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Self-triggered output feedback control of linear plants

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Abstract— **This paper addresses the problem of self-triggered output feedback control of linear time-invariant plants. In a self-triggered state feedback scenario, the controller is allowed to choose when the next sampling time should occur and does so based on the current sampled state and on a priori knowledge about the plant. The proposed solution extends previous results on state feedback stabilization to the case of dynamic output feedback, and uses a triggering mechanism based on the decrease of a Lyapunov function between sampling times. Since only the sampled output of the plant is available, a discrete time state observer is required, the stability properties of which are analyzed by resorting to key concepts on the observability of discretized switched linear systems.**

I. INTRODUCTION

In the design and analysis of controllers for continuous time systems, it is usually assumed that the state of the system (or its output) is available continuously. In practice, however, controllers must be implemented in digital devices. For this reason, there is a need to develop control techniques for systems whose state measurements are not available continuously. The special case where measurements are available periodically (periodic sampling) has been studied extensively in the literature (see, e.g, [13]). The relaxation of the periodic sampling restriction to accommodate sampled-data systems with non-uniform sampling has also been considered. Such systems arise for example in the study of networked control systems, where the sampling intervals are viewed as an exogenous signal that can be deterministic but bounded, or stochastic with a known distribution.

It is important to remark that while in some cases of practical interest the sampling times are not known in advance, in an event-based scenario the controller is often allowed to choose the next sampling time (also, know as update or release time, e.g., in the area of networked control systems), which effectively works as an extra degree of freedom in the design process. Controller design for such systems must therefore produce some kind of triggering or scheduling law. In an event-triggered control scenario (see Fig. 1), an event detector is responsible for triggering a sampling event, typically whenever some function of the plant's state or output exceeds a prescribed threshold. This generates a sequence of sampling intervals that in general is not periodic. Related work can be found in [1]–[5]. The advantage of this approach versus a periodic sampling strategy is that the

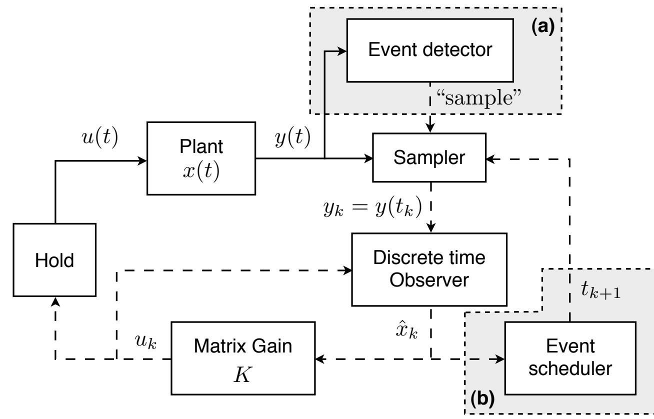

Fig. 1. Output feedback control: event-triggered (block **(a)** active); self-triggered (block **(b)** active). The state, the control input and the output of the plant are represented by x, u, and y, respectively. Solid lines denote continuous time signals while dashed lines denote sampled signals.

control input is only changed when some relevant change of the plant's state or output occurs and this typically leads to a reduction of the number of samples required. Nonetheless, the plant's state or output must be constantly monitored. In order to avoid this, self-triggered strategies have been proposed in [6]–[8] where, instead of continuously testing a triggering condition, an event scheduler (see Fig. 1) is responsible for computing when the next sampling event should occur, based on the current sampled state or an estimate of it and on knowledge about the plant dynamics.

To the best of the authors knowledge, this is the first time output feedback stabilization has been considered in the area of self-triggered control of linear plants. Previous work has been focused on state feedback stabilization. The paper's main contribution is the extension of previous results obtained in [8] to the case of dynamic output feedback. The proposed solution uses the triggering mechanism of [8] for control and borrows the ideas on observability of discretized switched linear systems presented in [9] to address the observability properties of a discrete time observer.

The paper is organized as follows. Section II introduces the class of plants under consideration and presents the control architecture proposed for self-triggered output feedback control. It also briefly reviews the self-triggering mechanism proposed in [8] for state feedback. The main contribution of this paper is developed in Section III. The main focus is on the observability properties of the discrete time equivalent of a continuous time plant and how they imply the existence of an asymptotic observer. In Section IV, an illustrative example with simulation results is provided. Finally, Section V contains concluding remarks.

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II. SELF-TRIGGERED CONTROL OF LINEAR PLANTS

Consider a linear time-invariant plant with state $x \in \mathbb{R}^n$ and initial state $x(t_0) = x_0 \in \mathbb{R}^n$ that satisfies, for all $t \geq t_0$,

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{1a}$$

$$y(t) = Cx(t) \tag{1b}$$

where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$ and $C \in \mathbb{R}^{p\times n}$ are the state, input and output matrices, respectively, $u \in \mathbb{R}^m$ is the control input and $y \in \mathbb{R}^p$ is the plant output. The pairs (A, B) and (A, C) are assumed to be controllable and observable, respectively (for the definition of these concepts see, e.g, [10, Chapter 9]).

In this paper we focus on sampled data control strategies that can be described by the block diagram of Fig. 1. In this setup, the plant's output y is sampled whenever $t = t_k$ and this information is sent to the discrete time observer on the controller side. The observer computes an estimate of the plant's state at the current sampling time $\hat{x}_k$ and feeds this estimate to the matrix gain and the event scheduler. The control input is kept constant and equal to $K\hat{x}_k$ between sampling times, where the gain matrix K is chosen so that a desired stability criterion is satisfied assuming continuous feedback. Based on the current estimated state and on knowledge about the plant dynamics, the event scheduler computes when the next sampling instant t_{k+1} should occur and communicates this information to the sampler. It is the goal of the scheduler to guarantee that certain stability conditions are verified by appropriately selecting the sequence of sampling intervals $(\tau_k)_{k=0}^{+\infty}$ where $\tau_k = t_{k+1} - t_k$ denote the kth sampling interval. This is done by defining a function $\tau : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ that maps states to time intervals so that $\tau_k = \tau(\hat{x}_k)$. The problem addressed in this paper is the following.

Problem 1: Given a linear plant (1) and the ability to sample its output at will, find a self-triggered dynamic output feedback controller that renders the closed loop system of Fig. 1 globally asymptotically stable (GAS) while trying to keep sampling events to a minimum.

Ideally, the last the part of the problem formulation should read "while minimizing the number of samples over some time interval". This is an harder problem, although for some linear scalar stochastic systems some results are reported in [11]. Note also that the problem formulation does not rule out periodic sampling since by increasing the sampling period the number of samples is reduced. However, with a fixed K, this leads to instability. In this paper we pursue a different strategy in order to reduce the sampling rate.

As mentioned in the introduction, the scheduling used in this paper is borrowed from [8] and will be explained next.

A. State feedback stabilization

To detail how the event scheduler generates sampling times, in this section we make a brief review of the results presented in [8] for the case of state feedback stabilization. Unlike [8], here we assume that there are no disturbances affecting the plant dynamics since our key goal is to extend the results to output feedback.

In this case $C = I$, the discrete time observer in Fig. 1 is omitted, and $\hat{x}_k = y_k = x_k$. Let the matrix gain K be such that $A + BK$ is Hurwitz. This is possible because the pair (A, B) is assumed controllable. Given a positive definite matrix Q, let P be the positive definite solution of the Lyapunov equation

$$(A + BK)^\top P + P(A + BK) + Q = 0. \tag{2}$$

Such a P always exists because $A+BK$ is Hurwitz. Consider the function $V : \mathbb{R}^n \to \mathbb{R}$ defined as

$$V(x) = (x^\top Px)^{\frac{1}{2}}, \tag{3}$$

where P is given by (2). Let $\lambda_0 = \frac{\lambda_{\min}(Q)}{2\lambda_{\max}(P)}$ where $\lambda_{\max}(X)$ and $\lambda_{\min}(X)$ denote the largest and the smallest eigenvalue of a symmetric matrix X, respectively. Under continuous feedback of the form $u(t) = Kx(t)$, we have that, for all $x_0 \in \mathbb{R}^n$ and all $t \geq t_0$,

$$V(x(t)) \leq V(x_0)e^{-\lambda_0(t-t_0)}, \tag{4}$$

that is, the closed loop system with continuous state feedback is globally exponentially stable (GES). We shall refer to λ_0 as the (continuous time) decay rate of V.

Consider now the case where the control input is kept constant between sampling times, that is,

$$u(t) = Kx_k \tag{5}$$

for all $t \in [t_k, t_{k+1})$ and all $k \geq 0$. Let $\lambda \in \mathbb{R}$ be a desired decay rate for V such that $0 < \lambda < \lambda_0$. If the sequence of sampling times $(t_k)_{k=0}^{+\infty}$ is such that

$$V(x(t)) \leq V(x_k)e^{-\lambda(t-t_k)} \tag{6}$$

holds for all $t \in [t_k, t_{k+1})$ and all $k \geq 0$, then the function V will satisfy

$$V(x(t)) \leq V(x_0)e^{-\lambda(t-t_0)} \tag{7}$$

for all $t \geq t_0$. To compute $(t_k)_{k=0}^{+\infty}$ such that (6) holds, the event scheduler simulates the evolution of $x(t)$ by using a copy of the plant's dynamics (1) with the control input as in (5). Given $x \in \mathbb{R}^n$, let $\xi_x \in \mathbb{R}^n$ satisfy

$$\dot{\xi}_x(t) = A\xi_x(t) + BKx, \quad \xi_x(0) = x \tag{8}$$

for all $t \geq 0$. Consider the function $h : \mathbb{R}_{\geq 0} \times \mathbb{R}^n \to \mathbb{R}$ defined as

$$h(\tau, x) = V(\xi_x(\tau)) - V(x)e^{-\lambda\tau}. \tag{9}$$

With *ideal* scheduling, the next sampling time is given by

$$t_{k+1} = t_k + \tau_{\text{ideal}}(x_k) \tag{10}$$

where the function $\tau_{\text{ideal}} : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ is defined as

$$\tau_{\text{ideal}}(x) = \max\{0 \leq \tau_2 \leq \tau_{\max} : h(\tau_1, x) \leq 0, \text{ for all } 0 \leq \tau_1 \leq \tau_2 \leq \tau_{\max}\}. \tag{11}$$

Here, $\tau_{\max}$ is a design parameter. This means that t_{k+1} is such that the interval $[t_k, t_{k+1})$ where (6) holds is of maximal length. It is shown in [8] that there exists a minimum time interval between consecutive sampling times, that is, for all $x \in \mathbb{R}^n$ there exists a positive constant $\tau_{\min}$ such that $\tau_{\text{ideal}}(x) \geq \tau_{\min}$. An implicit formula for the computation of $\tau_{\min}$ is given in [8, Lemma 4.1].

To evaluate (11), the solution or flow of (8) has to be computed, and this has to be done every time a sampling action is carried out. Hence, determining t_{k+1} can become computationally intensive. To mitigate this issue a gridding approach is employed. The *gridded* event scheduler computes the next sampling interval according to (10) where the function τ_{ideal} is replaced by the function $\tau_{\text{gridded}} : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ defined as

$$\tau_{\text{gridded}}(x) = \Delta \max\{1 \leq d_2 \leq M : h(d_1\Delta, x) \leq 0, \quad \text{for all } 1 \leq d_1 \leq d_2 \leq M\}. \quad (12)$$

Here, $0 < \Delta \leq \tau_{\min}$ and $M \in \mathbb{N}$ are design parameters. In [8] it is shown that no loss of asymptotic stability occurs if gridded scheduling is used instead of the ideal one, but there is a decrease in performance.

In summary, if the sampling intervals are selected using either (11) or (12), then (7) is satisfied and $(\tau_k)_{k=0}^{+\infty}$ is lower bounded by a positive constant.

III. SELF-TRIGGERED OUTPUT FEEDBACK STABILIZATION

When only the sampled output of the plant is available, we use the estimated state provide by the observer to replace the plant's actual state, that is,

$$u(t) = K\hat{x}_k, \forall t \in [t_k, t_{k+1}) \quad (13a)$$

$$t_{k+1} = t_k + \tau_{\text{gridded}}(\hat{x}_k). \quad (13b)$$

Let $\tilde{x}_k = x_k - \hat{x}_k$ for $k \geq 0$ denote observation errors. With the above changes, we have the following lemma.

Lemma 1: The self-triggered controller defined by (13) renders the plant (1) exponentially input-to-state stable (EISS) with respect to observation errors, that is, there exist positive constants σ and γ such that

$$\|x(t)\| \leq \sigma e^{-\lambda(t-t_0)}\|x_0\| + \gamma \max_{j\in\{0,1,\ldots,k\}} \|\tilde{x}_j\|$$

for all $t \geq t_0$, where $k = \max\{p \geq 0 : t_p \leq t\}$.

Proof: Omitted due to space constraints. ■

Lemma 1 clearly illustrates that because of the cascade interconnection between the observer and the subsystem formed by the plant and the self-triggered controller, asymptotic stability of the former implies asymptotic stability of the latter. The main focus of the paper is on guaranteeing the existence of an asymptotic discrete time observer for any sequence of sampling times generated by the event scheduler. Our proposed solution uses a general purpose time-varying observer for discrete time-varying linear systems. To guarantee asymptotical stability of this observer, an observability analysis of the discrete time equivalent of the continuous plant must first be performed.

A. Discrete time observer

The discrete time equivalent of (1) is

$$x_{k+1} = F_k x_k + G_k u_k \quad (14a)$$

$$y_k = Cx_k \quad (14b)$$

where $F_k = e^{A\tau_k}$ and $G_k = \int_0^{\tau_k} e^{As}\mathrm{d}sB$. In this paper, we consider discrete time observers whose dynamics are of the form

$$\hat{x}_{k+1} = F_k\hat{x}_k + G_k u_k + H_k(y_k - C\hat{x}_k), \quad (15)$$

where $H_k \in \mathbb{R}^{n\times p}$ are gain matrices to be selected. From (14) and (15), we can write the observation error dynamics as a discrete time switched system where τ_k is the switching signal, as follows:

$$\tilde{x}_{k+1} = [F(\tau_k) - H_k C]\,\tilde{x}_k, \quad \tau_k \in \mathcal{T}, \quad (16)$$

where $\mathcal{T}$ denotes the set of possible sampling intervals, that for a gridded event scheduler is a finite set of equally spaced values of the form

$$\mathcal{T} = \{d\Delta : d = 1, 2, \ldots, M\}. \quad (17)$$

The main issue at this point is how to produce a sequence of observer gain matrices $(H_k)_{k=0}^{+\infty}$ such that (16) is GAS for all possible sequences of sampling intervals $(\tau_k)_{k=0}^{+\infty} \in \mathcal{T}$. The first question that needs answering is whether the unforced ($u_k = 0$) discrete time equivalent of the plant is observable (in some sense). In this paper, we only analyze the observability properties of (14) when gridded scheduling is used. Although the continuous time plant is assumed observable, sampling may cause a loss observability for certain choices of Δ. Moreover, since we are dealing with a time-varying system, several types of observability are possible. Consider the unforced response of (1) with sampled output for $t \in [t_k, t_{k+1})$,

$$\dot{x}(t) = Ax(t) \quad (18a)$$

$$y(t_k) = Cx(t_k). \quad (18b)$$

Because of the structure of (17), the discrete time equivalent of (18) can be written as

$$x_{k+1} = E^{d_k} x_k \quad (19a)$$

$$y_k = Cx_k \quad (19b)$$

where $E = e^{A\Delta}$ and $d_k = \frac{t_{k+1}-t_k}{\Delta} \in \{1, 2, \ldots, M\}$. If for some reason the sequence of sampling intervals were formed by a repetitive pattern as, for example,

$$(\underbrace{\Delta, 4\Delta, 3\Delta, 2\Delta}_{\text{periodic frame}}, \Delta, 4\Delta, 3\Delta, 2\Delta, \ldots),$$

the discrete time system would be periodic and standard analysis tools could be used (see, e.g., [12]). In this paper, the sequence is generally not periodic.

Suppose (18) is discretized with a constant sampling period T. One question that is natural to ask is whether the discretized pair (e^{AT}, C) is observable when the original pair (A, C) is. The answer depends on the choice of T.

A sufficient condition to guarantee that observability is preserved is to prevent T from being a pathological sampling period of A (see, e.g, [13]). Let $\sigma(A)$ denote the spectrum of matrix A (the set of all eigenvalues of A). A sampling period T is called *nonpathological* if, for all $\lambda, \mu \in \sigma(A)$ and all $k \in \mathbb{Z}\backslash\{0\}$

$$\lambda - \mu \neq \mathrm{i}\frac{2\pi k}{T}.$$

As a special case, if A has only real eigenvalues, then all sampling periods are nonpathological.

In the following sections, we will show that an appropriate choice of Δ allows us to carry over the observability of the original continuous time-invariant plant (18) to the discrete time-varying system (19).

B. Observability of linear systems

In this section, we briefly present the tools required to analyse the observability properties of (19). Further details may be found in [10, Chapters 25 and 29] or in any textbook on linear system theory.

Consider the following discrete time-varying linear system

$$x_{k+1} = A_k x_k \tag{20a}$$

$$y_k = C x_k, \tag{20b}$$

where $x_0 \in \mathbb{R}^n$ is a given initial condition, $A_k \in \mathbb{R}^{n\times n}$ is a time-varying matrix and, to simply matters, the matrix $C \in \mathbb{R}^{p\times n}$ is constant but all definitions also hold for the time-varying case. The observability matrix associated with (20) or with the pair (A_k, C) on an interval $[k_o, k_f]$, with $k_f \geq k_o + 1$, is defined as

$$\mathcal{O}(k_o, k_f) = \begin{bmatrix} C \\ C\Phi(k_o+1, k_o) \\ C\Phi(k_o+2, k_o) \\ \vdots \\ C\Phi(k_f-1, k_o) \end{bmatrix} \in \mathbb{R}^{(k_f-k_o)p\times n},$$

where $\Phi(k, j)$ is the transition matrix associated with (20). A pair (A_k, C) is *observable* on $[k_o, k_f]$ if and only if $\operatorname{rank}\mathcal{O}(k_o, k_f) = n$. A pair (A_k, C) is said to be *l-step observable* if there exists a positive integer l such that, for all $k \geq 0$, $\operatorname{rank}\mathcal{O}(k, k+l) = n$. If $A_k = A$ for all $k \geq 0$, then $\mathcal{O}(k, k+n)$ is the standard observability matrix of linear time-invariant systems.

Another way of characterizing observability is by resorting to Gramian matrices. The system (20) is observable on $[k_o, k_f]$ if and only if the *observability Gramian*

$$M_{\mathcal{O}}(k_o, k_f) = \sum_{j=k_o}^{k_f-1} \Phi^\top(j, k_o) C^\top C \Phi(j, k_o)$$

is a positive definite matrix. Note that the observability Gramian can also be written as

$$M_{\mathcal{O}}(k_o, k_f) = \mathcal{O}^\top(k_o, k_f)\mathcal{O}(k_o, k_f). \tag{21}$$

If there exist a positive integer l, positive real numbers ϵ_1 and ϵ_2 such that, for all $k \geq 0$,

$$\epsilon_1 I \preceq M_{\mathcal{O}}(k-l+1, k+1) \preceq \epsilon_2 I,$$

then (20) is said to be *completely observable*. This kind of observability will be important later on to guarantee asymptotic stability of a discrete time observer. If A_k is bounded for all $k \geq 0$, then the existence of ϵ_2 is guaranteed. Note that, in general, l-step observability does not imply complete observability.

A complementary notion to observability is that of reconstructibility (sometimes simply constructibility). It suffices to say that if A_k is invertible for all $k \geq 0$, then observability is equivalent to reconstructibility.

C. Observability analysis of (19)

In [9], the authors resorted to the van der Waerden's Theorem to give a sufficient condition for observability of a discretized switched linear system under arbitrary switching. For positive integers n and M, the van der Waerden number $W(n, M)$ is the least integer w such that any partition of $[1, w]$ into M parts has a part that contains a n-term arithmetic progression. The celebrated theorem of van der Waerden proves the existence of $W(n, M)$.

Unlike [9], where the sampling period was fixed and the switching occurred in the plant's output matrix, our switching signal is the sequence of sampling intervals which induces a switching in the plant's state matrix while its output matrix is fixed. For this reason, we need to use a slightly different, yet equivalent, formulation of the van der Waerden's Theorem borrowed from [14]. Let $G(n, M)$ denote the smallest positive integer g such that if $(a_1, a_2, \ldots, a_g)$ is a strictly increasing sequence of integers with gaps bounded by M, that is, $1 \leq a_{j+1} - a_j \leq M$ for $j = 1, 2, \ldots g-1$, then $(a_1, a_2, \ldots, a_g)$ contains a n-term arithmetic progression.

Suppose k is fixed and let $l \geq 1$. The observability matrix associated with (19) on $[k, k+l]$ can be written as

$$\mathcal{O}(k, k+l) = \begin{bmatrix} C \\ CE^{d_k} \\ CE^{(d_k+d_{k+1})} \\ \vdots \\ CE^{\left(\sum_{j=k}^{k+l-2} d_j\right)} \end{bmatrix} = \begin{bmatrix} CE^{q_0} \\ CE^{q_1} \\ CE^{q_2} \\ \vdots \\ CE^{q_{l-1}} \end{bmatrix}, \tag{22}$$

where $(q_i)_{i=0}^{l-1}$ denotes the sequence of exponents of matrix E on $[k, k+l]$, that is,

$$q_0 = 0, q_1 = d_k, q_2 = d_k + d_{k+1}, \ldots, q_{l-1} = \sum_{j=k}^{k+l-2} d_j.$$

Notice that $q_{i+1} - q_i = d_{k+i}$ for $i = 0, 1, \ldots, l-2$. Hence, the sequence $(q_i)_{i=0}^{l-1}$ satisfies $1 \leq q_{i+1} - q_i \leq M$ and we can apply the van der Waerden's Theorem. If we pick $l \geq G(n, M)$, then $(q_i)_{i=0}^{l-1}$ will contain at least one arithmetic progression of length n. Select one such progression and let $0 \leq i_0 \leq l-n$ denote the index of the first term of that progression and r its rate. We have that

$$q_{i_j} = q_{i_0} + rj \text{ for } j = 0, 1, \ldots, n-1. \tag{23}$$

The rate r is between 1 and $R(n, M)$ where

$$R(n, M) = M\left(\left\lceil \frac{G(n, M)}{n-1} \right\rceil - 1\right). \tag{24}$$

TABLE I
KNOWN VALUES OF $G(n, M)$

n \ M	**1**	**2**	**3**	**4**	**5**	**6**	$\cdots$	M
2	2	2	2	2	2	2	$\cdots$	2
3	3	5	9	11	17	23		
4	4	10	26					
5	5	19						
6	6	37						
$\vdots$	$\vdots$							
n	n							

Because of (23), the matrix $\mathcal{O}(k, k+l)$ given in (22) contains the following submatrix:

$$\begin{bmatrix} CE^{q_{i_0}} \\ CE^{q_{i_1}} \\ CE^{q_{i_2}} \\ \vdots \\ CE^{q_{i_{n-1}}} \end{bmatrix} = \begin{bmatrix} C \\ CE^r \\ C(E^r)^2 \\ \vdots \\ C(E^r)^{n-1} \end{bmatrix} E^{q_{i_0}} = \mathcal{O}(E^r, C)E^{q_{i_0}}.$$

Since E is invertible and therefore also $E^{q_{i_0}}$, if $\operatorname{rank}\mathcal{O}(E^r, C) = n$ for all possible r, then $\operatorname{rank}\mathcal{O}(k, k+l) = n$, that is, the system is observable on $[k, k+l]$. Because the choice of l does not depend on k, the same conclusion is valid for all $k \geq 0$, which means that the system is $G(n, M)$-step observable. The condition "$\operatorname{rank}\mathcal{O}(E^r, C) = n$ for all possible r" is satisfied if, for $r = 1, 2, \ldots, R(n, M)$, $r\Delta$ is a nonpathological sampling period of A.

As an example, we consider two special cases.

- If the set $\mathcal{T}$ has only one element ($M = 1$), then $G(n, 1) = n$ and $R(n, 1) = 1$. We thus recover the usual criterion for preservation of observability, that is, Δ being a nonpathological sampling period of A. In this case the system is n-step observable.
- If the state of the plant has dimension $n = 2$, then $G(2, M) = 2$ and $R(2, M) = M$. Observability is preserved if, for $r = 1, 2, \ldots, M$, $r\Delta$ is a nonpathological sampling period of A. In this case the system is 2-step observable.

Unfortunately, only a few values of $G(n, M)$ are known exactly. These are given in Table I. Only upper bounds are known for the rest and these grow at an enormous rate with both n and M. This limits the applicability of the result for plants of large dimension or for scheduling with a high number of grid points. Nonetheless, the set of pathological sampling intervals is countable. Moreover, it allows us to prove a more conservative but simpler condition that guarantees the preservation of observability. Let

$$\mathcal{W}(A) = \big\{ \tfrac{1}{2} |\,\mathfrak{Im}\,\lambda - \mathfrak{Im}\,\mu| : \lambda, \mu \in \sigma(A), \\ \lambda \neq \mu, \mathfrak{Re}\,\lambda = \mathfrak{Re}\,\mu \big\} \quad (25)$$

represent the set of all natural frequencies of A, and let $\omega_s = \frac{2\pi}{\Delta}$. If for all $\omega \in \mathcal{W}(A)$ we have $\frac{\omega}{\omega_s} \notin \mathbb{Q}$, then $r\Delta$ is nonpathological for all $r \in \mathbb{N}$. That is, if the largest sampling frequency ω_s and all natural frequencies $\omega \in \mathcal{W}(A)$ are what is called *irrationally related*, then observability is preserved.

D. Asymptotic observer

We will use a generic result that applies to discrete time-varying linear systems and that provides a time-varying observer gain matrix. Let

$$M_{\mathcal{R}}^{\alpha}(k_o, k_f) = \sum_{j=k_o}^{k_f-1} \alpha^{4(j-k_f+1)} \Phi^\top(j, k_f) C^\top C \Phi(j, k_f)$$

where α is a positive constant.

Lemma 2 (Theorem 29.2 in [10]): If the system (19) is completely reconstructible, then the state observer (15) with time-varying gain matrix given by

$$H_k = \left[M_{\mathcal{R}}^{\alpha}(k - l + 1, k + 1)\right]^{-1} (A_k^{-1})^\top C^\top \qquad (26)$$

is GES with decay rate α.

In order to be able to use this result, we need to strengthen the observability properties of (19). We will do this by exploiting the fact that only a finite number of sampling intervals are possible. From the previous analysis, we know that (19) is $G(n, M)$-step observable. As mentioned before, l-step observability does not imply complete observability. However, in our case, due to the finiteness of the set of sampling intervals, this is in fact true. To see this, let $\mathcal{X} \subset \mathbb{R}^{np\times n}$ denote the set of all possible submatrices of $\mathcal{O}(k, k+l)$ that satisfy the arithmetic progression condition. Since M is finite, the set $\mathcal{X}$ is a finite set. Let

$$\epsilon_1 = \min_{X \in \mathcal{X}} \lambda_{\min}(X^\top X).$$

Since each matrix $X \in \mathcal{X}$ correspond to a submatrix of $\mathcal{O}(k, k+l)$ that verifies $\operatorname{rank} X = n$, we have that $\lambda_{\min}(X^\top X) > 0$ for all $X \in \mathcal{X}$, and therefore $\epsilon_1 > 0$. Without loss of generality, suppose $\mathcal{O}(k, k+l)$ can be written as $\mathcal{O}(k, k+l)^\top = [X^\top \;\; Y^\top]$ where the matrix $X \in \mathcal{X}$ denotes the observable submatrix and the matrix $Y \in \mathbb{R}^{(l-n)p\times n}$ represents the remaining entries (this can always be done by multiplying $\mathcal{O}(k, k+l)$ on the left by an appropriate permutation matrix, which is invertible). Thus, by (21), we have that

$$M_{\mathcal{O}}(k, k+l) = X^\top X + Y^\top Y \succeq X^\top X \succeq \epsilon_1 I.$$

Therefore, (19) is completely observable, or equivalently completely reconstructable. The preceding exposition proves the following lemma.

Lemma 3: If:

- (A, C) is an observable pair;
- the sequence of sampling intervals satisfies $(\tau_k)_{k=0}^{+\infty} \in \mathcal{T}$ where $\mathcal{T}$ is given by (17); and,
- $r\Delta$ is a nonpathological sampling period of A with $r = 1, 2, \ldots, R(n, M)$ where $R(n, M)$ is defined in (24);

then the discrete time-varying observer (15) with gain matrix given by (26) is GES.

E. Main Result

We are now ready to prove our main result.

Theorem 1: Under the assumptions of Lemma 3, the closed loop system formed by (1), (13) and (15) is GAS.

Proof: By Lemma 3, the error dynamics (16) are GES, that is, $\lim_{k\to\infty}\tilde{x}_k = 0$ for all $\tilde{x}_0 \in \mathbb{R}^n$. Since, by Lemma 1, the plant is EISS with respect to observation errors, resorting to [15, Lemma 4.7] we conclude that the closed loop system is GAS. ■

IV. AN ILLUSTRATIVE EXAMPLE

Consider the following third order linear system:

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -2 & 2 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, C^\top = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

The pairs (A, B) and (A, C) are controllable and observable, respectively. The spectrum of A is $\sigma(A) = \{1, \frac{1}{2} \pm \mathrm{i}\frac{\sqrt{3}}{2}\}$ and thus $\mathcal{W}(A) = \{\frac{\sqrt{3}}{2}\}$ by (25). We set the matrix $K = [-2 \quad 0 \quad -4]$ so that the eigenvalues of $A + BK$ are $\{-1, -\frac{1}{2} \pm \mathrm{i}\frac{\sqrt{3}}{2}\}$. The matrix P is obtained by solving (2) with $Q = I_3$, yielding a decay rate of $\lambda_0 = 0.1944$ and a minimum sampling interval of $\tau_{\min} = 439.94\,\mathrm{ms}$. We choose $\Delta = 400\,\mathrm{ms}$ and $M = 6$ that yields $G(3, 6) = 23$. Since $\omega_s = \frac{2\pi}{\Delta}$ and $\omega = \frac{\sqrt{3}}{2}$ are irrationally related, observability is preserved. The decay rate of the observer is set to $\alpha = 2$. We need to point out that to compute the time-varying gain matrix H_k in (26), an array with the last $G(n, M)$ values of d_k needs to be maintained. Therefore, an initialization phase is required to fill this array, a period during which the sampling interval is Δ and the observer runs with the constant gain matrix $H = [2 \quad 3 \quad 3]^\top$, that is stabilizing for periodic sampling with period Δ. In this example, this phase has a duration of $G(3, 6)\Delta = 9.2\,\mathrm{s}$.

The time evolution of $\|x(t)\|$ and $\|\tilde{x}_k\|$ is plotted in Fig. 2 for the initial conditions $x_0 = [-1 \quad 2 \quad 1]^\top$ and $\hat{x}_0 = [0 \quad 0 \quad 0]^\top$. As expected, both the plant's state and the observation error converge to zero. The sampling intervals generated by the event scheduler are shown in Fig. 3. One can clearly see the initialization phase where the sampling period is Δ. Once the self-triggering controller begins operating, the sampling intervals start showing some variability. A total of 86 samples were taken over the 50 s of simulation. Compared to periodic sampling with period Δ that would require 125 samples, the self-triggered strategy achieves a decrease of 31% in the number of samples taken.

V. CONCLUSION

In this paper we proposed a solution to the problem of self-triggered output feedback control of linear plants. The solution is based on the self-triggered state feedback controller presented in [8] in a cascade interconnection with a discrete time observer. The observability analysis of the system is based on work reported in [9] for discretized switched linear systems and the structure of the asymptotic observer is borrowed from [10]. Under some observability conditions, it is shown that the closed loop system is globally asymptotically stable.

Future work will focus on including in our observer based approach other scheduling methods proposed in the literature.

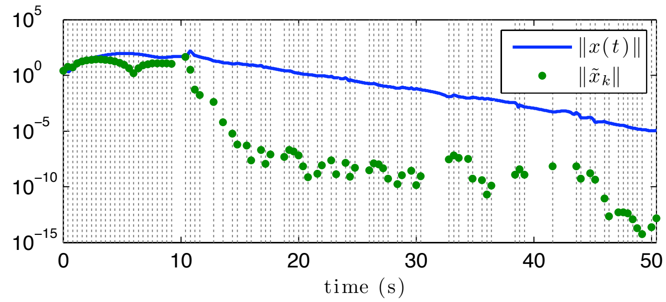

Fig. 2. Time evolution of the plant's state and the observation error norms on a logarithmic scale. Vertical dotted lines represent sampling times.

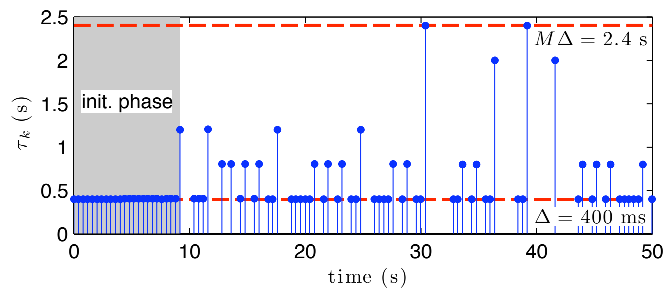

Fig. 3. Sequence of sampling intervals $(\tau_k)_{k\geq 0}$ with $\Delta = 400\,\mathrm{ms}$ and $M = 6$. During the first 9.2 s, the observer is being initialized.

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