2011 American Control Conference
on O'Farrell Street, San Francisco, CA, USA
June 29 - July 01, 2011

Experimental Evaluation of Model Predictive Control of Ball and Beam Systems

Naoyuki Hara, Masaaki Takahashi and Keiji Konishi

***Abstract*— In this paper, we discuss the control of a ball and beam system subject to an input constraint. Model predictive control (MPC) approaches are employed to derive a nonlinear control law satisfying the constraint. The control law is given by solving the optimization problem at each sample time, where the primal-dual interior point algorithm is implemented and used as the optimization solver. An experimental comparison of three control methods, two different MPCs and saturated LQR, has been presented for the control of the ball and beam system.**

I. INTRODUCTION

Almost all physical systems have constraints such as limitations on the input and output signals. Performance degradation or, at worst, instability might occur if these constraints are not taken into account in the design of control systems. Model predictive control (MPC) is one of the most advanced control methods that can deal with system constraints explicitly [1]. In MPC, the control input is calculated by solving an optimization problem on-line, and hence, MPCs have been traditionally used for systems with relatively slow dynamics. Recently, several studies have proposed various optimal and suboptimal MPC methods for reducing on-line computational burden of the optimization of MPC (e.g., [2], [3]). Using these methods, MPCs can be applied to a broader class of systems having fast dynamics, such as mechanical systems and electrical systems. Further, it is important that MPCs not only satisfy constraints but also provide robustness against uncertainty and disturbance; therefore, robust MPCs have also been studied (e.g., [4], [5], [6]).

In this paper, we consider the control of a ball and beam system (Fig. 1), which is a popular mechanical system in control laboratories. The system is driven by a DC motor and has a constraint on the input voltage reference signal. Various nonlinear control methods have been applied to ball and beam systems; however, very few studies have evaluated the use of MPCs. In [7], a nonlinear MPC was applied to the ball and beam systems; however, no constraints were imposed. In this paper, we employ two different MPC methods, standard MPC and MPC with disturbance attenuation. A comparison between these MPC methods is presented, and the performance of the MPC laws is evaluated experimentally. We implemented the infeasible primal-dual interior point method [8] in C language to solve the optimization problems on-line in MPC.

N. Hara and K. Konishi are with the Department of Electrical and Information Systems, Osaka Prefecture University, Osaka 599-8531, Japan. M. Takahashi is with Kawasaki Heavy Industries, Ltd. Corresponding author: N. Hara, n-hara@eis.osakafu-u.ac.jp

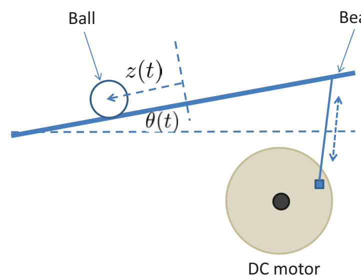

Fig. 1. Ball and beam system.

II. BALL AND BEAM SYSTEM

The state-space equation for the ball and beam system (Fig. 1) linearized around the origin is given by

$$\dot{x}(t) = Ax(t) + B_2u(t) + B_1w(t), \quad (1)$$
$$x(t) := [z(t), \dot{z}(t), \theta(t), \dot{\theta}(t)]^{\mathrm{T}}$$

where $x(t)$, $u(t)$, $w(t)$ denote the state, control input (voltage signal to the DC motor), and disturbance, respectively. The objective of the control is to regulate the ball to the origin by applying an appropriate control input to the DC motor. This should be achieved by satisfying constraint $u_{\min} \le u(t) \le u_{\max}$ on the control input.

III. MODEL PREDICTIVE CONTROL

We consider a discretized system of (1) with sample time T[s]:

$$x_{k+1} = Ax_k + B_2u_k + B_1w_k, \quad (2)$$
$$\text{s.t.} \quad u_{\min} \le u_k \le u_{\max}. \quad (3)$$

Here, we briefly review the MPC and MPC with disturbance attenuation.

A. MPC

Consider the finite-horizon constrained linear quadratic (LQ) control problem

$$\mathcal{P}_{LQ} : \min_{u_0,u_1,\ldots,u_{N-1}} J = \sum_{k=0}^{N-1} \{x_k^{\mathrm{T}}Qx_k + u_k^{\mathrm{T}}Ru_k\} + x_N^{\mathrm{T}}Px_N$$
$$\text{s.t.} \quad u_{\min} \le u_k \le u_{\max}, k = 0, 1, \ldots, N-1$$

where N is the prediction horizon, $Q > 0$, $R > 0$ are weighting matrices, and $P > 0$ is the stabilizing solution to the Riccati equation

$$P = A^{\mathrm{T}}PA + Q - A^{\mathrm{T}}PB(R + B^{\mathrm{T}}PB)^{-1}B^{\mathrm{T}}PA. \quad (4)$$

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This is a finite-horizon control problem and can be reformulated as an optimization problem (quadratic program). In MPC, the control input is calculated in the receding horizon manner as follows: 1) solve the optimization problem and obtain a control input sequence over the prediction horizon, 2) apply only the first control move to the system, and then, 3) based on the new state measurement and shifted prediction horizon, solve the optimization problem again and repeat the same procedure. Note that the control law is equivalent to the infinite horizon LQ control when no constraints are imposed because the terminal weight P is given by the solution to the Riccati equation (4).

B. MPC with disturbance attenuation

Although MPC is a feedback control method and has disturbance attenuation property to some extent, the effect of disturbance is not taken into account explicitly. Here, we present an MPC that aims at attenuating the effect of disturbance (or model uncertainty) explicitly [6]. For the cost functional

$$J_1 = \sum_{k=0}^{N-1} \{x_k^{\mathrm{T}} Q x_k + u_k^{\mathrm{T}} R u_k - \gamma^2 w_k^{\mathrm{T}} R_1 w_k\} + x_{k+N}^{\mathrm{T}} P x_{k+N}, \tag{5}$$

consider the following min-max control problem:

$$\begin{aligned} \mathcal{P}_1 : \min_{V} \ \ \max_{W} \ \ & J_1 \\ \text{s.t.} \ \ & u_{\min} \le K\tilde{x}_k + v_k \le u_{\max}, \\ & \tilde{x}_{k+1} = A_c \tilde{x}_k + B_2 v_k, \ \tilde{x}_0 = x_0 \\ & k = 0, 1, \ldots, N-1, \\ & A_c := A + B_2 K, \\ & V := [v_0^{\mathrm{T}}, \ldots, v_{N-1}^{\mathrm{T}}]^{\mathrm{T}}, \\ & W := [w_0^{\mathrm{T}}, \ldots, w_{N-1}^{\mathrm{T}}]^{\mathrm{T}}. \end{aligned}$$

The matrix K represents the pre-stabilizing feedback gain, which reduces the conservativeness of the resulting MPC law. For a small value of γ in (5), the control aims to attenuate the effect of disturbance w_k, whereas for a relatively large value of γ, the control becomes equivalent to the standard MPC. In order to guarantee the existence of a disturbance sequence maximizing J_1, the parameter γ should be chosen to satisfy the following condition:

$$\begin{aligned} \gamma > 0 : \hat{R}_1 > \bar{B}_\gamma^T \tilde{Q}_{\bar{A}_c} \bar{B}_\gamma + B_{\gamma\phi}^T P B_{\gamma\phi}, \\ B_\gamma := \gamma^{-1} B_1, \ B_{\gamma\phi} := \gamma^{-1} B_{1\phi}, \\ B_{i\phi} := [A_c^{N-1} B_i, \ A_c^{N-2} B_i, \ldots, B_i], i = 1, 2, \\ \tilde{Q}_{\bar{A}_c} := (I - \bar{A}_c)^{-T} \hat{Q}' (I - \bar{A}_c)^{-1}, \\ Q' := (Q + K^T R K), \end{aligned} \tag{6}$$

where the symbols $\bar{\ }$ and $\hat{\ }$ denote the matrix operations defined by

$$\bar{G} := \begin{bmatrix} 0 & 0 \\ \mathrm{diag}(G, \ldots, G) & 0 \end{bmatrix}, \ \hat{G} := \mathrm{diag}(G, \ldots, G)$$

for any matrix G. Under the condition (6), there exists the maximizing disturbance sequence, and the min-max optimization problem is then reformulated as the following minimization problem

$$\begin{aligned} \mathcal{P}_1' : \min_{V} \ & J_1' \\ \text{s.t.} \ \ & u_{\min} \le K\tilde{x}_k + v_k \le u_{\max}, \\ & \tilde{x}_{k+1} = A_c \tilde{x}_k + B_2 v_k, \ \tilde{x}_0 = x_0 \\ & k = 0, 1, \ldots, N-1, \\ J_1' := & V^{\mathrm{T}} H V + 2 V^{\mathrm{T}} F^{\mathrm{T}} x_0 + x_0^{\mathrm{T}} Y x_0, \end{aligned}$$

where H, F, Y are appropriate matrices [6]. This optimization problem yields the optimal sequence $v_0^*, v_1^*, \ldots, v_{N-1}^*$. We employ only the first move v_0^* for the current control input: $u_0 = Kx_0 + v_0^*$, and repeat the same procedure for the new state measurement at each sample time over a shifted horizon.

Remark 1: The closed-loop stability of an MPC is not necessarily guaranteed as MPC only solves the finite-horizon optimal control problem at each sample time. However, it is possible to ensure stability by including some additional conditions in optimization problem formulation or by employing sufficiently long prediction horizons [1].

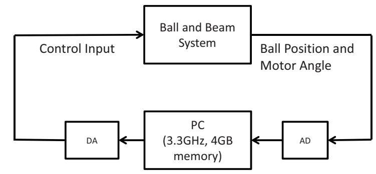

Fig. 2. Experimental setup.

IV. EXPERIMENTAL SETUP

Fig. 2 shows the experimental setup of the ball and beam system. We measure the ball position and motor angle of the system and a PC calculates the control law based on the control methods shown in the previous section (the states that are not directly measured are obtained by backward difference). We set the control interval (sample time) to 100 [ms]. In the case of MPC and MPC with disturbance attenuation, we need to solve the quadratic programming (QP) problem. We employ the infeasible primal-dual interior point algorithm [8] to solve the QP problem on-line. The algorithm was implemented in C, and the free GSL library (GNU Scientific Library) was used for matrix calculations.

V. EXPERIMENTAL RESULTS

We design the MPC control laws such that the imposed input constraint, $-1 \le u_k \le +1$, is satisfied. We first choose the design parameters for MPC and MPC with disturbance attenuation. For each control method, we use the weighting matrices, $Q = \mathrm{diag}(1000, 10, 10, 0.01)$, and $R = 1$. The prediction horizon for MPC and MPC with disturbance attenuation is $N = 10$, which indicates that the controller predicts the 1 [s] future behavior of the plant at each sampling instant and determines the control action. In the case of MPC with disturbance attenuation, the feedback

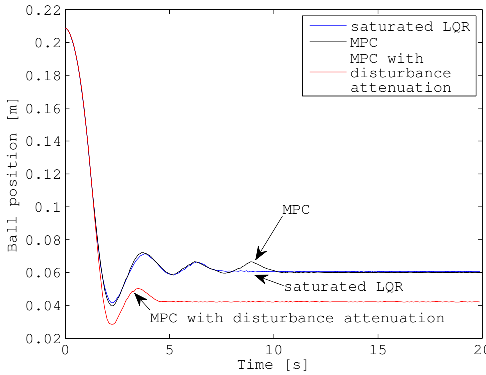

Fig. 3. Ball position z.

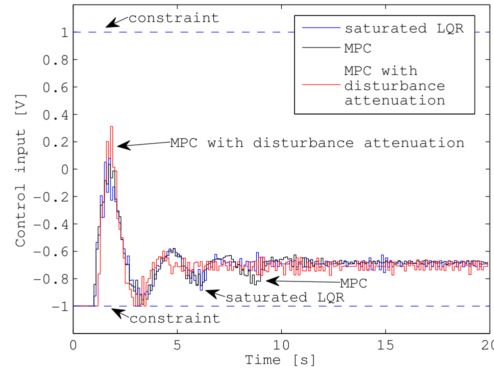

Fig. 4. Control input ($-1 \leq u_k \leq +1$).

gain is given by the infinite horizon LQR gain $K = K_{LQ} = -(R+B_2^{\mathrm{T}}PB_2)^{-1}B_2^{\mathrm{T}}PA$, and the minimum γ satisfying the inequality in (6) is calculated as 33.7316. For comparison, we also apply the saturated LQR control $u_k = \mathrm{sat}(K_{LQ}x_k)$, where sat is the saturation function. Note that when no constraints are imposed, MPC and saturated LQR yield the same control input. To investigate the disturbance attenuation performance of the cotrol methods, the base of the ball and beam system is inclined at an angle of 1° as a source of artificial disturbance.

Figs. 3–5 show the experimental results for three control methods, MPC, MPC with disturbance attenuation, and saturated LQR, for initial condition $z(0) = 0.21$ [m] (ball position). As can be inferred from Fig. 3, similar responses are obtained for MPC and saturated LQR. Indeed, the numerical simulation (this is not provided here due to the space limitation) shows that MPC and saturated LQR yield the same control law for the initial state considered here. The response of MPC with disturbance attenuation keeps the ball position near the origin. Note that because there is an artificial inclination angle of 1°, the ball position does not converge to the origin. Fig. 4 shows the control input; we observe that the imposed input constraint (± 1) is satisfied.

Although there are no significant differences between the results of saturated LQR and MPC, closed-loop stability can be guaranteed in the case of the MPC by imposing additional conditions such as terminal constraints. The saturated LQR is easy to implement; however, the state constraint cannot be handled. The deviation observed in Fig. 3 can be removed by incorporating the integral action into the controller. However, since the main objective of this study is to compare the fundamental performance of the three different type of controllers, we have not included the integral action.

VI. CONCLUDING REMARKS

In this paper, we discussed the control of a ball and beam system under the input constraint. A comparison of two different MPCs and saturated LQR was presented for the regulation control.

Various approaches to implement MPCs have been reported so far, such as in [2], [3]. Since each method has both

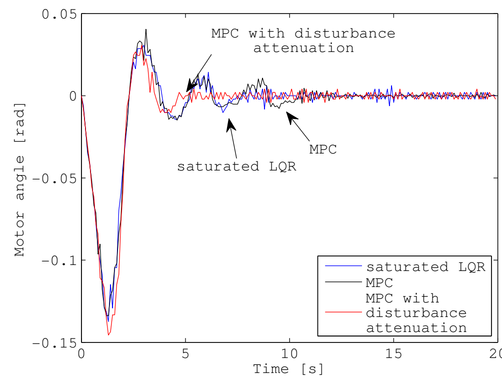

Fig. 5. Angle θ.

strengths and limitations, one of the future research direction is to provide criteria for selecting the most suitable MPC method while taking into account the available hardware and software resources, system size, required control interval, control performance, cost, etc.

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