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Sufficient Conditions for Closed-Loop Control Over a Gaussian Relay Channel

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Abstract— **The problem of remotely stabilizing a noisy first order linear time invariant system with an arbitrary distributed initial state over a general half-duplex white Gaussian relay channel is addressed. We propose to use linear and memoryless communication and control strategies which are based on the Schalkwijk-Kailath coding scheme. By employing the proposed scheme over the general half-duplex Gaussian relay channel, we derive sufficient conditions for mean square stability of the noisy first order linear time invariant dynamical system.**

I. Introduction

The problem of remotely controlling dynamical systems over communication channels has gained significant attention in recent years. Such problems ask for interaction between stochastic control theory and information theory [1, 2]. The minimum data rate below which the stability of an LTI system is impossible has been derived in stochastic and deterministic settings in [2–4], where they considered quantization errors and noise-free rate-limited channels. In [5, 6] are necessary rate conditions required to stabilize an LTI plant almost surely. However, from [7] we know that the characterization by Shannon capacity is not enough for sufficient conditions for moment stability in closed-loop control. In [8] a simple coding scheme is proposed to mean square stabilize an LTI plant over noise-free rate-limited channels. The mean square stability of discrete plant over signal-to-noise ratio constraint channels is addressed in [9, 10]. In [11] the authors considered noisy communication links between both observer–controller and controller–plant. In [12] the necessary and sufficient conditions are derived for mean square stability of an LTI system over time varying feedback channels.

In this paper we find sufficient conditions for stabilizing a scalar first order noisy LTI plant over half-duplex white Gaussian relay channels. The relay channel consists of one sender (source), one receiver (destination) and an intermediate node (relay) whose sole purpose is to help the communication between the source and the destination [13]. The achievable information rate over the relay channel depends on the processing strategy of the relay. The most well known relaying strategies are amplify-and-forward (AF), compress-and-forward, and decode-and-forward [14]. AF strategy is

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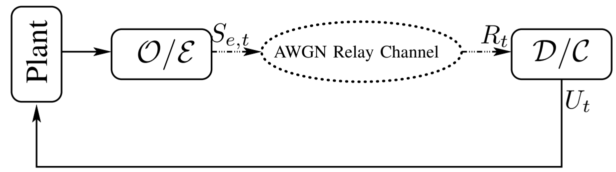


Fig. 1. The unstable plant has to be controlled by the actions of observer/encoder ($\mathcal{O}/\mathcal{E}$) and decoder/controller ($\mathcal{D}/\mathcal{C}$) over the AWGN relay channel.

well suited for delay sensitive closed-loop control applications and is therefore addressed in this paper. For communication and control we propose to use the Schalkwijk-Kailath based coding strategy [15] which is suitable for channels with feedback [16, 17]. We used the Schalkwijk-Kailath based scheme to obtain stability regions for control over multiple-access and broadcast channels in [18]. In [19] we derived rate sufficient conditions for stabilizability of a noiseless plant with Gaussian distributed initial state over non-orthogonal full-duplex and orthogonal half-duplex relay channels. In [20], the authors obtained sufficient conditions for stability over a Gaussian relay and cascade channels. In this paper we extend our results to stabilizability of a *noisy* plant with an *arbitrary distributed* initial state over a *general* half-duplex relay channel. The objective of this work is to derive sufficient conditions for stability of an LTI plant in mean square sense [3, 7, 8, 12] over half-duplex relay channels.

II. Problem Formulation

We consider a scalar discrete-time LTI system, whose state equation is given by

$$X_{t+1} = \lambda X_t + U_t + W_t, \tag{1}$$

where $\{X_t\} \subseteq \mathbb{R}, \{U_t\} \subseteq \mathbb{R}$, and $\{W_t\} \subseteq \mathbb{R}$ are state, control, and plant noise processes. The process noise $\{W_t\}$ is a zero mean white Gaussian noise sequence with variance n_w. We assume that the open-loop system is unstable $(\lambda > 1)$ and the initial state X_0 is a random variable with variance α_0 and an arbitrary probability distribution. We consider a remote control setup, where the observed state value is transmitted to the controller over an AWGN relay channel as shown in Fig. 1. We assume that there is no measurement noise. In order to communicate the observed state value X_t over the noisy channel, an encoder $\mathcal{E}$ is lumped with the observer $\mathcal{O}$ and a decoder $\mathcal{D}$ is lumped with the controller $\mathcal{C}$. In addition there is an intermediate relay node $\mathcal{R}$ within the channel to support communication from $\mathcal{E}$ to $\mathcal{D}$. At any time instant t, $S_{e,t}$ and R_t are the input and the output of the AWGN relay channel and U_t is the control action. Let f_t denote the observer/encoder policy, then we have $S_{e,t} = f_t(X_0, X_1, ..., X_t, U_1, U_2, ..., U_{t-1})$ which must satisfy an average power constraint $\lim_{T\to\infty} \frac{1}{T}\sum_{t=0}^{T-1} \mathbb{E}[S_{e,t}^2] \le P_S$. Further let γ_t denote the decoder/controller policy, then $U_t = \gamma_t(R_1, R_2, ..., R_t)$. The objective in this paper is to find a sufficient condition on the system parameters so that the system in (1) can be mean square stabilized.

Definition 2.1: A system is said to be *mean square stable* if there exists a constant $M < \infty$ so that $\mathbb{E}[X_t^2] < M$ for all t.

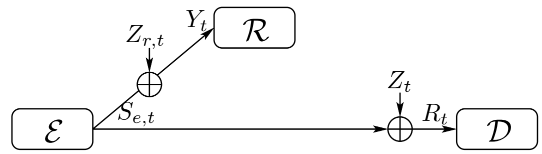
There can be various configurations of the relay channel. In this work we focus on a half-duplex relay channel where the relay cannot receive and transmit signals simultaneously. Moreover, we consider an instantaneous linear and memoryless (amplify-and-forward) relaying strategy which is in particular suitable for delay sensitive closed-loop control applications, although we know that linear strategies are not optimal in general for multi-sensor settings [21–23]. In amplify-and-forward strategy, the relay amplifies the received signal under an average power constraint P_R and forwards it to the decoder/controller. A general half-duplex AWGN relay channel is depicted in Fig. 2, where the variables $S_{e,t}$ and $S_{r,t}$ denote the transmitted signals from the encoder $\mathcal{E}$ and the relay $\mathcal{R}$ at any discrete time step t. The variables $Z_{r,t}$ and Z_t denote the mutually independent white noise components at the relay and at the decoder respectively with $Z_{r,t} \sim \mathcal{N}(0, N_r)$ and $Z_t \sim \mathcal{N}(0, N)$. The information transmission from the encoder consists of two phases as shown in Fig. 2. In the first transmission phase the encoder $\mathcal{E}$ transmits message with an average power $2\beta P_S$, where $0 < \beta \le 1$ is a parameter that allocates power to the two transmission phases. In this transmission phase the relay $\mathcal{R}$ receives a noisy signal Y_t from the encoder but it does not transmit any signal. In the second transmission phase both the encoder $\mathcal{E}$ and the relay $\mathcal{R}$ transmit with average powers $2(1-\beta)P_S$ and P_r respectively. The relay transmits an amplified version of the noisy signal Y_{t-1} received during the first transmission. The amplification at the relay is done under an average power constraint P_R. Therefore the relay transmit signal at the discrete time step t is given by

$$S_{r,t} = aY_{t-1} = \sqrt{\frac{P_r}{P_S + N_R}}(S_{e,t\text{-}1} + Z_{r,t\text{-}1}), \qquad (2)$$

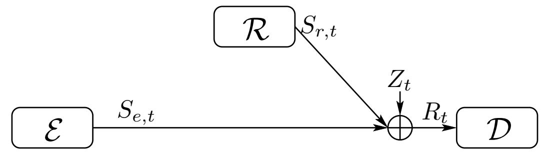
where the amplification factor a is chosen equal to $\sqrt{\frac{P_r}{P_S+N_R}}$ in order to satisfy the average power constraint i.e., $\mathbb{E}[S_{r,t}^2] = P_r \le P_R$. Accordingly the relay channel output at the decoder is R_t which is given by

$$R_t = hS_{e,t} + Z_t \qquad t = 1, 3, 5, \dots$$
$$R_t = hS_{e,t} + S_{r,t} + Z_t, \qquad t = 2, 4, 6, \dots$$

where $h \in \mathbb{R}$ denotes the gain of $\mathcal{E} - \mathcal{D}$ link. Generally speaking, the relay channel is more useful if the direct link is weak i.e., $|h|$ is small compared to the gains of $\mathcal{E} - \mathcal{R}$ and $\mathcal{R} - \mathcal{D}$ links.



(a) First transmission phase.



(b) Second transmission phase.

Fig. 2. Half-duplex AWGN relay channel.

III. Stability Results

We will first present our main results in a comprehensive fashion and then provide the proofs in the next section.

Theorem 3.1: The scalar linear time invariant system in (1) can be mean square stabilized over the *half-duplex* AWGN relay channel if

$$\log(\lambda) < \frac{1}{4} \max_{\substack{0<\beta\le 1\\ 0\le P_r\le P_R}} \left(\log\left(1 + \frac{2h^2\beta P_S}{N}\right) + \log\left(1 + \frac{\tilde{M}(\beta, P_r)}{\tilde{N}(\beta, P_r)}\right)\right) \qquad (3)$$

where $\tilde{N}(\beta, P_r) = \frac{P_r N_R}{2\beta P_S + N_R} + N$, $\tilde{M}(\beta, P_r) = \left(\sqrt{2h^2(1-\beta)P_S} + \sqrt{\frac{2\beta P_S P_r N}{(2\beta P_S + N_R)(2h^2\beta P_S + N)}}\right)^2$, and $\beta \in [0, 1]$.

Proof: The proof is given in Sec. IV. ■

Remark 3.1: Optimal choices of the power allocation parameter β at the encoder and the relay transmit power P_r which maximize the term on the right hand side of (3) depend on the quality (i.e., SNR) of $\mathcal{E}-\mathcal{D}$, $\mathcal{E}-\mathcal{R}$, and $\mathcal{R}-\mathcal{D}$ links.

Remark 3.2: The term on the right hand side of (3) is the information rate over the half-duplex AWGN relay channel with noiseless feedback. This is shown in Appendix I.

Remark 3.3: The condition in (3) does not depend on the process noise $\{W_t\}$. Further it is shown in Sec. IV that the sufficient condition for mean square stability of a system without process noise is identical to that with process noise. Although the sufficient conditions are identical, the second moment of the state process of a noiseless system ($n_w = 0$) converges to zero as time goes to infinity, i.e., $\lim_{t\to\infty} \mathbb{E}[X_t^2] = 0$, which is not possible in the noisy case.

By choosing certain values of the parameters β and h we get special cases of the general half-duplex relay channel. A relay channel is said to be *orthogonal* if the signal spaces of the encoder and the relay are orthogonal. The given half-duplex relay channel is orthogonal if $\beta = 1$, that is in the second transmission phase the encoder stays quiet and only the relay transmits. For this channel the noisy signal amplified and forwarded by the relay is not superimposed at the decoder to the signal coming directly from the encoder, therefore the optimal choice is to use the maximum available transmit power at the relay i.e., $P_r = P_R$.

Corollary 3.1: The scalar linear time invariant system in (1) can be mean square stabilized over an *orthogonal half-duplex* AWGN relay channel if

$$\log(\lambda) < \frac{1}{4}\left(\log\left(1+\frac{2h^2P_S}{N}\right)+\log\left(1+\frac{\tilde{M}(1,P_R)}{\tilde{N}(1,P_R)}\right)\right),$$

where the term on the right hand side of the inequality is an achievable information rate over the orthogonal half-duplex AWGN relay channel.

A relay channel is said to be *two-hop* when there is no direct communication link from the encoder to the decoder and the information can be communicated only via the relay. The given half-duplex relay channel becomes *two-hop* if $h = 0$. Naturally for this case we choose $\beta = 1$ and $P_r = P_R$.

Corollary 3.2: The scalar linear time invariant system in (1) can be mean square stabilized over a *two-hop* half-duplex AWGN relay channel if

$$\log(\lambda) < \frac{1}{4}\log\left(1+\frac{2P_SP_R}{P_RN_R+N(2P_S+N_R)}\right), \quad (4)$$

where the term on the right hand side of the inequality is an achievable information rate over the two-hop half-duplex AWGN relay channel.

For a setup which is equivalent to the *two-hop* relay channel, we find a necessary condition in [11, Theorem 4.1] which reads as

$$\log(\lambda) < \frac{1}{4}\min\left\{\log\left(1+\frac{2P_S}{N_R}\right), \log\left(1+\frac{P_R}{N}\right)\right\}.$$

Remark 3.4: The condition in (4) becomes both necessary and sufficient if either the $\mathcal{E}-\mathcal{R}$ link is noiseless ($N_R = 0$) or the $\mathcal{R}-\mathcal{D}$ link is noiseless ($N = 0$). That is the transmission scheme achieves capacity of a point-to-point channel.

Remark 3.5: Consider a *two-hop* relay channel with a causal noiseless feedback link from the controller to the relay. That is, the information at the decoder is nested with that of the encoder, therefore there is no dual effect. For this setup, the condition in (4) becomes necessary and sufficient if we restrict the encoder to be linear in the state. This can be observed from [24, 25].

IV. Proof

In order to prove Theorem 3.1 we propose a linear and memoryless communication and control scheme. This scheme is based on the well-known Schalkwijk-Kailath coding scheme [15, 16]. By employing the proposed linear scheme over the given *half-duplex* relay channel, we then find conditions on the system parameters λ which are sufficient to mean square stabilize the system in (1).

The control and communication scheme for the *half-duplex* relay channel works as follows.

Initial time step, $t = 0$: At time step $t = 0$, the encoder $\mathcal{E}$ observes X_0 and transmits $S_{e,0} = \sqrt{\frac{P_S}{\alpha_0}}X_0$. The decoder $\mathcal{D}$ receives $R_0 = hS_{e,0} + Z_0$. It then estimates X_0 as

$$\hat{X}_0 = \frac{1}{h}\sqrt{\frac{\alpha_0}{P_S}}R_0 = X_0 + \frac{1}{h}\sqrt{\frac{\alpha_0}{P_S}}Z_0.$$

The controller $\mathcal{C}$ then takes an action $U_0 = -\lambda\hat{X}_0$ which results in

$$X_1 = \lambda X_0 + U_0 + W_0 = -\frac{\lambda}{h}\sqrt{\frac{\alpha_0}{P_S}}Z_0 + W_0. \quad (5)$$

The new plant state $X_1 \sim \mathcal{N}(0,\alpha_1)$, where $\alpha_1 = \frac{\lambda^2 N}{h^2 P_S}\alpha_0 + n_w$.

First transmission phase, $t = 1,3,5,...$: The encoder $\mathcal{E}$ observes X_t and it then inputs $S_{e,t} = \sqrt{\frac{2\beta P_S}{\alpha_t}}X_t$ to the relay channel. The relay $\mathcal{R}$ listens but remains silent. The decoder $\mathcal{D}$ observes $R_t = hS_{e,t} + Z_t$ and computes the MMSE estimate of X_t, which is given by

$$\hat{X}_t = \mathbb{E}[X_t|R_1,R_2,...,R_t] \stackrel{(a)}{=} \mathbb{E}[X_t|R_t] \stackrel{(b)}{=} \frac{\mathbb{E}[X_tR_t]}{\mathbb{E}[R_t^2]}R_t \stackrel{(c)}{=} \left(\frac{h\sqrt{2\beta P_S\alpha_t}}{2h^2\beta P_S+N}\right)R_t,$$

where (a) follows from the *orthogonality principle* of MMSE estimation (that is $\mathbb{E}[X_tR_{t-j}] = 0$ for $j \geq 1$) [26]; (b) follows from the fact that the optimum MMSE estimator for a Gaussian variable is linear [26]; and (c) follows from $\mathbb{E}[X_tR_t] = \sqrt{2h^2\beta P_S\alpha_t}$ and $\mathbb{E}[R_t^2] = 2h^2\beta P_S + N$.
The controller $\mathcal{C}$ takes an action $U_t = -\lambda\hat{X}_t$ which results in $X_{t+1} = \lambda(X_t - \hat{X}_t) + W_t$. The new plant state X_{t+1} is linear combination of zero mean Gaussian variables $\{X_t, \hat{X}_t, W_t\}$, therefore it is also zero mean Gaussian with the following variance

$$\alpha_{t+1} \triangleq \mathbb{E}[X_{t+1}^2] = \lambda^2\mathbb{E}[(X_t-\hat{X}_t)^2] + \mathbb{E}[W_t^2] = \lambda^2\left(\frac{N}{2h^2\beta P_S+N}\right)\alpha_t + n_w, \quad (6)$$

where the last equality follows from $\mathbb{E}[X_t\hat{X}_t] = \mathbb{E}[\hat{X}_t^2] = \frac{2h^2\beta P_S\alpha_t}{2h^2\beta P_S+N}$ (by computation).

Second transmission phase, $t = 2,4,6,...$: The encoder $\mathcal{E}$ observes X_t and it then inputs $S_{e,t} = \sqrt{\frac{2(1-\beta)P_S}{\alpha_t}}X_t$ to the relay channel. The relay transmits $S_{r,t} = \sqrt{\frac{P_r}{(2\beta P_S+N_R)}}(S_{e,t-1} + Z_{r,t-1})$. The decoder $\mathcal{D}$ observes

$$R_t = hS_{e,t} + S_{r,t} + Z_t = L_1X_t + L_2X_{t-1} + \tilde{Z}_t, \quad (7)$$

where $L_1 = \sqrt{\frac{2(1-\beta)h^2P_S}{\alpha_t}}$, $L_2 = \sqrt{\frac{2\beta P_SP_r}{(2\beta P_S+N_R)\alpha_{t-1}}}$, and $\tilde{Z}_t = Z_t + \sqrt{\frac{P_r}{2\beta P_S+N_R}}Z_{r,t-1}$ with $\tilde{Z}_t \sim \mathcal{N}(0,\tilde{N}(\beta,P_r))$. The decoder then computes the MMSE estimate of X_t given all previous channel outputs $\{R_0,R_1,...,R_t\}$ in the following three steps:

1) Compute the MMSE prediction of R_t from $\{R_1,R_2,...,R_{t-1}\}$ as $\hat{R}_t = L_2\hat{X}_{t-1}$, where $\hat{X}_{t-1}$ is

the MMSE estimate of X_{t-1}.

2) Compute the innovation

$$I_t = R_t - \hat{R}_t = L_1 X_t + L_2(X_{t-1} - \hat{X}_{t-1}) + \tilde{Z}_t$$
$$= \left(\frac{\lambda L_1 + L_2}{\lambda}\right) X_t - \frac{L_2}{\lambda} W_{t-1} + \tilde{Z}_t, \quad (8)$$

which follows follows from $X_t = \lambda\left(X_{t-1} - \hat{X}_{t-1}\right) + W_{t-1}$.

3) Compute the MMSE estimate of X_t given $\{R_1, R_2, ..., R_{t-1}, I_t\}$. The state X_t is independent of $\{R_1, R_2, ..., R_{t-1}\}$, therefore we can compute the estimate $\hat{X}_t$ based on I_t only without any loss of optimality, that is

$$\hat{X}_t = \mathbb{E}[X_t|I_t] \overset{(a)}{=} \frac{\mathbb{E}[X_t I_t]}{\mathbb{E}[I_t^2]} I_t$$
$$\overset{(b)}{=} \frac{\lambda\left(\lambda L_1 + L_2\right)\alpha_t}{\left(\lambda L_1 + L_2\right)^2 \alpha_t + L_2^2 n_w + \lambda^2 \tilde{N}(\beta, P_r)} I_t, \quad (9)$$

where (a) follows from an MMSE estimation of a Gaussian variable; and (b) follows from $\mathbb{E}[X_t I_t] = \left(\frac{\lambda L_1 + L_2}{\lambda}\right)\alpha_t$ and $\mathbb{E}[I_t^2] = \left(\frac{\lambda L_1 + L_2}{\lambda}\right)^2 \alpha_t + \frac{L_2^2 n_w}{\lambda^2} + \tilde{N}(\beta, P_r)$.

The controller $\mathcal{C}$ takes an action $U_t = -\lambda \hat{X}_t$ which results in $X_{t+1} = \lambda(X_t - \hat{X}_t) + W_t$. The new plant state X_{t+1} is linear combination of zero mean Gaussian variables $\{X_t, \hat{X}_t, W_t\}$, therefore it is also zero mean Gaussian. The variance of the new plant state X_{t+1} is given in (10) on the top of the next page. In the computation of (10), (a) follows from $\mathbb{E}[X_t \hat{X}_t] = \mathbb{E}[\hat{X}_t^2] = \frac{(\lambda L_1 + L_2)^2 \alpha_t}{(\lambda L_1 + L_2)^2 \alpha_t + L_2^2 n_w + \lambda^2 \tilde{N}(\beta, P_r)}$; (b) follows by substituting the values of L_1 and L_2; and (c) by substituting $\frac{\alpha_t}{\alpha_{t-1}}$ from (6) and by defining $k \triangleq \frac{N}{2h^2 \beta P_S + N}$, $k_1 \triangleq \frac{2\beta P_S P_r}{2\beta P_S + N_R}$, $k_2 \triangleq \sqrt{2h^2(1 - \beta P_S)}$.

We want to find the values of the system parameter λ for which the second moment of the state remains bounded, i.e., the sequence $\{\alpha_t\}$ has to be bounded. Rewriting (6) and (10), the variance of the state at any time t is given by

$$\alpha_t = \lambda^2 \left(\frac{N}{2h^2 \beta P_S + N}\right) \alpha_{t-1} + n_w, \qquad t = 2, 4, 6, ... \quad (11)$$

$$\alpha_t = \lambda^2 \left(\lambda^2 k \alpha_{t-2} + n_w\right) f(\alpha_{t-2}) + n_w, \qquad t = 3, 5, 7, ... \quad (12)$$

where $\alpha_1 = \frac{\lambda^2 N}{h^2 P_S} \alpha_0 + n_w$ and $f(\alpha_{t-2}) \triangleq \left(\frac{\left(\frac{n_w k_1}{\lambda^2}\right)\frac{1}{\alpha_{t-2}} + \tilde{N}(\beta, P_r)}{(k_2 + \sqrt{k_1 k + (\frac{n_w k_1}{\lambda^2})\frac{1}{\alpha_{t-2}}})^2 + (\frac{n_w k_1}{\lambda^2})\frac{1}{\alpha_{t-2}} + \tilde{N}(\beta, P_r)}\right)$. If the odd indexed sub-sequence $\{\alpha_{2t+1}\}$ in (12) is bounded, then the even indexed sub-sequence $\{\alpha_{2t}\}$ in (11) is also bounded. Therefore it is sufficient to consider the odd indexed sub-sequence $\{\alpha_{2t+1}\}$. We will now construct a sequence $\{\alpha'_t\}$ which upper bounds the sub-sequence $\{\alpha_{2t+1}\}$. Then we will derive conditions on the system parameter λ for which the sequence $\{\alpha'_t\}$ stays bounded and consequently the boundedness of $\{\alpha_{2t+1}\}$ will be guaranteed. In order to construct the upper sequence $\{\alpha'_t\}$, we work on the term $f(\alpha_{t-2})$ in (12) and make use of the following lemma.

Lemma 4.1: Consider a function $f(x) = \frac{a + \frac{b}{x}}{(c + \sqrt{d + \frac{b}{x}})^2 + a + \frac{b}{x}}$ defined in the interval $[0, \infty)$, where $a, b, c, d \geq 0$. The function $f(x)$ can be upper bounded as $f(x) \leq f_\infty + \frac{m}{x}$ for some $m > 0$, where $f_\infty \triangleq \lim_{x \to \infty} f(x) = \frac{a}{(c + \sqrt{d})^2 + a}$.

Proof: The proof can be found in Appendix II. ■

Starting from (12) and by using the above lemma, we write the following series of inequalities

$$\alpha_t = \lambda^2 \left(\lambda^2 k \alpha_{t-2} + n_w\right) f(\alpha_{t-2}) + n_w$$
$$\overset{(a)}{\leq} \lambda^2 \left(\lambda^2 k \alpha_{t-2} + n_w\right) \left(f_\infty + \frac{m}{\alpha_{t-2}}\right) + n_w$$
$$= \lambda^4 k f_\infty \alpha_{t-2} + \frac{\lambda^2 n_w m}{\alpha_{t-2}} + n_w f_\infty + \lambda^4 m k + n_w$$
$$\overset{(b)}{\leq} \lambda^4 k f_\infty \alpha_{t-2} + \lambda^2 m + n_w f_\infty + \lambda^4 m k + n_w \triangleq g(\alpha_{t-2}), \quad (13)$$

where (a) follows from Lemma 4.1 and $f_\infty \triangleq \lim_{\alpha \to \infty} f(\alpha) = \left(\frac{\tilde{N}(\beta, P_r)}{(k_2 + \sqrt{k_1 k})^2 + \tilde{N}(\beta, P_r)}\right)$; and (b) follows from the fact that $\alpha_t \geq n_w$ for all t, which is obvious from (11) and (12) that the value of α_t can never be less than n_w. Since $g(\alpha)$ in (13) is a linearly increasing function, it can be used to construct the sequence $\{\alpha'_t\}$ which upper bounds the odd indexed sub-sequence $\{\alpha_{2t+1}\}$ given in (12). We construct the sequence $\{\alpha'_t\}$ as

$$\alpha_{2t+1} \leq \alpha'_{t+1} = g(\alpha'_t), \qquad \text{for all } t \geq 1$$
$$\overset{(a)}{=} \lambda^4 k f_\infty \alpha'_t + \lambda^2 m + n_w f_\infty + \lambda^4 m k + n_w$$
$$\overset{(b)}{=} \left(\lambda^4 k f_\infty\right)^t \alpha'_0$$
$$+ \left(\lambda^2 m + n_w f_\infty + \lambda^4 m k + n_w\right) \sum_{i=0}^{t-1} \left(\lambda^4 k f_\infty\right)^i, \quad (14)$$

where (a) follows from (13) and (b) follows by recursively applying (a). We observe from (14) that if $\left(\lambda^4 k f_\infty\right) = \left(\frac{\lambda^4 k \tilde{N}(\beta, P_r)}{(k_2 + \sqrt{k_1 k})^2 + \tilde{N}(\beta, P_r)}\right) < 1$, then the sequence $\{\alpha'_t\}$ converges to a limit point as $t \to \infty$ and consequently the original sequence $\{\alpha_t\}$ is guaranteed to stay bounded. Thus the system in (1) can be mean square stabilized over the *half-duplex* relay channel if

$$\lambda^4 < \left(\frac{(k_2 + \sqrt{k_1 k})^2 + \tilde{N}(\beta, P_r)}{k \tilde{N}(\beta, P_r)}\right) \quad (15)$$
$$\Rightarrow \log(\lambda) < \frac{1}{4} \log\left(\log\left(\frac{1}{k}\right) + \log\left(1 + \frac{(k_2 + \sqrt{k_1 k})^2}{\tilde{N}(\beta, P_r)}\right)\right)$$
$$= \frac{1}{4}\left(\log\left(1 + \frac{2h^2 \beta P_S}{N}\right) + \log\left(1 + \frac{\tilde{M}(\beta, P_r)}{\tilde{N}(\beta, P_r)}\right)\right), \quad (16)$$

where in the last equality we substituted $k = \frac{N}{2h^2 \beta P_S + N}$ and $M(\beta, P_r) = (k_2 + \sqrt{k_1 k})^2$ in order to show the dependencies on the average relay power P_r and the power allocation parameter β at the encoder. Since the relay node amplifies the desired signal as well as the noise which is then superimposed at the decoder to the signal coming directly from

$$\alpha_{t+1} \triangleq \mathbb{E}[X_{t+1}^2] = \lambda^2 \mathbb{E}[(X_t - \hat{X}_t)^2] + \mathbb{E}[W_t^2] \stackrel{(a)}{=} \lambda^2 \alpha_t \left(\frac{L_2^2 n_w + \lambda^2 \tilde{N}(\beta, P_r)}{(\lambda L_1 + L_2)^2 \alpha_t + L_2^2 n_w + \lambda^2 \tilde{N}(\beta, P_r)} \right) + n_w$$

$$\stackrel{(b)}{=} \lambda^2 \alpha_t \left(\frac{\left(\frac{2\beta P_S P_r}{2\beta P_S + N_R}\right) \frac{n_w}{\alpha_{t-1}} + \lambda^2 \tilde{N}(\beta, P_r)}{\left(\lambda\sqrt{2h^2(1-\beta P_S)} + \sqrt{\frac{2\beta P_S P_r}{2\beta P_S + N_R} \frac{\alpha_t}{\alpha_{t-1}}}\right)^2 + \left(\frac{2\beta P_S P_r}{2\beta P_S + N_R}\right) \frac{n_w}{\alpha_{t-1}} + \lambda^2 \tilde{N}(\beta, P_r)} \right) + n_w$$

$$\stackrel{(c)}{=} \lambda^2 \left(\lambda^2 k \alpha_{t-1} + n_w\right) \left(\frac{(n_w k_1) \frac{1}{\alpha_{t-1}} + \lambda^2 \tilde{N}(\beta, P_r)}{\left(\lambda k_2 + \sqrt{\frac{k_1}{\lambda^2}(\lambda^2 k + n_w \frac{1}{\alpha_{t-1}})}\right)^2 + (n_w k_1) \frac{1}{\alpha_{t-1}} + \lambda^2 \tilde{N}(\beta, P_r)} \right) + n_w$$

$$= \lambda^2 \left(\lambda^2 k \alpha_{t-1} + n_w\right) \left(\frac{\left(\frac{n_w k_1}{\lambda^2}\right) \frac{1}{\alpha_{t-1}} + \tilde{N}(\beta, P_r)}{\left(k_2 + \sqrt{k_1 k + \frac{n_w k_1}{\lambda^2} \frac{1}{\alpha_{t-1}}}\right)^2 + \left(\frac{n_w k_1}{\lambda^2}\right) \frac{1}{\alpha_{t-1}} + \tilde{N}(\beta, P_r)} \right) + n_w, \tag{10}$$

---

the encoder, an optimal choice of the relay transmit power $0 \leq P_r \leq P_R$ depends on the relay channel parameters $\{P_S, N_R, N, h, \beta\}$. Moreover an optimal choice of the power allocation factor β at the encoder also depends on the relay channel parameters $\{P_S, P_r, N_R, N, h\}$. Therefore we can rewrite (16) as (3.1), which completes the proof of Theorem 3.1. □

It is interesting to see that the sufficient condition for mean square stability does not depend on the process noise. This provides motivation to study stabilizability of the system in (1) without process noise, i.e., $W_t = 0$. In the absence of the process noise in (1), the state variance of the noiseless system at any time step t is then given by substituting $n_w = 0$ in (10), that is

$$\alpha_t = \left(\frac{\lambda^2 N}{2h^2\beta P_S + N}\right)\alpha_{t-1}, \qquad t = 2, 4, 6, ...$$

$$\alpha_t = \left(\frac{\lambda^4 k \tilde{N}(\beta)}{(k_2 + \sqrt{k_1 k})^2 + \tilde{N}(\beta)}\right)\alpha_{t-2}, \qquad t = 3, 5, 7, ...$$

Since $\alpha_1 = \frac{\lambda^2 N}{h^2 P_S}\alpha_0 + n_w$, the state variance $\alpha_t \to 0$ as $t \to \infty$ if $\left(\frac{\lambda^4 k \tilde{N}(\beta)}{(k_2+\sqrt{k_1 k})^2 + \tilde{N}(\beta)}\right) < 1$. This is the same condition as in (15). Thus by using the proposed linear coding and control scheme, we obtain identical sufficient conditions for mean square stability of noisy and noiseless first LTI system over *half-duplex* relay channel.

Appendix I
Information Rate

The given scheme can be seen as a point-to-point communication channel, where $R_{2t\text{-}1}$ is the channel output corresponding to the input $S_{e,2t\text{-}1}$ and I_{2t} is the channel output corresponding to the input $S_{e,2t}$. Since $P(I_{2t}, R_{2t\text{-}1}|S_{e,2t}, S_{e,2t\text{-}1}) = P(I_{2t}|S_{e,2t})P(R_{2t\text{-}1}|S_{e,2t\text{-}1})$, the channel is memoryless. The information rate is

$$\lim_{T\to\infty} \frac{1}{2T} I\left(\{S_{e,2t\text{-}1}, S_{e,2t}\}_{t=1}^T ; \{R_{2t\text{-}1}, I_{2t}\}_{t=1}^T\right)$$

$$\stackrel{(a)}{=} \lim_{T\to\infty} \frac{1}{2T}\left[\sum_{t=1}^T \Big(h(R_{2t\text{-}1}) + h(I_{2t}) - h(R_{2t\text{-}1}|S_{e,2t\text{-}1}) - h(I_{2t}|S_{e,2t})\Big)\right]$$

$$\stackrel{(b)}{=} \lim_{T\to\infty} \frac{1}{2T}\left[T\Big(h(R_{2t\text{-}1}) + h(I_{2t}) - h(R_{2t\text{-}1}|S_{e,2t\text{-}1}) - h(I_{2t}|S_{e,2t})\Big)\right]$$

$$= \frac{1}{2}\left(I(S_{e,2t\text{-}1}; R_{2t\text{-}1}) + I(S_{e,2t}; I_{2t})\right), \tag{17}$$

where (a) follows from the definition of mutual information and the fact that the channel is memoryless, $\mathbb{E}[R_{2l\text{-}1}R_{2k\text{-}1}] = \mathbb{E}[I_{2l}I_{2k}] = 0$ for $k \neq l$, and $\mathbb{E}[R_{2l\text{-}1}I_{2k}] = 0$ for all $l, k = 1, 2, 3, ..$; and (b) follows from the fact that $R_{2t\text{-}1}$ and I_{2t} are i.i.d. variables.

For the first transmission phase the mutual information between the transmitted variable and the received variable is given by

$$I(S_{e,2t\text{-}1}; R_{2t\text{-}1}) = h(R_{2t\text{-}1}) - h(R_{2t\text{-}1}|S_{e,2t\text{-}1})$$
$$= h(R_{2t\text{-}1}) - h(Z_{2t\text{-}1}) \stackrel{(a)}{=} \frac{1}{2}\log\left(1 + \frac{2h^2\beta P_S}{N}\right), \tag{18}$$

where (a) follows from $R_{2t\text{-}1} \sim \mathcal{N}(0, 2h^2\beta P_S + N)$ and $Z_{2t\text{-}1} \sim \mathcal{N}(0, N)$. In the second phase the decoder computes the innovation I_t according to (8). The mutual information between the transmitted variable and the innovation variable is then given by

$$I(S_{e,2t}; I_{2t}) = \frac{1}{2}\log\left(1 + \frac{\tilde{M}(\beta, P_r)}{\tilde{N}(\beta, P_r)}\right), \tag{19}$$

which follows from $I_{2t} \sim \mathcal{N}(0, \tilde{M}(\beta, P_r) + \tilde{N}(\beta, P_r))$ and $\tilde{Z}_{2t} \sim \mathcal{N}(0, \tilde{N}(\beta, P_r))$. From (18), (19), and (17) the information rate is equal to

$$\frac{1}{4}\left(\log\left(1 + \frac{2h^2\beta P_S}{N}\right) + \log\left(1 + \frac{\tilde{M}(\beta, P_r)}{\tilde{N}(\beta, P_r)}\right)\right). \tag{20}$$

For channels with feedback directed information is a useful quantity [27]. The directed information rate for the relay channel under discussion is given by

$$\begin{aligned}
&\lim_{T\to\infty}\frac{1}{2T}I\left(\{S_{e,2t\text{-}1},S_{e,2t}\}_{t=1}^{T}\to\{R_{2t\text{-}1},I_{2t}\}_{t=1}^{T}\right)\\
&\overset{(a)}{=}\lim_{T\to\infty}\frac{1}{2T}\left[\sum_{t=1}^{T}\left(I\left(S_{e,2t\text{-}1};R_{2t\text{-}1}\right)+I\left(S_{e,2t};I_{2t}\right)\right)\right]\\
&\overset{(b)}{=}\lim_{T\to\infty}\frac{1}{2T}\left[T\left(I\left(S_{e,2t\text{-}1};R_{2t\text{-}1}\right)+I\left(S_{e,2t};I_{2t}\right)\right)\right]\\
&=\frac{1}{2}\left(I\left(S_{e,2t\text{-}1};R_{2t\text{-}1}\right)+I\left(S_{e,2t};I_{2t}\right)\right), \qquad (21)
\end{aligned}$$

where (a) follows from the [27, Theorem 2] and orthogonality of the channel output sequence $\{R_{2t\text{-}1}, I_{2t}\}_{t=1}^{T}$; and (b) follows from the fact that $R_{2t\text{-}1}$ and I_{2t} are i.i.d. variables. Comparing (21) and (17), the directed information rate is equal to the information rate which is due to orthogonality of the channel output sequence.

Appendix II
Proof of Lemma 4.1

Consider the function $f(x)=\frac{a+\frac{b}{x}}{(c+\sqrt{d+\frac{b}{x}})^2+a+\frac{b}{x}}$ defined in the interval $[0,\infty)$, where $0\le a,b,c,d<\infty$. We want to show that $f(x)\le f_\infty+\frac{m}{x}$ in the interval $[0,\infty)$ for some $m>0$, where $f_\infty \triangleq \lim_{x\to\infty} f(x)=\frac{a}{(c+\sqrt{d})^2+a}$. In the following we show that $-f(x)+f_\infty+\frac{m}{x}$ is greater than or equal to zero for some $m\ge 0$.

$$\begin{aligned}
&-f(x)+f_\infty+\frac{m}{x}\\
&=-\frac{ax+b}{(c\sqrt{x}+\sqrt{dx+b})^2+ax+b}+\frac{a}{(c+\sqrt{d})^2+a}+\frac{m}{x}\\
&=\frac{1}{\left(\left(c+\sqrt{d}\right)^2+a\right)\left(\left(c\sqrt{x}+\sqrt{dx+b}\right)^2+ax+b\right)x}\\
&\times\Bigg[-(ax^2+bx)\left(\left(c+\sqrt{d}\right)^2+a\right)\\
&+ax\left(\left(c\sqrt{x}+\sqrt{dx+b}\right)^2+ax+b\right)\\
&+m\left(\left(c+\sqrt{d}\right)^2+a\right)\left(\left(c\sqrt{x}+\sqrt{dx+b}\right)^2+ax+b\right)\Bigg]
\end{aligned} \qquad (22)$$

The denominator term in (22) is always positive for $x\in[0,\infty)$, therefore we focus on the numerator term. The numerator term after simplification is equal to

$$\begin{aligned}
&m(c^4+c^2d+2c^3d+c^2d+d^2+2cd^{\frac{3}{2}}+2ad+ac^2\\
&+2ac\sqrt{d}+a^2)x-(bc^2+bd+2bc\sqrt{d})x\\
&+\underbrace{2ac(x\sqrt{dx^2+bx}-\sqrt{d}x^2)}_{\ge 0}+\underbrace{\vartheta}_{\ge 0}, \qquad (23)
\end{aligned}$$

where ϑ is the summation of the remaining terms, which are all non-negative and therefore their sum is also non-negative. If we choose $m \ge \frac{(bc^2+bd+2bc\sqrt{d})}{(c^4+c^2d+2c^3d+c^2d+d^2+2cd^{\frac{3}{2}}+2ad+ac^2+2ac\sqrt{d}+a^2)}$ in (23), the non-negativity of (22) in the interval $[0,\infty)$ will be guaranteed. □

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