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Modeling and Control of an Euler-Bernoulli Beam under Unknown Spatiotemporally Varying Disturbance

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Abstract— **In this paper, modeling and control of a vibrating Euler-Bernoulli beam is considered under the unknown external disturbances. The dynamics of the beam derived based on Hamilton's principle is represented by a partial differential equation (PDE) and four ordinary differential equations (ODEs) involving functions of both space and time. To deal with the system uncertainties and stabilize the beam, robust adaptive boundary control is developed at the tip of the beam based on Lyapunov's direct method. With the proposed boundary control, all the signals in the closed loop system are guaranteed to be uniformly bounded. The state of the system is proven to converge to a small neighborhood of zero by appropriately choosing the design parameters. The simulations are provided to illustrate the effectiveness of the proposed control.**

I. INTRODUCTION

Beam-type systems and their vibration suppression have received great attention due to large numbers of applications in industry. An Euler-Bernoulli beam is a model that can be used to describe many mechanical flexible systems such as flexible robotic manipulator [1], [2], flexible marine riser [3], [4] and moving strips [5]. The paper is motivated by the industrial applications in boundary control of vibrating flexible structure. Since the excessive vibration wastes energy, reduces the system quality, creates unwanted noise and results in premature fatigue failure, vibration suppression is well motivated to improve the performance of the system. Examples of practical applications where vibrating beams are exposed to undesirable spatiotemporally varying disturbances include flexible production risers used for offshore oil transportation, free hanging underwater pipelines, and drilling pipe for drilling mud transportation. Taking into account the unknown spatiotemporally varying disturbance of the beam leads to the appearance of oscillations, which makes the control problems of such systems relatively difficult.

Mathematically, the beam with a tip payload is represented by a set of infinite dimensional equations (i.e., PDEs describing the dynamics of the beam) coupled with a set of finite dimensional equations (i.e., ODEs describing the dynamics of the tip payload). The dynamics of the flexible mechanical system modeled by a set of PDEs is difficult to control due to the infinite dimensionality of the system. Approaches to control infinite dimensional flexible systems such as finite element method and assumed modes method [6] are based on the truncated finite-dimensional models of the system. The truncated models are obtained via the model analysis or spatial discretization, in which the flexibility is represented by an infinite number of modes by neglecting the higher frequency modes. Spillover [7] due to truncation of the model can lead to a unstable system, which should be avoided in practice.

In order to eliminate the spillover problem, boundary control [8], [9], [10], [11] combining with several other control techniques, such as variable structure control [12], sliding model control [2], energy-based robust control [13] have been developed for various infinite dimensional systems. Due to its advantages, boundary control has gained increasing attention in the literatures. A non-dissipative feedback that has been shown in [14] to exponentially stabilize an Euler-Bernoulli beam makes a Rayleigh beam and a Timoshenko beam unstable. Boundary control is developed in [15] to ensure a free vibrating beam exponentially stable. Fard and Sagatun construct a boundary control to exponentially stabilize a free transversely vibrating beam with axial tension in [16]. A boundary control is presented to stabilize beams by using active constrained layer damping [17]. In [18], backstepping boundary controller and observer are developed to stabilize the string and beam model respectively. However, in all the above works, control is designed with neglecting the unknown spatiotemporally varying disturbance. In this paper, we study the robust adaptive boundary control problem for an Euler-Bernoulli beam model with system parametric uncertainties and under both the unknown spatiotemporally varying distributed disturbance and unknown time-varying boundary disturbance. When bounds of the external disturbances are not available, the control problems become even more difficult. The main contributions of this paper include:

(i) A hybrid PDE/ODE model of the Euler-Bernoulli beam under unknown external disturbances is derived based on Hamilton's principle. The governing equations of system which can be used for the dynamic analysis of the beam-like structures are described as nonhomogeneous PDEs with the unknown disturbance terms.
(ii) A robust adaptive boundary control combined with a disturbance estimator for vibration suppression and uncertainties compensation is developed by using Lyapunov's direct method.
(iii) With the proposed boundary control, a new theorem is developed to illustrate that the uniform ultimate boundedness of the system. The closed loop system state will eventually converge to a compact set and the control performance of the system is guaranteed by suitably choosing the design parameters.

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II. Problem Formulation and Preliminaries

A. Dynamic analysis

For the beam-like systems shown in Fig. 1, $w(L,t)$, $\dot{w}(L,t)$ and $\ddot{w}(L,t)$ are the displacement, velocity and acceleration of the tip payload respectively, $u(t)$ is the boundary control force.

Remark 1: For clarity, notions $(\cdot)' = \partial(\cdot)/\partial x$ and $\dot{(\cdot)} = \partial(\cdot)/\partial t$ are used throught this paper.

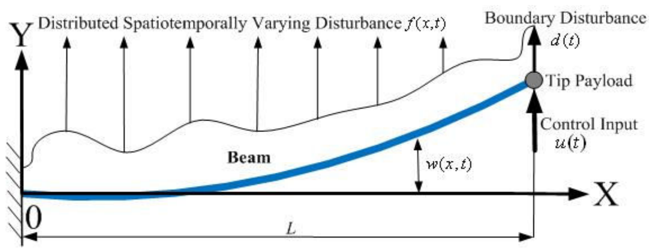

Fig. 1. A typical Euler-Bernoulli beam system.

The kinetic energy of the beam $E_k(t)$ can be represented as

$$E_k(t) = \frac{1}{2}m\left[\dot{w}(L,t)\right]^2 + \frac{1}{2}\rho\int_0^L \left[\dot{w}(x,t)\right]^2 dx, \quad (1)$$

where x and t represent the independent spatial and time variables respectively, m denotes the unknown mass of the payload at the right boundary of the beam, $w(x,t)$ is the displacement of the beam at the position x for time t, $\rho > 0$ is the uniform mass per unit length of the beam, and L is the length of the beam.

The potential energy $E_p(t)$ due to the bending can be obtained from

$$E_p(t) = \frac{1}{2}EI\int_0^L \left[w''(x,t)\right]^2 dx + \frac{1}{2}T\int_0^L \left[w'(x,t)\right]^2 dx, \quad (2)$$

where EI is the unknown bending stiffness which is the product of the elastic modulus E of the beam material and the area moment of inertia I of the beam cross-section. T is the unknown tension of the beam.

The virtual work done by disturbances including distributed disturbance $f(x,t)$ on the beam and boundary disturbance $d(t)$ on the tip payload is given by

$$\delta W_d(t) = \int_0^L f(x,t)\delta w(x,t)dx + d(t)\delta w(L,t). \quad (3)$$

The virtual work done by the control input force $u(t)$ which produces a transverse force for vibration suppression can be written as

$$\delta W_f(t) = u(t)\delta w(L,t). \quad (4)$$

Then, we have the total virtual work done on the system as

$$\begin{aligned}\delta W(t) &= \delta W_d(t) + \delta W_f(t)\\ &= \int_0^L f(x,t)\delta w(x,t)dx + \left[u(t)+d(t)\right]\delta w(L,t).\end{aligned} \quad (5)$$

Hamilton's principle [19] which permits the derivation of equations of motion from energy quantities in a variational form is represented by

$$\int_{t_1}^{t_2} \delta(E_k(t) - E_p(t) + W(t))dt = 0, \quad (6)$$

where t_1 and t_2 are two time instants, $t_1 < t < t_2$ is the operating interval and δ denotes the variational operator.

Applying Hamilton's principle Eq. (6), we derive the governing equations as

$$\rho\ddot{w}(x,t) + EIw''''(x,t) - Tw''(x,t) - f(x,t) = 0, \quad (7)$$

$\forall (x,t) \in (0,L)\times[0,\infty)$, and the boundary conditions of the system as

$$w'(0,t) = 0, \quad (8)$$
$$w''(L,t) = 0, \quad (9)$$
$$w(0,t) = 0, \quad (10)$$
$$-EIw'''(L,t) + Tw'(L,t) = u(t) + d(t) - m\ddot{w}(L,t), \quad (11)$$

$\forall t \in [0,\infty)$.

Assumption 1: For the unknown disturbances $f(x,t)$ and $d(t)$, we assume that there exists constants $\bar{f}, \bar{d} \in R^+$, such that $|f(x,t)| \le \bar{f}$, $\forall (x,t) \in [0,L]\times[0,\infty)$ and $|d(t)| \le \bar{d}$, $\forall (t) \in [0,\infty)$. Note that both the values of $\bar{f}$ and $\bar{d}$ are also unknown.

Remark 2: This is a reasonable assumption as the disturbances $f(x,t)$ and $d(t)$ have finite energy and hence are bounded, i.e., $f(x,t) \in \mathcal{L}_\infty([0,L])$ and $d(t) \in \mathcal{L}_\infty$.

B. Preliminaries

For the convenience of stability analysis, we present the following lemmas and properties for the subsequent development.

Lemma 1: [20] Let $\phi_1(x,t), \phi_2(x,t) \in R$ with $x \in [0,L]$ and $t \in [0,\infty)$, the following inequalities hold

$$\phi_1\phi_2 \le |\phi_1\phi_2| \le \phi_1^2 + \phi_2^2, \quad (12)$$
$$|\phi_1\phi_2| = \left|\left(\frac{1}{\sqrt{\delta}}\phi_1\right)(\sqrt{\delta}\phi_2)\right| \le \frac{1}{\delta}\phi_1^2 + \delta\phi_2^2, \quad (13)$$

$\forall \phi_1, \phi_2 \in R \quad and \quad \delta > 0$.

Lemma 2: [21] Let $\phi(x,t) \in R$ be a function defined on $x \in [0,L]$ and $t \in [0,\infty)$ that satisfies the boundary condition

$$\phi(0,t) = 0, \quad \forall t \in [0,\infty), \quad (14)$$

then the following inequalities hold:

$$\phi^2 \le L\int_0^L [\phi']^2 dx, \quad \forall x \in [0,L]. \quad (15)$$

If in addition to Eq. (14), the function $\phi(x,t)$ satisfies the boundary condition

$$\phi'(0,t) = 0, \quad \forall t \in [0,\infty), \quad (16)$$

then the following inequalities also hold:

$$[\phi']^2 \leq L\int_0^L [\phi'']^2 dx, \quad \forall x \in [0, L]. \tag{17}$$

Property 1: [4]: If the kinetic energy of the system (7) - (11), given by Eq. (1) is bounded $\forall t[0,\infty)$, then $\dot{w}(x,t)$, $\dot{w}'(x,t)$ and $\dot{w}''(x,t)$ are bounded $\forall(x,t) \in [0,L]\times[0,\infty)$.

Property 2: [4]: If the potential energy of the system (7) - (11), given by Eq. (2) is bounded $\forall t[0,\infty)$, then $w''(x,t)$, $w'''(x,t)$ and $w''''(x,t)$ are bounded $\forall(x,t) \in [0,L]\times[0,\infty)$.

III. Control Design

In this section, the adaptive boundary control $u(t)$ is designed at the right boundary of the flexible beam and the closed-loop stability of the system is proved by Lyapunov's direct method. The proposed control can compensate the system uncertainties when the system parameters EI, T, m and the bounds of the external disturbances $\bar{f}$ and $\bar{d}$ are all unknown. For the system given by governing Eq. (7) and boundary conditions Eqs. (8) - (11), the following robust adaptive control law is proposed as

$$u(t) = -P\hat{\Phi}(t) - ku_a(t) - \hat{d}(t), \tag{18}$$

where $P = [w'''(L,t) \quad -w'(L,t) \quad \dot{w}'(L,t) - \dot{w}'''(L,t)]$, parameter estimate vector $\hat{\Phi}(t) = [\widehat{EI}(t) \quad \widehat{T}(t) \quad \widehat{m}(t)]^T$, $\hat{d}(t)$ is the estimate of $\bar{d}$, k is a positive control parameter and the auxiliary signal $u_a(t)$ is defined as

$$u_a(t) = \dot{w}(L,t) - w'''(L,t) + w'(L,t). \tag{19}$$

We define parameter vector Φ, parameter error estimate vector $\tilde{\Phi}(t)$ and disturbance error estimate $\tilde{d}(t)$ as

$$\Phi = [EI \quad T \quad m], \tag{20}$$

$$\tilde{\Phi}(t) = \Phi - \hat{\Phi}(t) = [\widetilde{EI}(t) \quad \widetilde{T}(t) \quad \widetilde{m}(t)]^T, \tag{21}$$

$$\tilde{d}(t) = \bar{d} - \hat{d}(t). \tag{22}$$

After differentiating the auxiliary signal Eq. (19), multiplying the resulting equation by m, and substituting Eq. (11), we obtain

$$\begin{aligned} m\dot{u}_a(t) &= EIw'''(L,t) - Tw'(L,t) + d(t) \\ &\quad - m\dot{w}'''(L,t) + m\dot{w}'(L,t) + u(t) \\ &= P\Phi + u(t) + d(t). \end{aligned} \tag{23}$$

Substituting Eq. (18) into Eq. (23), we have

$$\begin{aligned} m\dot{u}_a(t) &= P\tilde{\Phi}(t) - ku_a(t) + d(t) - \hat{d}(t) \\ &\leq P\tilde{\Phi}(t) - ku_a(t) + \tilde{d}(t). \end{aligned} \tag{24}$$

The adaptation laws are designed as

$$\dot{\hat{\Phi}}(t) = \Gamma P^T u_a(t) - \zeta_1 \Gamma \hat{\Phi}(t), \tag{25}$$

$$\dot{\hat{d}}(t) = \gamma u_a(t) - \zeta_2 \gamma \hat{d}(t), \tag{26}$$

where $\Gamma \in R^{3\times3}$ is a diagonal positive-definite matrix, γ, ζ_1 and ζ_2 are positive constants. We define the maximum and minimum eigenvalue of matrix Γ as $\lambda_{\max}$ and $\lambda_{\min}$ respectively.

Remark 3: All the signals in the boundary control can be measured by sensors or obtained by a backwards difference algorithm. $w(L,t)$ can be sensed by a laser displacement senor at the right boundary of the beam, $w'(L,t)$ can be measured by an inclinometer and $w'''(L,t)$ can be obtained by a shear force sensor.

Remark 4: The control (18) is based on the original distributed parameter model Eqs. (7) - (11), and the spillover problems associated with traditional truncated model-based approaches caused by ignoring high-frequency modes in controller and observer design are avoided.

Consider the Lyapunov function candidate

$$\begin{aligned} V(t) &= V_1(t) + V_2(t) + \Delta(t) + \frac{1}{2}\tilde{\Phi}^T(t)\Gamma^{-1}\tilde{\Phi}(t) \\ &\quad + \frac{1}{2}\gamma^{-1}\tilde{d}^2(t), \end{aligned} \tag{27}$$

where the energy term $V_1(t)$ and the auxiliary term $V_2(t)$ and the small crossing term $\Delta(t)$ are defined as

$$\begin{aligned} V_1(t) &= \frac{\beta}{2}\rho\int_0^L [\dot{w}]^2 dx + \frac{\beta}{2}EI\int_0^L [w'']^2 dx \\ &\quad + \frac{\beta}{2}T\int_0^L [w']^2 dx, \end{aligned} \tag{28}$$

$$V_2(t) = \frac{1}{2}mu_a^2(t), \tag{29}$$

$$\Delta(t) = \alpha\rho\int_0^L x\dot{w}w' dx, \tag{30}$$

where α and β are two positive weighting constants. Note that the terms $V_1(t)$ and $V_2(t)$ are positive semi-definite while the term $\Delta(t)$ is arbitrary.

Lemma 3: The Lyapunov function candidate given by Eq. (27), can be upper and lower bounded as

$$\begin{aligned} &\lambda_1(V_1(t) + V_2(t) + ||\tilde{\Phi}(t)||^2 + \tilde{d}^2(t)) \leq V(t) \\ &\leq \lambda_2(V_1(t) + V_2(t) + ||\tilde{\Phi}(t)||^2 + \tilde{d}^2(t)), \end{aligned} \tag{31}$$

where λ_1 and λ_2 are two positive constants.

Proof: Substituting of Ineq. (12) into Eq. (30) yields

$$\begin{aligned} |\Delta(t)| &\leq \alpha\rho L\int_0^L ([w']^2 + [\dot{w}]^2)dx \\ &\leq \alpha_1 V_1(t), \end{aligned} \tag{32}$$

where

$$\alpha_1 = \frac{2\alpha\rho L}{\min(\beta\rho, \beta T)}. \tag{33}$$

Then, we obtain

$$-\alpha_1 V_1(t) \leq \Delta(t) \leq \alpha_1 V_1(t). \tag{34}$$

Considering α is a small positive weighting constant satisfying $0 < \alpha < \frac{\min(\beta\rho,\beta T)}{2\rho L}$, we can obtain

$$\alpha_2 = 1 - \alpha_1 = 1 - \frac{2\alpha\rho L}{\min(\beta\rho, \beta T)} > 0, \tag{35}$$

$$\alpha_3 = 1 + \alpha_1 = 1 + \frac{2\alpha\rho L}{\min(\beta\rho, \beta T)} > 1. \tag{36}$$

Then, we further have

$$0 \le \alpha_2 V_1(t) \le V_1(t) + \Delta(t) \le \alpha_3 V_1(t), \tag{37}$$

Given the Lyapunov function candidate Eq. (27), we obtain

$$\begin{aligned} 0 \le \gamma_1(V_1(t) + V_2(t)) \;&\le\; V_1(t) + V_2(t) + \Delta(t) \\ &\le\; \gamma_2(V_1(t) + V_2(t)), \end{aligned} \tag{38}$$

where $\gamma_1 = \min(\alpha_2, 1) = \alpha_2$ and $\gamma_2 = \max(\alpha_3, 1) = \alpha_3$ are positive constants. From the properties of matrix Γ, we obtain

$$\frac{1}{2\lambda_{\max}}||\tilde{\Phi}(t)||^2 \le \frac{1}{2}\tilde{\Phi}^T(t)\Gamma^{-1}\tilde{\Phi}(t) \le \frac{1}{2\lambda_{\min}}||\tilde{\Phi}(t)||^2. \tag{39}$$

Combining Eq. (27) and Ineqs. (38), (39), we have

$$\begin{aligned} &\lambda_1(V_1(t) + V_2(t) + ||\tilde{\Phi}(t)||^2 + \tilde{d}^2(t)) \le V(t) \\ &\le \lambda_2(V_1(t) + V_2(t) + ||\tilde{\Phi}(t)||^2 + \tilde{d}^2(t)), \end{aligned} \tag{40}$$

where $\lambda_1 = \min(\alpha_2, \frac{1}{2\lambda_{\max}}, \frac{1}{2\gamma})$ and $\lambda_2 = \max(\alpha_3, \frac{1}{2\lambda_{\min}}, \frac{1}{2\gamma})$ are two positive constants. ∎

Lemma 4: The time derivative of the Lyapunov function candidate Eq. (27) can be upper bounded with

$$\dot{V}(t) \;\le\; -\lambda V(t) + \psi, \tag{41}$$

where $\lambda > 0$ and $\psi > 0$.

Proof: Differentiating Eq. (27) with respect to time leads to

$$\begin{aligned} \dot{V}(t) \;=&\; \dot{V}_1(t) + \dot{V}_2(t) + \dot{\Delta}(t) + \tilde{\Phi}^T(t)\Gamma^{-1}\dot{\tilde{\Phi}}(t) \\ &+\gamma^{-1}\tilde{d}(t)\dot{\tilde{d}}(t). \end{aligned} \tag{42}$$

The first term of the Eq. (42)

$$\begin{aligned} \dot{V}_1(t) \;=&\; \beta\rho\int_0^L \dot{w}\ddot{w}dx + \beta EI\int_0^L w''\dot{w}''dx \\ &+\beta T\int_0^L w'\dot{w}'dx. \end{aligned} \tag{43}$$

Integrating Eq. (43) by part and using the boundary conditions, we have

$$\begin{aligned} \dot{V}_1(t) \;=&\; \beta\left[-EIw'''(L,t) + Tw'(L,t)\right]\dot{w}(L,t) \\ &+\beta\int_0^L f(x,t)\dot{w}dx. \end{aligned} \tag{44}$$

Substituting the Eq. (19) into Eq. (44) and using Ineqs. (13), we obtain

$$\begin{aligned} \dot{V}_1(t) \;\le&\; \frac{\beta EI}{2}u_a^2 - \frac{\beta EI}{2}\{[\dot{w}(L,t)]^2 + [w'''(L,t)]^2 \\ &+\; [w'(L,t)]^2\} + \frac{\beta}{\delta_1}|T - EI|[\dot{w}(L,t)]^2 \\ &+\; \beta\delta_1|T - EI|[w'(L,t)]^2 + \beta EIw'(L,t)w'''(L,t) \\ &+\; \beta\delta_2\int_0^L[\dot{w}]^2dx + \frac{\beta}{\delta_2}\int_0^L f^2(x,t)dx, \end{aligned} \tag{45}$$

where δ_1 and δ_2 are two positive constants. Substituting Eq. (24) into the second term of the Eq. (42), we have

$$\begin{aligned} \dot{V}_2(t) \;=&\; mu_a(t)\dot{u}_a(t) \\ =&\; -ku_a^2(t) + P\tilde{\Phi}(t)u_a(t) + \tilde{d}(t)u_a(t) \end{aligned} \tag{46}$$

The third term of the Eq. (42)

$$\begin{aligned} \dot{\Delta}(t) \;=&\; \alpha\rho\int_0^L(x\ddot{w}w' + x\dot{w}\dot{w}')dx \\ =&\; \alpha\int_0^L xw'\left[-EIw'''' + Tw'' + f(x,t)\right]dx \\ &+\alpha\rho\int_0^L x\dot{w}\dot{w}'dx. \end{aligned} \tag{47}$$

After integrating Eq. (47) by parts and applying the Ineqs. (13) and (17), we have

$$\begin{aligned} \dot{\Delta}(t) \;\le&\; -\alpha EILw'(L,t)w'''(L,t) - \frac{3\alpha EI}{2}\int_0^L[w'']^2dx \\ &+\frac{\alpha TL^2}{2}\int_0^L[w'']^2dx - \frac{\alpha T}{2}\int_0^L[w']^2dx \\ &+\frac{\alpha L}{\delta_3}\int_0^L f^2(x,t)dx + \alpha L\delta_3\int_0^L[w']^2dx \\ &+\frac{\alpha\rho L}{2}[\dot{w}(L,t)]^2 - \frac{\alpha\rho}{2}\int_0^L[\dot{w}]^2dx, \end{aligned} \tag{48}$$

where δ_3 is a positive constant. Substituting Eqs. (45), (46) and (48) into Eq. (27), and using Ineqs. (13) and (17), we obtain

$$\begin{aligned} \dot{V}(t) \;\le&\; -\left(\frac{\alpha\rho}{2} - \beta\delta_2\right)\int_0^L[\dot{w}]^2dx - (\frac{\beta LEI}{2} \\ &+\; \frac{3\alpha EI}{2} - \frac{\alpha TL^2}{2} - \delta_4 EI|\beta - \alpha L|L \\ &-\; \beta\delta_1 L|T - EI|)\int_0^L[w'']^2dx \\ &-\; \left(\frac{\alpha T}{2} - \alpha L\delta_3\right)\int_0^L[w']^2dx - \left(k - \frac{\beta EI}{2}\right)u_a^2 \\ &-\; \left(\frac{\beta EI}{2} - \frac{\beta}{\delta_1}|T - EI| - \frac{\alpha\rho L}{2}\right)[\dot{w}(L,t)]^2 \\ &-\; \left(\frac{\beta EI}{2} - \frac{EI}{\delta_4}|\beta - \alpha L|\right)[w'''(L,t)]^2 \\ &+\; \left(\frac{\beta}{\delta_2} + \frac{\alpha L}{\delta_3}\right)\int_0^L f^2(x,t)dx + \tilde{\Phi}^T(t)\Gamma^{-1}\dot{\tilde{\Phi}}(t) \\ &+\; \gamma^{-1}\tilde{d}(t)\dot{\tilde{d}}(t) + P\tilde{\Phi}(t)u_a(t) + \tilde{d}(t)u_a(t), \end{aligned} \tag{49}$$

where δ_4 is a positive constant. Substituting Eqs. (25) and (26) into Ineq. (49), we have

$$\begin{aligned} \dot{V}(t) \;\le&\; -\lambda_3[V_1(t) + V_2(t)] + \zeta_1\tilde{\Phi}^T(t)\hat{\Phi}(t) \\ &+\zeta_2\tilde{d}(t)\hat{d}(t) + \varepsilon \\ \le&\; -\lambda_3[V_1(t) + V_2(t)] - \frac{\zeta_1}{2}||\tilde{\Phi}(t)||^2 + \frac{\zeta_1}{2}||\Phi||^2 \\ &-\frac{\zeta_2}{2}\tilde{d}^2(t) + \frac{\zeta_2}{2}\bar{d}^2 + \varepsilon \\ \le&\; -\lambda_4[V_1(t) + V_2(t) + ||\tilde{\Phi}(t)||^2 + \tilde{d}^2(t)] + \psi, \end{aligned} \tag{50}$$

where $\lambda_4 = \min(\lambda_3, \frac{\zeta_1}{2}, \frac{\zeta_2}{2})$ is a positive constant and other constants k, α, β, δ_1, δ_2, δ_3 and δ_4 are chosen to satisfy the following conditions:

$$\alpha < \frac{\min(\beta\rho, \beta T)}{2\rho L}, \tag{51}$$

$$\frac{\beta EI}{2} - \frac{\beta}{\delta_1}|T - EI| - \frac{\alpha\rho L}{2} \geq 0, \tag{52}$$

$$\frac{\beta EI}{2} - \frac{EI}{\delta_4}|\beta - \alpha L| \geq 0, \tag{53}$$

$$\sigma_1 = \frac{\alpha\rho}{2} - \beta\delta_2 > 0, \tag{54}$$

$$\sigma_2 = \frac{\beta LEI}{2} + \frac{3\alpha EI}{2} - \frac{\alpha TL^2}{2} - \delta_4 EI|\beta - \alpha L|L - \beta\delta_1 L|T - EI| > 0, \tag{55}$$

$$\sigma_3 = \frac{\alpha T}{2} - \alpha L\delta_3 > 0, \tag{56}$$

$$\sigma_4 = k - \frac{\beta EI}{2} > 0, \tag{57}$$

$$\lambda_3 = \min\left(\frac{2\sigma_1}{\beta\rho}, \frac{2\sigma_2}{\beta EI}, \frac{2\sigma_3}{\beta T}, \frac{2\sigma_4}{m}\right) > 0, \tag{58}$$

$$\varepsilon = \left(\frac{\beta}{\delta_2} + \frac{\alpha L}{\delta_3}\right)\int_0^L \bar{f}^2 dx \in \mathcal{L}_\infty, \tag{59}$$

$$\psi = \varepsilon + \frac{\zeta_1}{2}||\Phi||^2 + \frac{\zeta_2}{2}\bar{d}^2. \tag{60}$$

Combining Ineqs. (38) and (50), we have

$$\dot{V}(t) \leq -\lambda V(t) + \psi, \tag{61}$$

where $\lambda = \lambda_4/\lambda_2 > 0$. ■

With the above lemmas, we are ready to present the following stability theorem of the closed-loop beam system.

Theorem 1: For the system dynamics described by (7) and boundary conditions (8) - (11), under Assumption 1, and the control law Eq. (18), given that the initial conditions are bounded, we can conclude that uniform boundedness (UB): the state of the closed loop system $w(x,t)$ will remain in the compact set Ω defined by

$$\Omega := \left\{ w(x,t) \in R \middle| \quad |w(x,t)| \leq D, \forall (x,t) \in [0,L] \times [0,\infty) \right\}, \tag{62}$$

where constant $D = \sqrt{\frac{2L}{\beta T\lambda_1}\left(V(0) + \frac{\psi}{\lambda}\right)}$.

Proof: Multiplying Eq. (41) by $e^{\lambda t}$ yields

$$\frac{\partial}{\partial t}(V(t)e^{\lambda t}) \leq \psi e^{\lambda t}. \tag{63}$$

Integrating of the above inequality, we obtain

$$V(t) \leq \left(V(0) - \frac{\psi}{\lambda}\right)e^{-\lambda t} + \frac{\psi}{\lambda} \leq V(0)e^{-\lambda t} + \frac{\psi}{\lambda} \in \mathcal{L}_\infty. \tag{64}$$

which implies $V(t)$ is bounded. Utilizing Ineq. (15) and Eq. (28), we have

$$\frac{\beta}{2L}Tw^2(x,t) \leq \frac{\beta}{2}T\int_0^L [w'(x,t)]^2 dx \leq V_1(t) + V_2(t) \leq \frac{1}{\lambda_1}V(t) \in \mathcal{L}_\infty. \tag{65}$$

Appropriately rearranging the terms of the above inequality, we obtain $w(x,t)$ is uniformly bounded as follows

$$|w(x,t)| \leq \sqrt{\frac{2L}{\beta T\lambda_1}\left(V(0)e^{-\lambda t} + \frac{\psi}{\lambda}\right)}, \ \forall x \in [0,L]. \tag{66}$$

■

Remark 5: From Eqs. (40), (64), and (65), we can obtain that $\tilde{\Phi}(t)$, $\tilde{d}(t)$, $V_1(t)$, and $V_2(t)$ are bounded $\forall t \in [0,\infty)$. Thus, $\dot{w}(x,t)$, $w''(x,t)$ and $w'(x,t)$ are bounded $\forall (x,t) \in [0,L] \times [0,\infty)$, and u_a is bounded $\forall t \in [0,\infty)$. Then, we can obtain that the kinetic energy Eq. (1) and the potential energy Eq. (2) of the system are also bounded. Using Properties 1 and 2, $\dot{w}'(x,t)$, $\dot{w}'''(x,t)$, $w'''(x,t)$ and $w''''(x,t)$ are bounded $\forall (x,t) \in [0,L] \times [0,\infty)$. Using Assumption 1, Eq. (7) and the above statements, we can state that $\ddot{w}(x,t)$ is also bounded $\forall (x,t) \in [0,L] \times [0,\infty)$. From the above information, it is shown that the proposed control Eq. (18) ensures all internal system signals uniformly bounded and the boundary control Eq. (18) is also bounded $\forall t \in [0,\infty)$.

Remark 6: It is shown that the increase in the control gain k will result in a larger σ_4, which will lead a greater λ_3. Then the value of λ will increase, which will reduce the size of Ω and produce a better vibration suppression performance. We can conclude that the bound of the system state $w(x,t)$ can be made arbitrarily small provided that the design control parameters are appropriately selected. However, increasing k will bring a high gain control scheme. Therefore, in practical applications, the design parameters should be adjusted carefully for achieving suitable transient performance and control action.

IV. Numerical Simulations

Consider a beam initially at rest $w(x,0) = \dot{w}(x,0) = 0$, and then is excited by the disturbances $f(x,t)$ and $d(t)$. The parameters of the beam is listed below:

Table 1: Parameters of the beam

Parameter	Description	Value
L	Length of beam	$100m$
EI	Bending stiffness	$5Nm^2$
T	Tension	$10N$
ρ	Mass per unit length	$0.1kg/m$
m	Mass of the tip payload	$1kg$

The boundary disturbance $d(t)$ on the tip payload and the distributed disturbance $f(x,t)$ on the beam generated by the following equations

$$d(t) = 1 + \sin(\pi t) + \sin(2\pi t) + \sin(3\pi t), \tag{67}$$

$$f(x,t) = 1[1 + \sin(0.1\pi xt) + \sin(0.2\pi xt) \; 1 + \sin(0.3\pi xt)] \times \frac{x}{1000L}. \tag{68}$$

Displacement of the beam for free vibration, i.e., $u(t) = 0$, under the external disturbances is shown in Fig. 2. Displacement of the beam with the proposed adaptive boundary control (18), by choosing $k = 100000$, $\zeta_1 = \zeta_2 = 0.001$, $\gamma = 1$ and $\Gamma = \mathrm{diag}\{1,1,1\}$ subjected to the disturbances is shown in Fig. 3. Figs. 2 and 3 illustrate that the proposed adaptive boundary control is able to stabilize the beam at the small neighborhood of its equilibrium position. The corresponding boundary control $u(t)$ is shown in Fig. 4.

V. Conclusion

In this paper, robust adaptive boundary control has been studied for a vibrating beam under unknown spatiotemporally varying distributed disturbance and unknown time-varying boundary disturbance. Both the parametric uncertainties and disturbances uncertainties have been compensated. The proposed control has been proved to ensure all the signals of the closed-loop system uniformly bounded despite the presence of an unknown payload mass, stiffness and tension. Numerical simulations have been provided to verify the effectiveness of the proposed boundary control.

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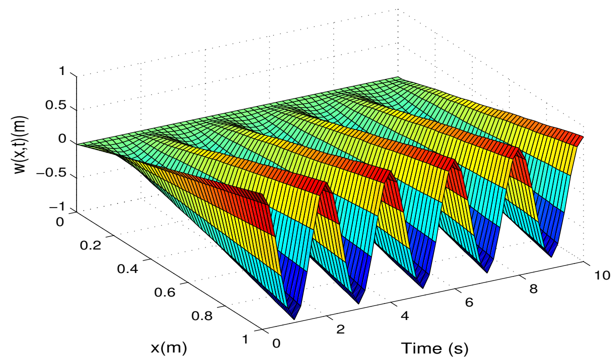

Fig. 2. Displacement of the beam without control.

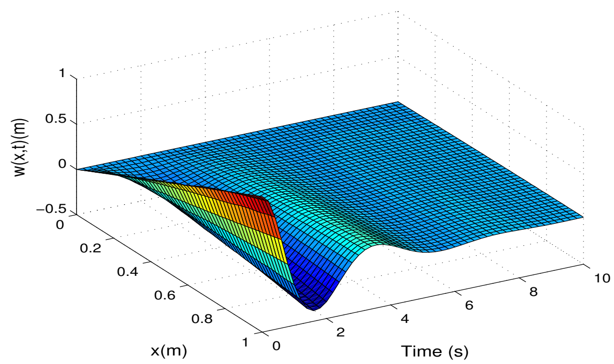

Fig. 3. Displacement of the beam with adaptive boundary control.

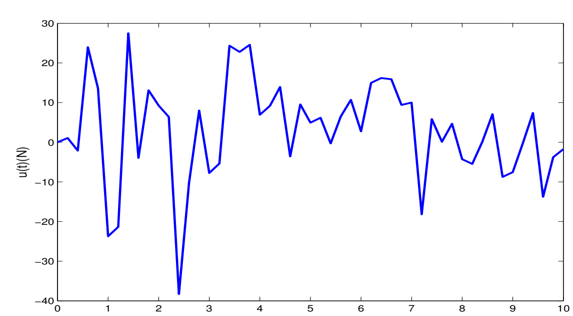

Fig. 4. Control input $u(t)$.