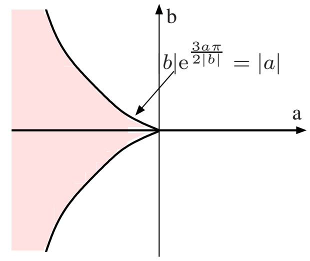

Fig. 2. Illustration of the set $\mathcal{P}$; the set includes the real axis (axis a) and the pink area.

and the time-varying multiplication operator $\boldsymbol{\Delta}(A, \theta) : v \mapsto w$, $w[k] = \Delta(\theta[k])v[k]$. Figure 1 illustrates the feedback system.

In [14], it is shown that l_2 stability of the $(\Sigma, \boldsymbol{\Delta})$ feedback system shown in Figure 1 implies exponential stability of the aperiodic sampled-data system (1). A small gain condition, $\|\Sigma\|_{l_2} \cdot \|\boldsymbol{\Delta}(A, \theta)\|_{l_2} < 1$, is proposed to verify l_2 stability of the $(\Sigma, \boldsymbol{\Delta})$ feedback system. Furthermore, an estimate of $\|\boldsymbol{\Delta}(A, \theta)\|_{l_2}$ is given and a procedure is provided for checking exponential stability of (1).

The goal of this manuscript is aimed at refining the stability condition proposed in [14]. More specifically, we are interested in identifying the conditions under which $\boldsymbol{\Delta}(A, \theta)$ is positive real. By definition, positive realness of $\boldsymbol{\Delta}(A, \theta)$ implies that $\boldsymbol{\Delta}(A, \theta)$ satisfies certain IQCs. With these integral quadratic constraints identified, the general IQC stability theory can be applied to derive stability conditions for the $(\Sigma, \boldsymbol{\Delta})$ feedback system.

III. Characterizations of the Spectrum of $\Delta(\theta)$

In this section, we characterize the spectrum of the matrix $\Delta(\theta)$ defined in (5), and identify the conditions under which the spectrum of $\Delta(\theta)$ belongs to $\mathbb{C}_+$. The following lemma shows the relationship between the spectrums of matrices A and $\Delta(\theta)$.

Lemma 1: $\mathrm{eig}\,(\Delta(\theta)) = \mathcal{S}_1 \cup \mathcal{S}_2$, where

$$\mathcal{S}_1 = \left\{ \frac{\mathrm{e}^{\lambda\theta} - 1}{\lambda} : \lambda \in \mathrm{eig}(A), \lambda \neq 0 \right\}, \tag{7}$$

and

$$\mathcal{S}_2 = \begin{cases} \{\theta\} & \text{if } 0 \in \mathrm{eig}(A) \\ \emptyset & \text{otherwise} \end{cases} \tag{8}$$

Proof: Let $J_p(\lambda)$ denote the $p \times p$ Jordan block with λ being the value of its diagonal elements. Consider the Jordan canonical form of A

$$S^{-1}AS = \mathcal{J} = \begin{bmatrix} J_{n_1}(\lambda_1) & & \\ & \ddots & \\ & & J_{n_q}(\lambda_q) \end{bmatrix},$$

where $n_i \geq 0$, $i = 1, \cdots, q$, and $n_1 + \cdots + n_q = n$. Without loss of generality, we assume $\lambda_1 = 0$ and $\lambda_i \neq 0$, $i =$

$2, \cdots, q$. Clearly,

$$S^{-1}\Delta(\theta)S = \int_0^\theta e^{S^{-1}AS\eta} d\eta = \int_0^\theta e^{\mathcal{J}\eta} d\eta$$
$$= \begin{bmatrix} \int_0^\theta e^{J_{n_1}(\lambda_1)\eta} d\eta & & \\ & \ddots & \\ & & \int_0^\theta e^{J_{n_q}(\lambda_q)\eta} d\eta \end{bmatrix}.$$

Furthermore, each $\int_0^\theta e^{J_{n_i}(\lambda_i)\eta} d\eta$ is a upper triangular matrix, whose diagonal entries are equal to θ if $i = 1$, and equal to $(e^{\lambda_i \theta} - 1)/\lambda_i$ if $2 \le i \le q$. Finally, since the spectrum of a matrix is invariant under similarity transformations, we see that the spectrum of $\Delta(\theta)$ consists of θ, which corresponds to the zero eigenvalue of A, and $(e^{\lambda_i \theta} - 1)/\lambda_i$, which corresponds to the eigenvalue λ_i of A, $i = 2, \cdots, q$. This concludes the proof.

Now consider the following subset of $\mathbb{C}$

$$\mathcal{P} := \{a + jb : |b| e^{\frac{3a\pi}{2|b|}} \le |a|, a < 0\} \cup \mathbb{R}_+ \tag{9}$$

An illustration of the set can be found in Figure 2. The conditions under which $\text{eig}(\Delta(\theta)) \in \mathbb{C}_+$ are given in the following propositions.

Proposition 1: $\text{eig}(\Delta(\theta)) \subset \mathbb{C}_+$ for all $\theta \in \mathbb{R}_+$ if and only if $\text{eig}(A) \subset \mathcal{P}$.

Proof: Let $\lambda := a + jb \in \text{eig}(A)$. Following Lemma 7 we know that the corresponding eigenvalue of $\Delta(\theta)$ is either equal to θ if $\lambda = 0$, or equal to $\frac{e^{(a+bi)\theta} - 1}{a+bi}$ if $\lambda \neq 0$. In the latter case, the real part of the complex eigenvalue is equal to

$$\mathcal{R}(\theta) = \frac{1}{a^2 + b^2} \left(a(e^{a\theta} \cos b\theta - 1) + b e^{a\theta} \sin b\theta \right).$$

Note that $\mathcal{R}(0) = 0$ and $\frac{d\mathcal{R}}{d\theta}(\theta) = e^{a\theta} \cos b\theta$. Hence, if $b = 0$, $\frac{d\mathcal{R}}{d\theta}(\theta) \ge 0$ for all $\theta \in \mathbb{R}_+$ which in turn implies that $\mathcal{R}(\theta) \ge 0$ for all $\theta \in \mathbb{R}_+$. This proves that $\text{eig}(\Delta(\theta)) \in \mathbb{C}_+$ if $\text{eig}(A) \in \mathbb{R}$.

Suppose $b \neq 0$. It is then easy to verify that $\mathcal{R}(\theta)$ has local minimums at $\theta_k = \frac{3+4k\pi}{2b}$ if $b > 0$ or $\theta_k = \frac{-(3+4k\pi)}{2b}$ if $b < 0$, $k \in \mathbb{Z}_+$. The corresponding local minimums are

$$\mathcal{R}(\theta_k) = -a + b e^{a\theta_k} \sin b\theta_k = -a - |b| e^{\frac{3a\pi}{2|b|}} e^{\frac{2a\pi}{|b|}k}$$

Clearly, if $a > 0$, $\mathcal{R}(\theta_k) < 0$ for all $k \in \mathbb{Z}_+$. This proves that, if $a > 0$ and $b \neq 0$, then the real part of $\frac{e^{(a+bi)\theta} - 1}{a+bi}$ is negative for some $\theta \in \mathbb{R}_+$.

Finally, suppose $b \neq 0$ and $a < 0$. Then clearly

$$\min_{k \in \mathbb{Z}_+} \mathcal{R}(\theta_k) = \mathcal{R}(\theta_0) = |a| - |b| e^{\frac{3a\pi}{2|b|}}$$

and therefore $\mathcal{R}(\theta) \ge 0$ for all $\theta \in \mathbb{R}_+$ if and only if $|b| e^{\frac{3a\pi}{2|b|}} \le |a|$. Combining all cases together, we conclude that $\mathcal{R}(\theta) \ge 0$ for all $\theta \in \mathbb{R}_+$ if and only if $\lambda \in \mathcal{P}$. This concludes the proof.

Proposition 2: Suppose $\text{eig}(A) \cap \mathcal{P}^c \neq \emptyset$, where $\mathcal{P}^c = \mathbb{C} \backslash \mathcal{P}$. Then $\text{eig}(\Delta(\theta)) \subset \mathbb{C}_+$ for all $\theta \in [0, \theta_0]$, where

$$\theta_0 = \min_{\lambda \in \text{eig}(A) \cap \mathcal{P}^c} \breve{\theta}(\text{Re}\lambda, \text{Im}\lambda) \tag{10}$$

and $\breve{\theta}(a, b)$ is the greatest lower bound of the following set

$$\left\{ \theta : \theta > 0, \frac{1}{a^2 + b^2} \left(a(e^{a\theta} \cos b\theta - 1) + b e^{a\theta} \sin b\theta \right) < 0 \right\}.$$

Proof: Let $\lambda := a + jb \in \text{eig}(A) \cap \mathcal{P}^c$. Denote the eigenvalue of $\Delta(\theta)$ corresponding to λ as $\lambda_\Delta(\theta)$. By Proposition 1, we know that

$$\text{Re}\lambda_\Delta(\theta) = \frac{1}{a^2 + b^2} \left(a(e^{a\theta} \cos b\theta - 1) + b e^{a\theta} \sin b\theta \right)$$

and $\text{Re}\lambda_\Delta(\theta) < 0$ for some positive θ. Hence $\breve{\theta}(\text{Re}\lambda, \text{Im}\lambda)$ is well defined. Furthermore, from the proof of Proposition 1, we also know that $\text{Re}\lambda_\Delta(0) = 0$ and the derivative of $\text{Re}\lambda_\Delta(\theta)$,

$$\frac{d\text{Re}\lambda_\Delta}{d\theta}(\theta) = e^{a\theta} \cos b\theta,$$

which is nonnegative for $\theta \in \left[0, \frac{\pi}{2|b|}\right]$. Thus, by continuity we know that $\text{Re}\lambda_\Delta(\theta) \ge 0$ for $\theta \in [0, \breve{\theta}(\text{Re}\lambda, \text{Im}\lambda)]$. Having established this, it is now clear that $\text{eig}(\Delta(\theta)) \subset \mathbb{C}_+$ for all $\theta \in [0, \theta_0]$, where

$$\theta_0 = \min_{\lambda \in \text{eig}(A) \cap \mathcal{P}^c} \breve{\theta}(\text{Re}\lambda, \text{Im}\lambda).$$

A conservative estimate of θ_0 is given in the following proposition.

Proposition 3: Suppose $\text{eig}(A) \cap \mathcal{P}^c \neq \emptyset$, where $\mathcal{P}^c = \mathbb{C} \backslash \mathcal{P}$. Then $\text{eig}(\Delta(\theta)) \subset \mathbb{C}_+$ for all $\theta \in \left[0, \frac{\pi}{2b_0}\right]$, where $b_0 = \max_{\lambda \in \text{eig}(A) \cap \mathcal{P}^c} |\text{Im}\lambda|$.

Proof: Following the arguments in the proof of Proposition 2, we know that $\text{Re}\lambda_\Delta(0) = 0$ and

$$\frac{d\text{Re}\lambda_\Delta}{d\theta}(\theta) \ge 0, \ \forall \theta \in \left[0, \frac{\pi}{2|b|}\right].$$

Thus, $\text{Re}\lambda_\Delta(0) \ge 0$ for all $\theta \in \left[0, \frac{\pi}{2|b|}\right]$. This concludes the proof.

Proposition 2 provides a mean to estimate an upper bound of θ for which $\Delta(\theta)$ is passive. The algorithm involves finding all eigenvalues of A which locate in $\mathcal{P}^c$ and the corresponding $\breve{\theta}$. For a $\lambda \in \text{eig}(A) \cap \mathcal{P}^c$, a good approximation of $\breve{\theta}(\text{Re}\lambda, \text{Im}\lambda)$ can be easily found by sweeping θ over a dense grid of $\left[\frac{\pi}{2|b|}, \frac{3\pi}{2|b|}\right]$. Actually, numerical experiments indicate that $\breve{\theta}(\text{Re}\lambda, \text{Im}\lambda)$ is usually just slightly larger than $\frac{\pi}{2|b|}$.

IV. Positive Realness of $\mathbf{\Delta}(A, \theta)$ and the Corresponding Integral Quadratic Constraints

In this section, we consider the multiplication operator $\mathbf{\Delta}(A, \theta) : v \mapsto w$ defined in Section II

$$w[k] = (\mathbf{\Delta}(A, \theta) v)[k] = \Delta(\theta[k]) v[k] \tag{11}$$

and show under what conditions the operator is positive real. Furthermore, based on positive real property, characterizations in the form of integral quadratic constraints are identified for $\mathbf{\Delta}(A,\theta)$. Note that if $|\theta[k]|$ is upper bounded by a constant for all k, then $\mathbf{\Delta}(A,\theta)$ is a bounded operator on l_2.

Theorem 1: Consider the multiplication operator $\mathbf{\Delta}(A,\theta)$ defined in (11). Suppose A can be diagonalized by the similarity matrix T, such that

$$T^{-1}AT = \begin{bmatrix} \lambda_1 I_{n_1} & & \\ & \ddots & \\ & & \lambda_q I_{n_q} \end{bmatrix}, \tag{12}$$

where n_i, $i=1,\cdots,q$, are positive integers and $n_1+\cdots+n_q=n$. If $0\le\theta[k]\le\theta_0$ for all k, where θ_0 is defined as in (10), then $T^{-1}\mathbf{\Delta}(A,\theta)T$ is positive real and $\mathbf{\Delta}(A,\theta)$ satisfies integral quadratic constraint defined by the multiplier

$$\Pi_1 = \begin{bmatrix} T^{-1} & 0 \\ 0 & T^{-1} \end{bmatrix}^* \begin{bmatrix} 0 & X \\ X^* & 0 \end{bmatrix} \begin{bmatrix} T^{-1} & 0 \\ 0 & T^{-1} \end{bmatrix} \tag{13}$$

where X has the form

$$X = \begin{bmatrix} X_1 & & \\ & \ddots & \\ & & X_q \end{bmatrix},$$

and $X_i = X_i^* \in \mathbb{C}^{n_i\times n_i}$, $i=1,\cdots,q$, is any positive semidefinite matrix.

Proof: Given $v\in l_2$, let $w=\mathbf{\Delta}(A,\theta)v$, $v_T=T^{-1}v$, and $w_T=T^{-1}w$. One can verify that

$$w_T[k] = T^{-1}\mathbf{\Delta}(A,\theta)Tv_T[k] = \left(\int_0^{\theta[k]} \mathrm{e}^{T^{-1}AT\eta}\mathrm{d}\eta\right) v_T[k]$$
$$= \begin{bmatrix} \delta_1[k] I_{n_1} & & \\ & \ddots & \\ & & \delta_q[k] I_{n_q} \end{bmatrix} v_T[k] = \begin{bmatrix} \delta_1[k]v_{T,1}[k] \\ \vdots \\ \delta_q[k]v_{T,q}[k] \end{bmatrix}, \tag{14}$$

where $\delta_i[k] = \frac{\mathrm{e}^{\lambda\theta[k]}-1}{\lambda_i}$ if $\lambda_i\neq 0$, or $\delta_i[k]=\theta[k]$ if $\lambda_i=0$. Furthermore, since $0\le\theta[k]\le\theta_0$ for all k, $\mathrm{Re}\delta_i[k]\ge 0$ for all $i=1,\cdots,q$ and for all k. Therefore, let $w_{T,i}[k]=\delta_i[k]v_{T,i}[k]$ and we have

$$(Xw_T[k])^* v_T[k] + v_T[k]^*(Xw_T[k]) = \sum_{i=1}^{q} (2\mathrm{Re}\delta_i[k])\, v_{T,i}[k]^* X_i v_{T,i}[k] \ge 0,\ \forall\ k.$$

This proves that

$$\left\langle \begin{bmatrix} v_T \\ w_T \end{bmatrix}, \begin{bmatrix} 0 & X \\ X^* & 0 \end{bmatrix} \begin{bmatrix} v_T \\ w_T \end{bmatrix} \right\rangle_{l_2} \ge 0,$$

which in turn implies that $\mathbf{\Delta}(A,\theta)$ satisfies IQC defined by Π_1. Finally, by taking X as the identify matrix, we prove that $T^{-1}\mathbf{\Delta}(A,\theta)T$ is positive real.

Theorem 2: Consider the multiplication operator $\mathbf{\Delta}(A,\theta)$ defined in (11). Suppose A satisfies (12) and all λ_i, $i=1,\cdots,q$, are real. Also suppose there exists a constant $\bar\theta$ such that $0\le\theta[k]\le\bar\theta$ for all k. Then $T^{-1}\mathbf{\Delta}(A,\theta)T$ is positive real and $\mathbf{\Delta}(A,\theta)$ satisfies integral quadratic constraint defined by the multiplier

$$\Pi_2 = \begin{bmatrix} T^{-1} & 0 \\ 0 & T^{-1} \end{bmatrix}^* \begin{bmatrix} 0 & X \\ X^* & 0 \end{bmatrix} \begin{bmatrix} T^{-1} & 0 \\ 0 & T^{-1} \end{bmatrix} \tag{15}$$

where X has the form

$$X = \begin{bmatrix} X_1\mathrm{e}^{j\psi_1} & & \\ & \ddots & \\ & & X_q\mathrm{e}^{j\psi_q} \end{bmatrix},$$

$X_i = X_i^* \in \mathbb{C}^{n_i\times n_i}$, $i=1,\cdots,q$, is any positive semidefinite matrix, and $\psi_i\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$.

Proof: The proof is similar to that of Theorem 1. The crucial difference here is that, since all eigenvalues of A are real, $\delta_i[k]$, $i=1,\cdots,q$, are real and positive for all k. Thus,

$$(Xw_T[k])^* v_T[k] + v_T[k]^*(Xw_T[k]) = \sum_{i=1}^{q} (2\delta_i[k]\cos\psi_i)\, v_{T,i}[k]^* X_i v_{T,i}[k] \ge 0,\ \forall\ k.$$

This implies that $\mathbf{\Delta}(A,\theta)$ satisfies IQC defined by Π_2. Finally, taking X as the identify matrix (all $X_i=I$ and all $\psi_i=0$) shows that $T^{-1}\mathbf{\Delta}(A,\theta)T$ is positive real.

Theorem 3: Consider the multiplication operator $\mathbf{\Delta}(A,\theta)$ defined in (11). Suppose A satisfies (12) and all λ_i, $i=1,\cdots,q$, are real. Also suppose all $\theta[k]$ have the same value. Then $\mathbf{\Delta}(A,\theta)$ satisfies integral quadratic constraint defined by the multiplier

$$\Pi_3 = \begin{bmatrix} T^{-1} & 0 \\ 0 & T^{-1} \end{bmatrix}^* \begin{bmatrix} 0 & X \\ X^* & 0 \end{bmatrix} \begin{bmatrix} T^{-1} & 0 \\ 0 & T^{-1} \end{bmatrix} \tag{16}$$

where $X:[-\pi,\pi]\to\mathbb{C}^{n\times n}$ has the form

$$X(\omega) = \begin{bmatrix} X_1(\omega)\mathrm{e}^{j\psi_1(\omega)} & & \\ & \ddots & \\ & & X_q(\omega)\mathrm{e}^{j\psi_q(\omega)} \end{bmatrix},$$

and X_i, $i=1,\cdots,q$, is a $\mathbb{C}^{n_i\times n_i}$ matrix-valued function which satisfies $X_i(\omega)=X_i(\omega)^*\ge 0$ for all $\omega\in[-\pi,\pi]$. The function $\psi_i:[-\pi,\pi]\to\mathbb{R}$, $i=1,\cdots,q$, satisfies $\psi_i(\omega)\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$ for all $\omega\in[-\pi,\pi]$.

Proof: The crucial point here is that all $\theta[k]$ have the same value. This implies that each $\delta_i[k]$, as defined in (14), has the same value for all k. In other words, the operator $\mathbf{\Delta}(A,\theta)$ is time invariant. Let $v\in l_2$, $w=\mathbf{\Delta}(A,\theta)v$, $v_T=T^{-1}v$, $w_T=T^{-1}w$, and $\hat v$, $\hat w$, $\hat v_T$, $\hat w_T$ are Fourier transforms of v, w, v_T, and w_T, respectively. Since $\mathbf{\Delta}(A,\theta)$ is time invariant, we have

$$\hat w_T = \mathbf{\Delta}(A,\theta)\hat v_T,\ \Leftrightarrow\ \begin{bmatrix} \hat w_{T,1} \\ \vdots \\ \hat w_{T,q} \end{bmatrix} = \begin{bmatrix} \delta_1\hat v_{T,1} \\ \vdots \\ \delta_q\hat v_{T,q} \end{bmatrix}.$$

Moreover, note that

$$\begin{aligned}
&\left\langle \begin{bmatrix} v \\ w \end{bmatrix}, \Pi_3 \begin{bmatrix} v \\ w \end{bmatrix} \right\rangle_{l_2} \\
&= \int_{-\pi}^{\pi} \begin{bmatrix} \hat{v}_T(\mathrm{e}^{j\omega}) \\ \hat{w}_T(\mathrm{e}^{j\omega}) \end{bmatrix}^* \begin{bmatrix} 0 & X(\omega) \\ X^*(\omega) & 0 \end{bmatrix} \begin{bmatrix} \hat{v}_T(\mathrm{e}^{j\omega}) \\ \hat{w}_T(\mathrm{e}^{j\omega}) \end{bmatrix} \mathrm{d}\omega \\
&= \sum_{i=1}^{q} \int_{-\pi}^{\pi} (2\delta_i \cos\psi_i(\omega)) \hat{v}_{T,i}(\mathrm{e}^{j\omega})^* X_i(\omega) \hat{v}_{T,i}(\mathrm{e}^{j\omega}) \mathrm{d}\omega
\end{aligned} \tag{17}$$

Since the eigenvalues of A are all real, by Lemma 1 all δ_i are larger than or equal to zero. Furthermore, $X_i(\omega) = X_i(\omega)^* \geq 0$ and $\psi_i(\omega) \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ for all i and for all $\omega \in [-\pi, \pi]$. Therefore, the integrant of integral (17) is greater than or equal to zero for all $\omega \in [-\pi, \pi]$, which in turn implies that

$$\left\langle \begin{bmatrix} v \\ w \end{bmatrix}, \Pi_3 \begin{bmatrix} v \\ w \end{bmatrix} \right\rangle_{l_2} \geq 0$$

and $\mathbf{\Delta}(A, \theta)$ satisfies IQC defined by Π_3. Finally, again, taking X as the identify matrix (all $X_i(\omega) = I$ and all $\psi_i(\omega) = 0 \ \forall \omega$) shows that $T^{-1}\mathbf{\Delta}(A, \theta)T$ is positive real.

To end this section, we provide for $\mathbf{\Delta}(A, \theta)$ an integral quadratic constraint characterization that comes from the upper bound of $\|\mathbf{\Delta}(A, \theta)\|_{l_2}$, and the general IQC theory by which one can use to derive stability conditions for the $(\Sigma, \mathbf{\Delta})$ feedback system.

Theorem 4: Suppose $0 < \theta[k] \leq \bar{\theta}$. Then $\mathbf{\Delta}(A, \theta)$ satisfies integral quadratic constraint defined by the multiplier

$$\Pi_4 = \begin{bmatrix} \gamma^2 \mathrm{e}^{-A^*\xi} \mathrm{e}^{-A\xi} & 0 \\ 0 & -\mathrm{e}^{-A^*\xi} \mathrm{e}^{-A\xi} \end{bmatrix} \tag{18}$$

where ξ is any real number and

$$\gamma = \int_0^{\bar{\theta}} \mathrm{e}^{\mu(A)\eta} \mathrm{d}\eta, \quad \mu(A) := \lambda_{\max}\left(\frac{A + A^*}{2}\right).$$

Proof: See [14].

Theorem 5 (IQC): Consider the $(\Sigma, \mathbf{\Delta})$ feedback system shown in Figure 1. Assume Σ is stable and there exists a constant $\bar{\theta}$ such that $0 < \theta[k] \leq \bar{\theta}$. Suppose

1) $\mathbf{\Delta}(A, \theta)$ satisfies the IQC defined by Π;
2) $\Pi_{11} \geq 0$ and $-\Pi_{22} \geq 0$;
3) there exists $\epsilon > 0$ such that

$$\begin{bmatrix} \Sigma(\mathrm{e}^{j\omega}) \\ I \end{bmatrix}^* \Pi(\mathrm{e}^{j\omega}) \begin{bmatrix} \Sigma(\mathrm{e}^{j\omega}) \\ I \end{bmatrix} \leq -\epsilon I, \ \forall\ \omega \in [-\pi, \pi].$$

Then the feedback interconnection of Σ and $\mathbf{\Delta}$ is stable.

Proof: The theorem presented above is completely analog to that of [15], where the continuous-time systems were considered.

V. Examples

In this section, we show how helpful the positive real property is in reducing conservatism of stability analysis by two examples.

Example 1: Let $A = 0$, $B = 1$, $F = -\frac{\varepsilon}{h_0}$, where $0 < \varepsilon \ll 1$. Let $\tau[k+1] - \tau[k] = h_0 + \theta[k]$, where $0 \leq \theta[k] \leq \bar{\theta}$. With these values, we have

$$\begin{aligned}
\Phi(h_0) &= \mathrm{e}^{Ah_0} + \left(\int_0^{h_0} \mathrm{e}^{A\eta} \mathrm{d}\eta\right) BF = 1 - \varepsilon \\
\Psi(h_0) &= A\Phi(h_0) + BF = \mathrm{e}^{Ah_0}(A + BF) = -\frac{\varepsilon}{h} \\
\Delta(\theta_k) &= \theta_k
\end{aligned}$$

and the discrete-time system Σ has the form

$$\Sigma[z] = -\frac{\varepsilon}{h_0} \frac{1}{z - 1 + \varepsilon}.$$

Note that

$$\begin{aligned}
&\Sigma[\mathrm{e}^{\mathrm{j}0}] = -\frac{1}{h_0}, \\
&\Sigma[\mathrm{e}^{\mathrm{j}\phi_0}] = \mathrm{j}\frac{\varepsilon}{h_0} \frac{1}{\sin\phi_0}, \quad \cos\phi_0 = 1 - \varepsilon, \quad 0 < \phi_0 < \pi, \\
&\Sigma[\mathrm{e}^{\mathrm{j}\pi}] = \frac{\varepsilon}{h_0} \frac{1}{2 - \varepsilon}.
\end{aligned}$$

Thus we have

$$\begin{aligned}
&\mathrm{Re}\Sigma[\mathrm{e}^{\mathrm{j}\omega}] < 0,\ |\Sigma[\mathrm{e}^{\mathrm{j}\omega}]| > \frac{\varepsilon}{h_0} \frac{1}{\sin\phi_0},\ \forall\omega \text{ s.t. } \cos\omega > 1 - \varepsilon, \\
&\mathrm{Re}\Sigma[\mathrm{e}^{\mathrm{j}\omega}] \geq 0,\ |\Sigma[\mathrm{e}^{\mathrm{j}\omega}]| \leq \frac{\varepsilon}{h_0} \frac{1}{\sin\phi_0},\ \forall\omega \text{ s.t. } \cos\omega \leq 1 - \varepsilon.
\end{aligned}$$

Hence by the small gain condition proposed in [14], which is equivalent by applying Theorem 5 with IQC defined by Π_4, one obtain the upper bound $\bar{\theta}$ as

$$\|\Sigma\|_{l_2}^{-1} = h_0.$$

On the other hand, with the positive real property of $\mathbf{\Delta}(A, \theta)$, we may apply Theorem 5 with IQC defined by $\Pi_1 + \Pi_4$ (with $T = I$ in Π_1). In this case, the stability condition becomes: there exists $x \geq 0$ and $\epsilon > 0$ such that

$$\bar{\theta}^2 |\Sigma[\mathrm{e}^{\mathrm{j}\omega}]|^2 - 1 + 2x\mathrm{Re}\Sigma[\mathrm{e}^{\mathrm{j}\omega}] \leq -\epsilon,\ \forall\ \omega \in [-\pi, \pi].$$

It can be readily verified that, for the above stability condition to hold, the upper bound $\bar{\theta}$ must satisfy the constraint

$$\bar{\theta}^2 \left(\frac{\varepsilon}{h_0} \frac{1}{\sin\phi_0}\right)^2 < 1,\ \Leftrightarrow\ \bar{\theta} < \frac{h_0 \sin\phi_0}{\varepsilon} = h_0\sqrt{\frac{2 - \varepsilon}{\varepsilon}}.$$

Since $\varepsilon \ll 1$, clearly, one obtain a much better bound for $\theta[k]$ if the positive real property of $\mathbf{\Delta}(A, \theta)$ is taken into account.

Example 2: Consider the following third order unstable system:

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1010 & 99 & -10 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

with the feedback gain

$$F = \begin{bmatrix} -1016.9 & -116.1 & 3.2 \end{bmatrix}$$

and the nominal sampling period $h_0 = 0.01$ assuming that $\underline{\theta} = 0$. One can verify that

$$\|\Psi(h_0)(zI - \Phi(h_0))^{-1}\|_{l_2} = 517.49$$

and hence the gain information of $\mathbf{\Delta}(A, \theta)$ guarantees the stability if

$$0 \leq \theta[k] \leq 0.0013, \ \forall k,$$

but not for $\theta[k] \geq 0.0014$. In other words, the small gain condition

$$\|\mathbf{\Delta}(A, \theta)\|_{l_2} \cdot \|\Psi(h_0)(zI - \Phi(h_0))^{-1}\|_{l_2} < 1$$

holds for $\theta \in [0, 0.0013]$, and fails to hold if $\theta \geq 0.0014$.

On the other hand, applying Theorem 5 with IQC defined by Π:

$$\Pi := \begin{bmatrix} \gamma^2 I & 0 \\ 0 & -I \end{bmatrix} + \zeta \begin{bmatrix} 0 & T^{-*}T^{-1} \\ T^{-*}T^{-1} & 0 \end{bmatrix},$$

where T is the similarity transformation matrix which diagonalizes A and ζ is a positive real parameter, one obtains that the stability is guaranteed for

$$\|\mathbf{\Delta}(A, \theta)\|_{l_2} < 0.0029,$$

which is satisfied if

$$0 \leq \theta[k] \leq 0.00179.$$

In other words, by utilizing the gain information together with the positive real property of $\mathbf{\Delta}(A, \theta)$, the stability margin is improved by more than 37%.

VI. Concluding Remarks

Robustness analysis of aperiodic sampled-data systems against variation of sampling intervals is considered. The methodology adopted here follows that of [14]. The key idea is to transform the aperiodic sampled-data system into a feedback interconnection of an LTI discrete-time system and an uncertain multiplication operator; conditions for stability are then obtained by applying IQC theory to the transformed system. The new technical contribution of this manuscript is to identify the conditions under which the uncertain multiplication operator is positive real, and to derive the corresponding integral quadratic constraint characterizations for the operator. Two examples are given to illustrate that the new IQCs indeed can reduce conservatism of stability analysis. Detailed comparison to other recent articles on this subject, e.g., [16], [17], remains open and is a subject for future research.

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