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Optimization of Multiagent Systems with Increasing State Dimensions: Hybrid LQ Approach

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Abstract— **In this paper, we study a specific optimal control problem associated with a multiagent dynamic system. The interest is placed on minimization of the tracking error in the multiagent leader-follower model. We replace this problem by a specific hybrid optimal control problem. In particular, we consider control systems with monotonically increasing dimensions of the state vector. The change of the state dimension has the character of a jump and is modeled by an impulsive hybrid system. The paper proposes an effective computational procedure for the above optimal tracking control problem in a multiagent setting. The theoretical and numerical approaches obtained in this contribution are tested on a practically motivated example.**

I. INTRODUCTION

Traditional control theory usually deals with system dynamics of a fixed dimension (see e. g., [24], [14], [10]). Lately, there has been an increasing interest in practically relevant interconnected systems that can be formalized as dynamical models with variable state dimensions. In fact, the evolution of the dimension on complex engineering models is a general effect that can appear in many applications (see e.g., [17], [26], [27]). These application range from robot dynamics [17], [18], [22], to networked control [16], [18], [22], and control of autonomous vehicles [15]. One of the possible formal approaches to take these impulsive discrete event dynamics into consideration is related to a suitable extension of the original state space. Moreover, an extension of the system dimension has consequences for the resulting optimal control problem and the generation of a effective, numerical solution procedures constitute a new challenging task. Note that a comparatively small number of papers are devoted to the control design problem of systems with variable (time-dependent) dimensions. We refer to [17], [26], [27] for some partial results.

Motivated by the above-mentioned tracking (path-following) optimization problem of a variable dimension we represent the given multiagent systems as a hybrid system. This interpretation gives rise to a constructive formal description of the discrete event effects caused by changes in the state dimension. The initial multiagent model is replaced by an auxiliary hybrid system that not only has the discrete-continuous parts, but also contains subsystems of different state dimensions. Note that the impulsive character of the changes of dimension contributes to the general complexity of the optimal control design for the multiagent system under consideration. We concentrate on dynamic systems with monotonically increasing dimensions. The last situation corresponds to a modeling framework that describe a possible behavior of some groups (networks) of interconnected intelligent machines. The dimension of a robots-network will be growing if a new agent is associated with the network at any time instant t (see [18]). Note that an example of this specific behavior is a vacuum cleaner type problem where there are some agents doing certain tasks in a given area and we need to collect them all in an ordered manner, i.e. the agent are sparse in an area therefore we can send an agent to collect them following a planned path. An additional example of a system with variable state dimension is provided by a mechanical "trampoline" model that describes the periodical trampoline jumping [26].

In this paper, we apply the hybrid LQ-type technique to the optimal control design of a class of multiagent systems with increasing state dimensions. Hybrid optimal control processes and in particular, the hybrid optimal LQ dynamics with a fixed dimension have been extensively studied over the past few years (see e.g., [2], [4], [9], [7], [8], [12], [20], [21], [23], [25], [29]). Note that these investigations are recently extended by the hybrid optimal control problem methodology in the presence of the additional impulsive effects [1], [3], [5]. Some alternative variational techniques related to the optimization of complex multidimensional systems have been proposed in [17], [26]. Applications of the hybrid LQ methodology is associated with a specific hybrid Riccati-type formalism that describes the optimal feedback control strategy in the case of the increasing systems dimension. In this context we are also interested in constructive solution procedures for the above optimal control problem.

The paper is organized as follows: Section II deals with a formal description of a specific multiagent dynamic model and contains the main concepts and facts. Section III is devoted to the equivalent representation of the initial leader-followers model in the form of a hybrid dynamic system. We also consider an optimal tracking problem associated with this hybrid model. Section IV proposes a LQ-based control design in the auxiliary hybrid setting for the tracking problem stated in the previous section. In Section V we develop a concrete solution procedure for the hybrid LQ regulator problem in the case of a growing dimensional space. In Section VI we applied the proposed hybrid methodology to the initial multiagent problem. The applicability of the the-

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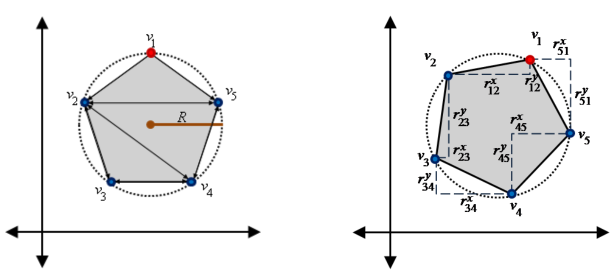
oretical schemes and computational algorithms is illustrated by the numerical simulations of an optimal behavior of the multiagent robots-network. Section VII concludes the paper.

II. A Multiagent Leader-Follower Modelling Framework

Consider a leader-followers model that is composed as a network of N_0 dynamical agents $v_l,\ l = 1, ..., N_0$. The leader (a selected element of this network) is denoted here by v_1. The dynamic behavior of the leader constitutes an optimal strategy that preserve an adequate geometrical configuration of the given followers, while the leader ensures a prescribed smooth path $r_e(\cdot)$. In assure linearity of the system dynamics we assume that the dynamics of the above multiagent system are described by the consensus protocol

$$\begin{aligned} \dot{y}_1(t) &= u(t), \\ \dot{y}_l(t) &= \sum_{j \in \mathcal{Q}_l} (y_j - y_l - r_{lj}) \quad l = 2, \ldots, N_0, \end{aligned} \tag{1}$$

where $\mathcal{Q}_l$ is an index set or neighbors to agent v_l, $y_1(t), y_l(t) \in \mathbb{R}^m$ specifies a position of the agent v_l at $t \in [0, t_f]$, $u(t)$ constitute the control input of the leader.The initial conditions for (1) are given as $y_1(0),\ \ y_l(0)$ for $l = 2, \ldots, N_0$. We refer to [18], [19] for some additional details related to the consensus protocols. By r_{lj}, where $l = 2, ..., N_0,\ j \in \mathcal{Q}_l$, we denote a collection of suitable displacements between agents v_l and v_j that helps ensure preservation of the given formation (see Fig. 1(b)). The initial



(a) Convex configuration of a multi agent system

(b) Distances associated with a configuration

Fig. 1. Geometrical parameters of a leader-followers configuration

formation of the given group of agents (the leader-followers configuration) is determined as follows: every agent is located at the vertex of a regular convex N_0-polygon in a two-dimensional space (as presented in Fig. 1(b)). At times $t_i \in [0, t_f],\ i = 1, ..., L$, the multiagent system detect d_i additional agents, i.e. the distance between the newly selected followers and the leader v_1 is less or equal to a prescribed detecting distance δ. For simplify of the mathematical formulas, we assume in this manuscript $d_i = d,\ i = 1, \ldots, L$; but this assumption does not affect the applicability, in the general case, of the obtained results. The new system now contains some additional agents denoted by $v_{N_0+\nu},\ \nu = 1, ..., d$ and is now characterized by a $N_1 = N_0 + d$-polygonal formation. Note that we need to know the final number of found agents in order to have a system with L switches, note also that in every time interval $[t_{i-1}, t_i)$ the dimension of the system is $N_i = N_{i+1} + d$, for $i = 1, ..., L$. The agents selection process described above can be represented by a dynamic oriented graph $G_{[t_{i-1}, t_i)}$. The initial conditions $y_1(0),\ y_l(0)$ represent the vertices of an initial graph $G_{[t_0, t_1)}$. The directed graph $G_{[t_{i-1}, t_i)}$ is associated with a set $\mathcal{V}(G_{[t_{i-1}, t_i)})$ of nodes and a set $\mathcal{E}(G_{[t_{i-1}, t_i)}) \subset \mathcal{V}(G_{[t_{i-1}, t_i)}) \times \mathcal{V}(G_{[t_{i-1}, t_i)})$ of edges of $G_{[t_{i-1}, t_i)}$. The nodes $v_l, v_j \in \mathcal{V}(G_{[t_{i-1}, t_i)})$ are called neighbors if $(v_l, v_j) \in \mathcal{E}(G_{[t_{i-1}, t_i)})$. Evidently, the set $\mathcal{E}(G_{[t_{i-1}, t_i)})$ is also a dynamic set corresponding to the dynamics of the leader-followers configuration described above. We are now ready to specify concretely the index set $\mathcal{Q}_l$ in (1) introduced above. Let us consider the set of all the neighbors of an agent v_l and put

$$\mathcal{Q}_l(G_{[t_{i-1}, t_i)}) = \{j \in \mathbb{N} | (v_l, v_j) \in \mathcal{E}(G_{[t_{i-1}, t_i)})\}.$$

Note that the dimensions of the above sets are

$$\dim(\mathcal{V}(G_{[t_{i-1}, t_i)})) = N_i,\ \dim(\mathcal{E}(G_{[t_{i-1}, t_i)})) \leq N_i^2,$$
$$\dim(\mathcal{Q}_l(G_{[t_{i-1}, t_i)})) \leq N_i.$$

Fig. 2 illustrates this dynamical behavior of the multiagent system in the case $d = 1$. The $(N_i \times \rho_i)$-incidence matrix

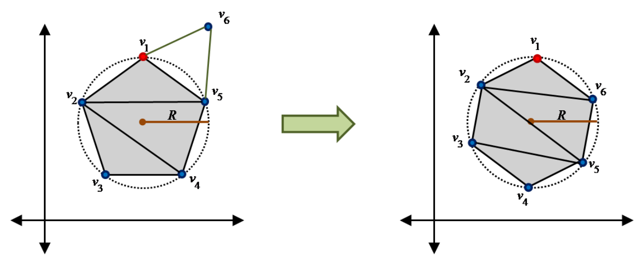


Fig. 2. An additional follower detected

associated with $G_{[t_{i-1}, t_i)}$ is determined as $\mathcal{D}(G_{[t_{i-1}, t_i)}) = [\mathcal{D}_{lj}]_{lj}$, where ρ_i is the number of edges of the Graph $G_{[t_{i-1}, t_i)}$ and

$$\mathcal{D}_{lj} = \begin{cases} \ \ 1 & \text{if } (v_l, v_j) \in \mathcal{E}(G_{[t_{i-1}, t_i)}), \\ -1 & \text{if } (v_j, v_l) \in \mathcal{E}(G_{[t_{i-1}, t_i)}), \\ \ \ 0 & \text{otherwise} \end{cases}$$

This matrix not only captures the adjacency relationships in the graph, but also the orientation of the graph. Let us define the graph Laplacian $\mathcal{L}(G_{[t_{i-1}, t_i)}) \in \mathbb{R}^{N_i \times N_i}$ as

$$\mathcal{L}(G_{[t_{i-1}, t_i)}) = \mathcal{D}(G_{[t_{i-1}, t_i)}) \mathcal{D}^T(G_{[t_{i-1}, t_i)})$$

which is a symmetric positive semidefinite matrix. Without loss of generality assume that the incidence matrix and the graph Laplacian for the leader-follower framework can be rewritten as

$$\mathcal{D}(G_{[t_{i-1}, t_i)}) = \begin{bmatrix} \mathcal{D}_{leader}(G_{[t_{i-1}, t_i)}) \\ \mathcal{D}_{follower}(G_{[t_{i-1}, t_i)}) \end{bmatrix}$$

$$\mathcal{L}(G_{[t_{i-1}, t_i)}) = \begin{bmatrix} \mathcal{L}_{leader}(G_{[t_{i-1}, t_i)}) \\ \mathcal{L}_{follower}(G_{[t_{i-1}, t_i)}) \end{bmatrix}$$

where the matrices $\mathcal{D}_{leader}(G_{[t_{i-1},t_i)}) \in \mathbb{R}^{1\times\rho_i}$, and $\mathcal{L}_{leader}(G_{[t_{i-1},t_i)}) \in \mathbb{R}^{1\times N_i}$ are the incidence matrix and the Laplacian's rows associated with the leader. Here $\mathcal{L}_{follower}(G_{[t_{i-1},t_i)}) \in \mathbb{R}^{(N_i-1)\times N_i}$ and $\mathcal{D}_{follower}(G_{[t_{i-1},t_i)}) \in \mathbb{R}^{(N_i-1)\times\rho_i}$ are composed with the rows corresponding to the followers. The dynamics of the multiagent system 1) can be written in a compact form using those matrices and we refer to [18] for the details.

III. The Hybrid LQ Optimal Tracking Problem Associated with the Leader-Followers Dynamics

In this section we deal with an equivalent representation of the multiagent system from the previous section in the form of an auxiliary hybrid system. This representation makes it possible to consider an associated optimal control problem in order to optimize the dynamic behavior of the initial leader-followers model.

The dynamical change of the number of agents (vertexes of the polygonal formation) constitutes a discrete event effect in the model. This effect can also be described in the framework of the hybrid systems theory. Let i be a discrete i-state of the multiagent system that is determined by N_i-polygonal formation, where $i = 1,...,L$. Clearly, $N_i - N_{i-1} = d$. Note that this uniform evolution of the system dimension can be easily generalized to the case of different increments of dimension in every discrete i-state. As mentioned above the dynamics of this leader-followers configuration with a N_i-polygonal formation is given by (1). Let $r_i := (r_{lj})^T_{(l=1,...,N_i,\ j\in\mathcal{Q}_l)} \in \mathbb{R}^{\rho_i}$. The complete evolution of the above group of agents can now be interpreted as a hybrid linear control system with L locations

$$\dot{x}_i(t) = A_i x_i(t) + B_i u(t) + C_i r_i, \tag{2}$$

$x_1(0) = (y_1(0), y_l(0))^T$. Here $i = 1,...,L$ represents the current N_i-polygonal formation and also the active subsystem in (2).

$$A_i = \begin{bmatrix} 0 \\ -\mathcal{L}_{follower}(G_{[t_{i-1},t_i)}) \end{bmatrix} \in \mathbb{R}^{\chi_i\times\chi_i},$$
$$B_i = \begin{bmatrix} I_d \\ 0 \end{bmatrix} \in \mathbb{R}^{\chi_i\times m},$$
$$C_i = \begin{bmatrix} 0 \\ \mathcal{D}_{follower}(G_{[t_{i-1},t_i)}) \end{bmatrix} \in \mathbb{R}^{\chi_i\times\rho_i}.$$

and $\chi_i := N_i m = (N_0 + d\,(i-1))\,m$. Moreover, $x_i(t) := (y_1^T(t), y_l^T(t))^T \in \mathbb{R}^{\chi_i}$ is the new state vector, $\mathcal{L}_{follower}(G_{[t_{i-1},t_i)}) \in \mathbb{R}^{(\chi_i-dm)\times\chi_i}$ and $\mathcal{D}_{follower}(G_{[t_{i-1},t_i)}) \in \mathbb{R}^{(\chi_i-dm)\times\rho_i}$ are the Laplacian-type and the incidence matrices related to the followers agents v_l, $l = 2,\dots,L$ defined in Section II. Note that $x_i(t) \in \mathbb{R}^{\chi_{i+1}}$ and is different from the vector $x_i(t)$.

Evidently, the resulting hybrid system is characterized by the monotonically increasing dimension of the subsystems associated with the given sequence of locations. Note that the above-mentioned mechanism of the agents detection defines the switching rules for the location transitions in the resulting hybrid system (2). This switching rules can be analytically expressed by the norm-inequalities:

$$||y_1(t_i) - y_{N_i+\nu}(t_i)|| \le \delta,\ \nu = 1,...,d$$

where $t_i,\ i = 1,...,L$ denotes a switching time between locations $(i-1)$ and i. Note that we are assuming that the switching times t_i are uniquely defined by the switching rule, i.e. we assume that our system does not have zeno behavior. As a consequence of the above switching mechanism we obtain the following "continuity" property of the resulting trajectory generated by (2): $x_i(t_i) = \mathrm{Pr}_{\mathbb{R}^{\chi_i}}\big(x_{i+1}(t_i)\big)$, where $\mathrm{Pr}_{\mathbb{R}^{\chi_i}}\big(x\big)$ is a projection of the vector x on the space $\mathbb{R}^{\chi_i}$.

Note that in every location we will have a corresponding graph, and the new connections should be assigned such that every subsystem be controllable (see e.g. [22], [18], [16] for a detail definition of multiagent controllability).

A possible treatment of the above leader-followers problem can be based on the newly elaborated theory of hybrid LQ optimal control (see e.g. [4], [5]). Let us firstly introduce, some necessary matrices associated with a hybrid LQ-type cost functional. Assume $S_f \in \mathbb{R}^{\chi_i\times\chi_i}$, $R_i : \mathbb{R} \to \mathbb{R}^{m\times m}$, $S_i : \mathbb{R} \to \mathbb{R}^{\chi_i\times\chi_i}$ where $i = 1,2,\dots,L$, be symmetric matrices. In addition we assume that S_f is positive semidefinite, and that for every time instant $t \in [0,t_f]$ and every i every $S_i(t)$ is also positive semidefinite. Moreover, let $R_i(t)$ be symmetric and positive definite for every $t \in [0,t_f]$ and every i. Additionally the given matrix-functions $S_i(\cdot),\ R_i(\cdot)$ are continuously differentiable. Our aim is to minimize the following cost function.

$$\begin{aligned} J(\cdot) = &\frac{1}{2}(D_L x_L(t_f) - E_L\bar{r}_L(t_f))^T S_f (D_L x_L(t_f) - E_L\bar{r}_L(t_f)) + \\ &\sum_{i=1}^{L}\int_{t_{i-1}}^{t_i}\frac{1}{2}((D_i x_i(t) - E_i\bar{r}_i(t))^T S_i(t)(D_i x_i(t) - E_i\bar{r}_i(t)) + \\ &u^T(t)R_i(t)u(t))dt \end{aligned} \tag{3}$$

with respect to (2). Here

$$D_i = \begin{bmatrix} \begin{bmatrix} I_d & 0 \end{bmatrix} \\ -\mathcal{L}_{follower}(G_{[t_{i-1},t_i)}) \end{bmatrix},\quad E_i = \begin{bmatrix} I_d & 0 \\ 0 & C_i \end{bmatrix}$$

are weight matrices of the corresponding dimension and $\bar{r}_i(t) := (r_e^T(t), r_i^T)^T$ is a reference vector of the leader-follower configuration. Note that r_e es the leader's reference path defined in the previous section.

We now assume that the above reference vector can be chosen from the following linear (hybrid) model:

$$\dot{\bar{r}}_i = \Gamma_i(t)\bar{r}_i(t),\ i = 1,...,L \tag{4}$$

where $\Gamma_i(t) \in \mathbb{R}^{(\rho_i+md)\times(\rho_i+md)}$ are some reference matrices associated with locations $i = 1,...,L$. Note that in the case of a nonlinear reference dynamics the linear model (4) is a result of a suitable linear approximation (see [13] for the linear neuronal network approximation approach). Note that the LQ-type optimal control problem (3) represents a special optimal tracking problem in the form of the hybrid optimal control problem for the initial multiagent system

(2). A non-standard character of the above tracking problem is characterized by different dimensions of the state and references. Moreover, the current dimension of the state vector $x_i(t)$ and the reference vector $\bar{r}_i(t)$ is a monotonically increasing function of the index i.

IV. Riccati-Based Techniques for Analysis of the Hybrid Tracking Problem

As we have seen in the previous section the original optimal control problem (3) has two specific characteristics in comparison to a conventional hybrid LQ optimization problem (see e.g., [4], [5]). These specific points, namely, the different dimensions of the states-reference processes and the growing dimensions of subsystems from (2) (described in section III) can be analyzed separately. In this section we focus our attention on the first phenomena and reduce the specific optimal tracking problem 3 to a hybrid LQ regulator. Note that we consider here (3) under assumption of identical subsystems dimensions (no dimension evolution).

Recall that $\dim(\bar{r}_i(t)) \neq \dim(x_i(t))$. The first m component of the above reference vector $\bar{r}(t)$, $t \in [0, t_f]$ represents a required state $r_e(t)$ of the leader. The next components of $\bar{r}(t)$ formalize the condition of a desirable geometrical configuration of followers. Our aim is to design an optimal control strategy in (3) in the absence of the dimensions evolution of the states in every location $i = 1, ..., L$. The optimal control in (3) can be obtained using the hybrid Maximum Principle and the associated Riccati formalism (see [4], [5]). We firstly introduce the auxiliary variable $z_i(t) := x_i(t) - D_i^{-1}E_i\bar{r}_i$ that satisfies the following differential equation

$$\begin{aligned} \dot{z}_i(t) = A_i z_i(t) + B_i u(t) + \\ \left(C_i r_i + A_i D_i^{-1} E_i \bar{r}_i(t) - D_i^{-1} E_i \dot{\bar{r}}_i(t)\right) \end{aligned} \tag{5}$$

The weight matrix D_i is assumed to be invertible. Note that this invertibility condition conforms with the dynamics of the initial multiagent system (1). Using (5) and the dynamical reference model (4), we deduce the differential equation for the new vectors $z_i(t)$

$$\dot{z}_i(t) := A_i z_i(t) + B_i u(t) + \omega_i(t) \tag{6}$$

where $\omega_i(t) := \left(A_i D_i^{-1} E_i - D_i^{-1} E_i \Gamma(t)\right) r_i(t)$. Note that in general $z_i(t_i) \neq z_{i+1}(t_i)$ and the resulting system (6) can be interpreted as an impulsive hybrid system (see [2], [3], [4]). This impulsive character of (6) is a consequence of the dynamics of the reference vectors $\bar{r}_i$. The corresponding interpretation leads to the hybrid LQ-type optimal control problem

$$\begin{aligned} &\text{minimize} \\ &J(\cdot) = \frac{1}{2} z_L^T(t_f) D_L^T S_f D_L z_L(t_f) + \\ &\sum_{i=1}^{L} \int_{t_{i-1}}^{t_i} \frac{1}{2}\left(z_i^T(t) D_i^T S_i D_i z_i(t) + u^T(t) R_i(t) u(t)\right) dt \end{aligned} \tag{7}$$

over all admissible trajectories of (6).

Note that (7) is a general impulsive LQ optimization problem for a linear system with an additive (external) input. Following the general approach to impulsive hybrid systems developed in [5] we now solve the above optimal control problem, and obtain the optimal piecewise linear feedback in the form

$$u(t) = -R_i^{-1}(t) B_i^T(t) P_i(t) z_i(t), \ t \in [t_{i-1}, t_i), \tag{8}$$

where P_i is the solution of the differential Riccati equation

$$\begin{aligned} &\dot{P}_i(t) = -P_i(t)A_i(t) - A_i^T(t)P_i(t) + \\ &P_i(t)B_i(t)R_i^{-1}(t)B_i^T(t)P_i(t) - S_i(t) \\ &\forall t \in (t_{i-1}, t_i), i = 1, 2, \ldots, L \end{aligned} \tag{9}$$

with a boundary (terminal) $P_L(t_f) = S_f$. The difference between the Riccati matrices at the switching time instances t_i, $i = 1, 2, \ldots, L$ are given as solutions of the specific system algebraic Riccati equation [4], [5]

$$\begin{aligned} &A_i^T P_i + P_i A_i - P_i B_i R_i^{-1} B_i^T P_i + S_i - A_{i+1}^T P_{i+1} + \\ &P_{i+1} A_{i+1} - P_{i+1} B_{i+1} R_{i+1}^{-1} B_{i+1}^T P_{i+1} + S_{i+1} = 0. \end{aligned} \tag{10}$$

Theorem 1: Under assumptions of Section III the optimal feedback control in problem (7) is given by (8).

Proof: Let $H_{ND}(z_i, u_i, \phi_i)$ be a Hamiltonian associated with the specific variant of (7) determined by $\omega_i(t) \equiv 0$. The Hamiltonian $H_D(z_i, u_i, \psi_i)$ for the general problem (7) (with a nontrivial $\omega_i(t)$) can be written as

$$H_D(z_i, u_i, \psi_i) = H_{ND}(z_i, u_i, \psi_i) + \psi_i(t)\omega_i(t),$$

where $i = 1, ..., L$. Evidently, the systems of adjoint equations and the corresponding boundary value problems for the above two variants of the basic problem (7) have the same form. The maximization conditions from the hybrid Maximum Principle also lead to the same result in both cases. These facts imply the conformity of the Riccati matrices for the general optimal control problem (7) and for the specific case of (7) indicated above. The Riccati matrix can now uniquely determined form the differential-algebraic system (9)-(10) and the optimal piecewise feedback in (7) is given by (8) with $z_i(\cdot)$, where $i = 1, ..., L$, are solutions to the general systems (6). The proof is completed. ■

Theorem 1 provides an analytical basis for the effective treatment of the specific LQ-type optimal control problem (7) that represents a formalization of the non-standard quadratic optimal tracking problem with different dimensions of the states and the reference processes.

V. Optimal Control of the Multiagent Leader-Follower Model

This section deals with the constructive analysis of the growing dimensions of subsystems in the optimal control problem (7). The dimension of the reference trajectory is assumed to be increased. This fact is indicated in model (4) by the variable dimensions of vectors $r_i(t)$. Also the dimensions of the matrices $\Gamma_i(t)$ are correspondingly evolved. Theorem 1 from the previous section make it possible to compute the gain matrix in (8) using the Riccati formalism for a non-disturbed model (6) for $\omega_i(\cdot) \equiv 0,\ i = 1, ..., L$. The increasing dimension of $x_i(t_i)$ causes the equivalent change of dimension of the vector $z_i(t_i)$. Note that this evolution of dimensions happens at the switching times $t_i,\ i = 1, ..., L$ and has an impulsive nature. We can interpret system (6) as an impulsive linear hybrid system from [1], [2], [5] and also rewrite the above algebraic Riccati equation

$$A_i^T P_i + P_i A_i - P_i B_i R_i^{-1} B_i^T P_i + \Delta_i = 0 \tag{11}$$

where

$$\Delta_i = S_i - F^T \left(A_{i+1}^T P_{i+1} + P_{i+1} A_{i+1} - \right. \\ \left. P_{i+1} B_{i+1} R_{i+1}^{-1} B_{i+1}^T P_{i+1} + S_{i+1} \right) F$$

and $F = [I, 0]^T$. Note that the product $\Delta_i - S_i$ determines a projection $\mathrm{Pr}_{\mathbb{R}^{\chi_i \times \chi_i}}(P_{i+1}(t))$ of the matrix $P_{i+1}(t)$ on the space $\mathbb{R}^{\chi_i \times \chi_i}$. Resulting from the dynamics of dimension this projection associated with a matrix $\Omega \in \mathbb{R}^{\chi_{i+1} \times \chi_{i+1}}$ can be realized as $F^T \Omega F$ with the above matrix F. Let us now summarize the above facts in the form of a theorem.

Theorem 2: The Riccati matrices $P_i(\cdot)$ in (8) for optimal control problem (7) with increasing dimension of the state vectors $z_i(t)$ are determined by (9) and (11).

Proof: Every subsystem related to the location i can be embedded in a space of dimension χ_{i+1} such that all the relevant vectors and matrices are represent as appropriate projection. $z_i(t) = \mathrm{Pr}_{\mathbb{R}^{\chi_i \times \chi_i}}(\hat{z}_i(t)),\ A_i(t) = \mathrm{Pr}_{\mathbb{R}^{\chi_i \times \chi_i}} \hat{A}_i(t)$. Moreover $B_i(t) = \mathrm{Pr}_{\mathbb{R}^{\chi_i \times \chi_i}} \hat{B}_i(t),\ P_i(t) = \mathrm{Pr}_{\mathbb{R}^{\chi_i \times \chi_i}} \hat{P}_i(t)$, where $\hat{z}_i(t) = \left(z_i^T(t), 0\right)^T$,

$$\hat{A}_i(t) = \begin{bmatrix} A_i(t) & 0 \\ 0 & 0 \end{bmatrix},\ \hat{B}_i(t) = \begin{bmatrix} B_i(t) \\ 0 \end{bmatrix}.$$

The vectors and matrices of the $i+1$ subsystem have the following structure: $z_{i+1}(t) = (z_{i+1,k}^T(t), \zeta_{i+1}^T(t))^T$, where $z_{i+1,k}$ denotes the first k-elements of the vector z_{i+1},

$$A_{i+1}(t) = \begin{bmatrix} A_{11,i+1}(t) & A_{12,i+1}(t) \\ A_{21,i+1}(t) & A_{22,i+1}(t) \end{bmatrix},$$
$$B_{i+1}(t) = \begin{bmatrix} B_{1,i+1}(t) \\ B_{2,i+1}(t) \end{bmatrix},$$
$$P_{i+1}(t) = \begin{bmatrix} P_{11,i+1}(t) & P_{12,i+1}(t) \\ P_{12,i+1}^T(t) & P_{22,i+1}(t) \end{bmatrix}.$$

where the $A_{11,i+1},\ P_{11,i+1} \in \mathbb{R}^{\chi_i \times \chi_i}$ and

$$A_{12,i+1},\ S_{12,i+1},\ P_{12,i+1},\ \hat{P}_{1,i} \in \mathbb{R}^{\chi_i \times dm},$$
$$A_{22,i+1},\ S_{22,i+1},\ P_{22,i+1},\ \hat{P}_{3,i} \in \mathbb{R}^{dm \times dm},$$
$$A_{21,i+1} \in \mathbb{R}^{dm \times \chi_i},\ \ B_{1,i+1} \in \mathbb{R}^{\chi_i \times m},\ B_{2,i+1} \in \mathbb{R}^{dm \times m}.$$

Moreover $\dim(\zeta_i(t)) = dm,\ \ t \in [t_{i-1}, t_i),\ i = 1, ..., L$ and the matrix $\hat{P}_i(t),\ t \in [t_{i-1}, t_i)$ is computed from (9). As a consequence of the hybrid Maximum Principle ([2], [25], [4]) we obtain an explicit form of the Hamiltonians associated with the locations i and $i+1$

$$H_i(t, \hat{z}, u, \hat{\psi}) := \langle \hat{\psi}_i, \hat{A}_i(t)\hat{z}_i + \hat{B}_i(t)u \rangle - \\ \frac{1}{2}\left(\hat{z}_i^T \hat{S}_i(t)\hat{z}_i + u^T R_i(t) u\right),$$
$$H_{i+1}(t, z, u, \psi) := \langle \psi_{i+1}, A_{i+1}(t) z_{i+1} + B_{i+1}(t) u \rangle - \\ \frac{1}{2}\left(z_{i+1}^T S_{i+1}(t) z_{i+1} + u^T R_{i+1}(t) u\right),$$

where $\langle \cdot, \cdot \rangle$ denotes the scalar product in $\mathbb{R}^{\chi_i}$,

$$\hat{S}_i(t) = \begin{bmatrix} S_i(t) & 0 \\ 0 & 0 \end{bmatrix},\ S_{i+1}(t) = \begin{bmatrix} S_{11,i+1}(t) & S_{12,i+1}(t) \\ S_{12,i+1}(t) & S_{22,i+1}(t) \end{bmatrix}$$

and $\hat{\psi}_i(t) = -\hat{P}_i(t)\hat{z}_i(t)\ \psi_{i+1} = -P_{i+1}\hat{z}_{i+1}$. Note that

$$S_{11,i+1} \in \mathbb{R}^{\chi_i \times \chi_i},\ S_{12,i+1} \in \mathbb{R}^{\chi_i \times dm},\ S_{22,i+1} \in \mathbb{R}^{dm \times dm}$$

The embedded state vector $\hat{z}_i(t_i)$ is different to $z_{i+1}(t_i)$ and the hybrid system (6) in the location $i+1$ has a dimension χ_{i+1}. Therefore, we consider the dynamics of the extended vector $\hat{z}_i(t)$ in the hybrid framework of the χ_{i+1}-dimensional system (6) Evidently, the dynamics of the "zero"-extensions of the original state vector $z_i(t)$ is irrelevant. Following the methodology developed in [2], [5], we now introduce the auxiliary variable $\hat{z}_i(t) = \tilde{z}_i(t) + \left(0, \zeta_{i+1}^T(t_i)\right)^T$. We now deal with a new family of continuous Hamiltonian (see [2], [5]) of the form

$$\begin{aligned} H_i(t, \hat{x}, u, \hat{\psi}) &= \begin{bmatrix} z_i^T \\ \zeta_{i+1}^T \end{bmatrix}^T \begin{bmatrix} H_{11,i} & H_{12,i} \\ H_{21,i} & H_{22,i} \end{bmatrix} \begin{bmatrix} z_i \\ \zeta_{i+1} \end{bmatrix} \\ H_{i+1}(t, \hat{x}, u, \psi) &= \begin{bmatrix} z_i^T \\ \zeta_{i+1}^T \end{bmatrix}^T \begin{bmatrix} H_{11,i+1} & H_{12,i+1} \\ H_{21,i+1} & H_{22,i+1} \end{bmatrix} \begin{bmatrix} z_i \\ \zeta_{i+1} \end{bmatrix} \end{aligned} \tag{12}$$

where $H_{11,i+1}, H_{11,i} \in \mathbb{R}^{\chi_i \times \chi_i}$, $H_{12,i+1}, H_{12,i} \in \mathbb{R}^{\chi_i \times dm}$, $H_{21,i+1}, H_{21,i} \in \mathbb{R}^{dm \times \chi_i}$ and $H_{22,i+1}, H_{22,i} \in \mathbb{R}^{dm \times dm}$. Using the continuity of Hamiltonians we deduce for every location the modified Riccati equations that contain jumps

$$-A_i^T(t_i)P_i(t_i) + \frac{1}{2}P_i(t_i)B_i(t_i)R_i^{-1}(t_i)B_i^T(t_i)P_i(t_i) - \\ \frac{1}{2}S_i(t_i) - H_{11,i}(t_i) = 0,$$

The last (non-symmetrical) equation can be finally rewritten as (11). The complete system that allows characterize the dynamics of the Riccati matrices. ∎

Theorem 2 represents our final result that provides a basis for a constructive computational approach to the general LQ-type optimal control problem (3). Recall that the general optimal control problem (3) was characterized by two non-standard phenomena, namely, the different dimensions of the state and reference processes and the growing dimension of the full system.

VI. A Computational Example of the Optimal Leader-Followers Behavior

We now show the effectiveness of the proposed approach and apply the theory developed in Sections III, IV and V to a particular case of the general multiagent system from Section II. Let us consider a leader-followers model with total numbers of agents equal to 10. Assume that an initial system with $N_0 = 5$ agents forms a (regular) pentagon that is inscribed in a circle of radius $R = 1$. To assure that the controllability of the system remains we use the dynamical graph presented in Fig. 3. Note that those graphs

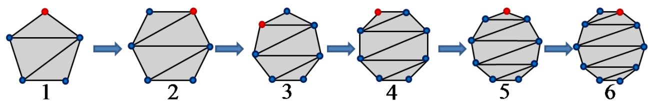


Fig. 3. Located graphs

are directed and the connections between the nodes are ruled by the dynamics presented in (1). Recall that our aim is to preserve a configuration of the system in which the leader follows a reference trajectory given by a circle with radius $\rho = 10$. Moreover, there are 5 additional agents that extend the current system configuration. In the context of the leader-followers strategy discussed in Section II a current configuration is extended by a further agent if the Euclidean distance is less or equal to $\delta = 1$ (see Section II). Then the auxiliary hybrid system is characterized by $L = 6$ locations.

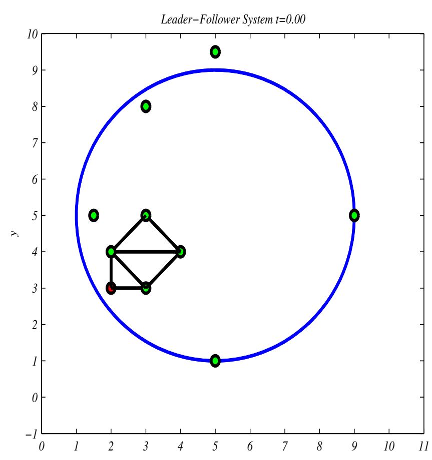


Fig. 4. Leader–Followers system simulation: Initial formation

For a concrete computation of the impulses and the distances we use the following scheme.

Computational Algorithm 1:

i) For every location i a circle with radius R is centered at the origin and collocate the leader in any point along the circumference of this circle $y_{i,1} = y_{i,0}$.

ii) Calculate the internal angle of the current N_i-polygon formation: $\gamma_i = \frac{2\pi}{N_i}$;

iii) For $k = 2, 3, \ldots, N_i$ determine the running position of the agents included into the current polygon

$$y_{k,i} = \begin{bmatrix} \cos\gamma_i & \sin\gamma_i \\ -\sin\gamma_i & \cos\gamma_i \end{bmatrix} y_{k-1,i};$$

iv) for $l = 1, 2, \ldots, N_{i-1}$, $j \in \mathcal{Q}_l$, and $l \neq j$ compute the actual distances between the agents

$$r_{lj,i_x} = r_{jl,i_x} = y_{l,i_x} - y_{j,i_x},$$
$$r_{lj,i_y} = r_{jl,i_y} = y_{l,i_y} - y_{j,i_y}.$$

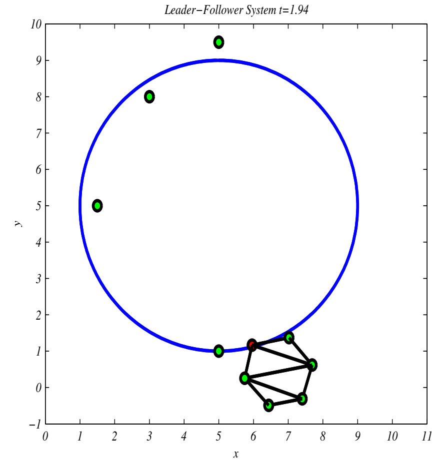


Fig. 5. Leader–Followers system simulation: Last second location 2

v) Calculate the value of the impulses in the switching time t_i: $\theta_i = \mathrm{Pr}_{\mathbb{R}^{\rho_i}} r_{i+1} - r_i$.

Following the theoretic approach developed in Sections III IV and V we obtain the optimal input in a closed form

$$u_i(t) = -R_i^{-1}(t) B_i^T(t) P_i(t) \left(x_i(t) - D_i^{-1} E_i \bar{r}_i(t) + \theta_i\right)$$

where $t \in [t_{i-1}, t_i)$ and P_i is the solution of (9). Let us note that jumps of the differential Riccati equations (9) at switching times t_i, $i = 1, \ldots, L$ are governed by the algebraic Riccati equation (11). Using the above algorithm, we establish the dynamical configuration of the followers with the leader's coordinate equal to $(0.6503, -0.7597)$ (see Figs. 4 – 7).

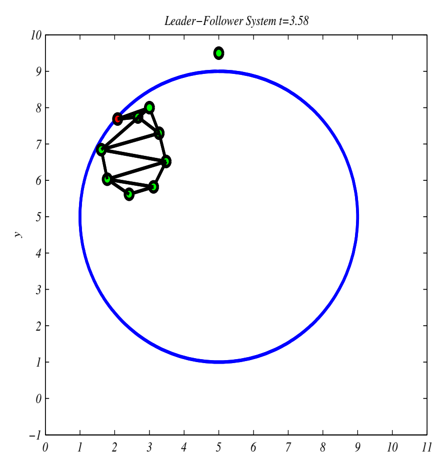


Fig. 6. Leader–Followers system simulation: Initial time location 5

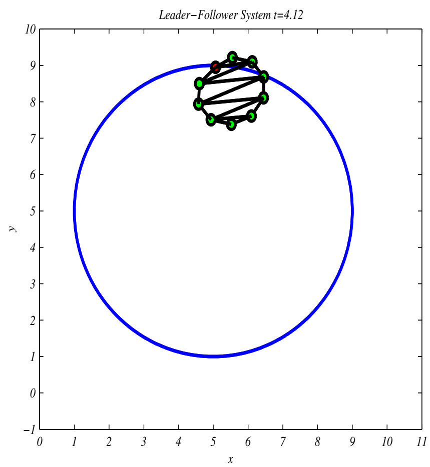


Fig. 7. Leader–Followers system simulation: Final configuration

VII. Conclusions

In this paper we proposed an optimal tracking methodology for a leader-followers problem in a multiagent setting. The dynamic systems under consideration are characterized by two additional formal effects, namely, by the evolution of the dimensions of the state and references vectors and by the monotonically increasing dimensions of the current state space.

The resulting gap of dimensions and the dimension evolution mentioned above are due to the concrete nature of the leader-followers multiagent model. In our contribution we deal with two partial problems: a multiagent optimal tracking problem and a hybrid LQ-type optimal regulator problem. Both of these sub-problems incorporate theoretical aspects related to the effects of the dimension evolution. The hybrid LQ-based optimization techniques are considered here as an auxiliary method associated with the constructive solution procedure for the initial multiagent tracking problem.

Let us note that the theoretical and computational approaches proposed in this paper can also provide a conceptual basis for the optimal control design in some practical applications associated with the multiagent dynamic systems. Also the multiagent result presented can be generalized to more general formations, where the agents are not uniformly distributed on a ring. Finally the approach presented in the paper can be extended to multiagent models with potentials and collision avoidance.

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