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Truncated Unscented Particle Filter

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Abstract— **The problem of state estimation of nonlinear stochastic dynamic systems with nonlinear inequality constraints is treated. The paper focuses on a particle filtering approach, which provides an estimate of the state in the form of a probability density function. A new computationally efficient particle filter for the constrained estimation problem is proposed. The importance function of the particle filter is generated by the unscented Kalman filter that is supplemented with a designed truncation technique to accommodate the constraint. The proposed filter is illustrated in a numerical example.**

I. INTRODUCTION

State estimation of dynamic stochastic systems is of extreme importance in fields such as automatic control [1], system identification, signal processing [2], navigation, position tracking [3], fault diagnosis, communication systems, bioengineering, geophysics and econometrics.

Its goal is to find an estimate of a state, using a set of measurements. The dynamics of the state and the relation between the state and measurement are described by a discrete time state-space model.

Due to stochastic nature of the system, the estimate is given in the form of a conditional probability density function (pdf) of the state conditioned by the measurement. Calculation of the conditional pdf [4] for nonlinear or non-Gaussian systems is an intricate functional problem. Usually a simpler concept approximating the system and providing only point estimates of the state is preferred. The methods implicit in this concept are called *local* as the state estimate, which they provide, is valid only within a relatively small vicinity of a point. The methods providing the conditional pdf of the state are called *global* as the estimate is valid in almost whole state space.

In general, the state estimation problem can be solved using the Bayesian recursive relations (BRR's). Unfortunately, their closed-form solution can be obtained in a few special cases such as a linear Gaussian system where the solution corresponds to the famous Kalman filter (KF). In other cases an approximate solution must be searched.

A natural way to obtain an approximate solution is to simplify the system to achieve analytical solvability of the BRR's. Traditionally, the approximation of a nonlinear system is performed by the Taylor series expansion of nonlinear functions in the system description and considering only the first few terms. This linearization approach leads to the extended Kalman filter (EKF) or iterated EKF [5].

Lately, a great deal of effort has been dedicated to using stochastic and polynomial linearization instead. The idea of the stochastic linearization is to approximate a random variable by a set of points which are transformed through nonlinear mappings (i.e. mappings in the system description) [6], [7]. The idea of the polynomial linearization is to approximate a nonlinear mapping by a polynomial interpolation [7]. The methods are called σ-point or derivative-free methods and their main advantage over the traditional methods is that they do not require computation of the Jacobi matrix of the nonlinear functions in the state and measurement equations of the system [8]. The unscented Kalman filter (UKF) and the divided difference filter are the main representatives of these methods.

The above mentioned filtering methods provide for nonlinear and/or non-Gaussian systems local estimates only. To enlarge the estimate validity area and provide a global estimate, the Gaussian mixture filter has been proposed [9], which is based on the approximation of all random pdf's of interest by weighted mixtures of Gaussian pdf's and using multiple local estimators of the Kalman filtering framework.

In the 1960's solving the integrals appearing in the filtering problem by means of stochastic Monte Carlo (MC) integration has appeared [10]. But it was not until the 1990's when the paper of [11] has been published proposing the bootstrap filter as an efficient solution to the filtering problem in very general settings. Popularity of the filter, belonging to a group later called particle filters (PF's), was caused by cheap and formidable computational power which is vital for application of a PF. The idea of the PF [12]–[14] is to compute the conditional pdf of the state in the form of an empirical pdf consisting of a finite set of random samples (also called particles) and corresponding weights. The samples and weights define the shape of the conditional pdf of the state. The central part of the PF's is the importance sampling technique which uses an importance function (IF) for drawing samples. The respective weights are computed so that the samples and weights together correspond to the conditional pdf of the state.

Design of the IF is a critical part of the PF. The IF determines quality of the samples that are generated and, consequently, efficiency of the PF. Two different approaches to the IF design can be found. The former, which can be called *direct*, focuses on developing the original concepts of the IF design and proposes enhancements of a prior IF [15], [16]. The latter, which can be called the *composite*, comes out of utilization of another filtering technique to obtain a

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filtering pdf approximation which subsequently used as the IF, e.g. [17].

Majority of the above mentioned filtering methods largely have been developed under a condition that system is described by the state and measurement equations and pdf's of the uncertainties only. But in some cases, an additional information about the state is known. This information appears as a constraint for the state variable which often describes a physical quantity present within the system. The constraints arise due to physical laws, technological limitations, kinematic constraints or geometric considerations of the system [18], [19]. Mathematically, the constraints are given by a set of linear or nonlinear equalities or inequalities.

In the last decade, several approaches to solve the constrained estimation problem have been proposed. Among others, the following can be mentioned: reparametrizing and pseudo-measurement approaches [20], [21], optimization approaches [22], and projection and truncation approaches [5], [23]–[25]. Most frequently, local filtering methods are utilized within these approaches. A nice survey of the approaches was published in [26].

If a global solution to a filtering problem with an inequality constraint is considered, an ensemble KF based solution [27] and few PF based techniques have been proposed so far, mainly based on the optimization [28] or truncation approach [29]. The truncation approach is more promising for such a constrained filtering problem due to its lower computational demands as opposed to the optimization approach [28].

For *nonlinear* inequality constraints only a simple technique has been proposed based on drawing samples until a pre-specified number of samples meet the constraint [29]. This may be inefficient, especially if the probability that a sample drawn from the IF meets the constraint is very low.

Thus, the goal of the paper is to propose an IF for efficient sampling for the filtering problem with generally nonlinear inequality constraint. The aim of the IF design is to follow the composite approach and use the UKF to provide an unconstrained IF. Consequently, the unconstrained IF should be truncated to comply with the constraint.

The paper is organized as follows: Specification of the constrained state estimation problem is presented in Section II. In Section III, the UKF will be introduced and the truncation technique will be proposed. The truncated unscented PF will be designed in Section IV. In Section V, a numerical illustration of the new filter will be given and concluding remarks are drawn in Section VI.

II. State Estimation with Inequality Constraints

Suppose the system is given by the following state and measurement equations

$$\mathbf{x}_{k+1} = \mathbf{f}_k(\mathbf{x}_k, \mathbf{u}_k) + \mathbf{w}_k, \tag{1}$$

$$\mathbf{z}_k = \mathbf{h}_k(\mathbf{x}_k) + \mathbf{v}_k, \tag{2}$$

where $\mathbf{x}_k \in \mathrm{R}^{n_x}$, $\mathbf{u}_k \in \mathrm{R}^{n_u}$ and $\mathbf{z}_k \in \mathrm{R}^{n_z}$ are the state, input and measurement, respectively at time instant k, $\mathbf{f}_k: \mathrm{R}^{n_x} \times \mathrm{R}^{n_u} \to \mathrm{R}^{n_x}$ and $\mathbf{h}_k: \mathrm{R}^{n_x} \to \mathrm{R}^{n_z}$ are known mappings, $\mathbf{w}_k$ and $\mathbf{v}_k$ are state and measurement white noises, described by known pdf's $p(\mathbf{w}_k)$ and $p(\mathbf{v}_k)$, respectively. The noises are mutually independent and independent of the initial condition of the state $\mathbf{x}_0$ given by a known $p(\mathbf{x}_0)$.

Filtering aims at finding the state $\mathbf{x}_k$ based on measurements up to the time instant k, which will be denoted as $\mathbf{z}^k = [\mathbf{z}_0^{\mathrm{T}}, \mathbf{z}_1^{\mathrm{T}}, \ldots \mathbf{z}_k^{\mathrm{T}}]^{\mathrm{T}}$. Due to the stochastic nature of the system, the state estimate is a random variable described by the conditional pdf $p(\mathbf{x}_k|\mathbf{z}^k)$.

As was mentioned above, in some cases besides the system description, given by (1) and (2) and distributions of the noises and initial condition, also an additional information about the state is at disposal. As an important feature of the state-space description is that the state often corresponds to some physical quantities, a validity region of the state enforced by the physical representation may be an important supplementary information for the filter designer.

In this paper the information will be considered in the form of a generally nonlinear inequality constraint

$$\mathbf{a}_k \leq \mathbf{C}_k(\mathbf{x}_k) \leq \mathbf{b}_k, \tag{3}$$

where $\mathbf{C}_k: \mathrm{R}^{n_x} \to \mathrm{R}^{n_c}$, $\mathbf{a}_k, \mathbf{b}_k \in \mathrm{R}^{n_c}$, and the inequality $\leq$ holds for all elements of the vectors and $\mathbf{a}_k \neq \mathbf{b}_k$, $\forall k$.

The aim of the constrained filtering problem is to find the conditional pdf $p(\mathbf{x}_k|\mathbf{z}^k)$ considering (1) and (2) and respecting (3). For convenience, let $\mathcal{C}_k$ be a set of all states satisfying the inequality constraint (3):

$$\mathcal{C}_k = \{\mathbf{x}_k : \mathbf{x}_k \in \mathrm{R}^{n_x}, \mathbf{a}_k \leq \mathbf{C}_k(\mathbf{x}_k) \leq \mathbf{b}_k\}. \tag{4}$$

Then, the aim of constrained state estimation is to find the pdf $p_C(\mathbf{x}_k|\mathbf{z}^k)$ given by

$$p_C(\mathbf{x}_k|\mathbf{z}^k) \propto \begin{cases} p(\mathbf{x}_k|\mathbf{z}^k), & \text{if } \mathbf{x}_k \in \mathcal{C}_k, \\ 0, & \text{otherwise.} \end{cases} \tag{5}$$

III. UKF for Constrained State Estimation Problem

As has been mentioned in the introduction, in this paper the IF of the PF will be generated by a truncated UKF. First, the UKF will be introduced for an unconstrained filtering problem and consequently truncation of the filtering estimate to comply with the constraint will be proposed. This truncated UKF will later be used in Section IV as an IF generator.

The UKF is rooted in the unscented transformation (UT) which computes the first two moments of a random variable transformed through a nonlinear mapping using a set of deterministically chosen points (called σ-points). For the sake of presentation clarity, the procedure to obtain the σ-points, $\{\mathcal{X}_i\}$ with corresponding weights $\{\mathcal{W}_i\}$ based on the first two moments $\hat{\mathbf{x}}$ and $\mathbf{P}_x$ of a random variable will be denoted as

$$[\{\mathcal{X}_i\}, \{\mathcal{W}_i\}] = \mathrm{UTpoints}(\hat{\mathbf{x}}, \mathbf{P}_x, \kappa), \tag{6}$$

and represents the following relations

$$\mathcal{X}_0 = \hat{\mathbf{x}}, \mathcal{W}_0 = \frac{\kappa}{n_x + \kappa}, \tag{7}$$

$$\mathcal{X}_i = \hat{\mathbf{x}} + \left(\sqrt{(n_x+\kappa)\mathbf{P}_x}\right)_i, \mathcal{W}_i = \frac{1}{2(n_x+\kappa)}, \tag{8}$$

$$\mathcal{X}_{n_x+i} = \hat{\mathbf{x}} - \left(\sqrt{(n_x+\kappa)\mathbf{P}_x}\right)_i, \mathcal{W}_{n_x+i} = \mathcal{W}_i, \tag{9}$$

with $i = 1, 2, \ldots, n_x$,where the notation $(\mathbf{A})_i$ denotes the i-th column of the matrix $\mathbf{A}$, the parameter κ is usually selected in accord with the recommendation given in the case of the UT. One time-step of the UKF algorithm, which will later be used for generation of the IF, can be summarized as follows:

Step 1: Suppose, the filtering pdf $p(\mathbf{x}_{k-1}|\mathbf{z}^{k-1})$ is given by its first two moments, i.e. mean $\hat{\mathbf{x}}_{k-1|k-1} = \mathsf{E}[\mathbf{x}_{k-1}|\mathbf{z}^{k-1}]$ and covariance matrix $\mathbf{P}_{k-1|k-1} = \mathsf{cov}[\mathbf{x}_{k-1}|\mathbf{z}^{k-1}]$.
Step 2: Then, the predictive statistics are given by

$$\hat{\mathbf{x}}_{k|k-1} = E[\mathbf{x}_k|\mathbf{z}^{k-1}] = E[\mathbf{f}_{k-1}(\mathbf{x}_{k-1}, \mathbf{u}_{k-1})|\mathbf{z}^{k-1}] \approx \sum_{i=0}^{2n_x} \mathcal{W}_i \mathcal{X}_{i,k|k-1}, \tag{10}$$

$$\mathbf{P}_{k|k-1} = E[(\mathbf{x}_k - \hat{\mathbf{x}}_{k|k-1})(\mathbf{x}_k - \hat{\mathbf{x}}_{k|k-1})^T|\mathbf{z}^{k-1}] \tag{11}$$

$$\approx \sum_{i=0}^{2n_x} \mathcal{W}_i (\mathcal{X}_{i,k|k-1} - \hat{\mathbf{x}}_{k|k-1})(\mathcal{X}_{i,k|k-1} - \hat{\mathbf{x}}_{k|k-1})^T + \mathbf{Q}_{k-1},$$

$$\mathcal{X}_{i,k|k-1} = \mathbf{f}_k(\mathcal{X}_{i,k-1|k-1}, \mathbf{u}_{k-1}), \forall i, \tag{12}$$

where $[\{\mathcal{X}_{i,k-1|k-1}\}, \{\mathcal{W}_i\}] = \text{UTpoints}(\hat{\mathbf{x}}_{k-1|k-1}, \mathbf{P}_{k-1|k-1}, \kappa)$.
Step 3: The state predictive estimate is updated with respect to the last measurement $\mathbf{z}_k$ according to

$$\hat{\mathbf{x}}_{k|k} = \hat{\mathbf{x}}_{k|k-1} + \mathbf{K}_{k|k}(\mathbf{z}_k - \hat{\mathbf{z}}_{k|k-1}), \tag{13}$$

$$\mathbf{P}_{k|k} = \mathbf{P}_{k|k-1} - \mathbf{K}_{k|k}\mathbf{P}_{z,k|k-1}\mathbf{K}_{k|k}^T, \tag{14}$$

where $\mathbf{K}_{k|k} = \mathbf{P}_{xz,k|k-1}(\mathbf{P}_{z,k|k-1})^{-1}$ is the filter gain and

$$\hat{\mathbf{z}}_{k|k-1} = E[\mathbf{z}_k|\mathbf{z}^{k-1}] = E[\mathbf{h}_k(\mathbf{x}_k)|\mathbf{z}^{k-1}] \approx \sum_{i=0}^{2n_x} \mathcal{W}_i \mathcal{Z}_{i,k|k-1}, \tag{15}$$

$$\mathbf{P}_{z,k|k-1} = E[(\mathbf{z}_k - \hat{\mathbf{z}}_{k|k-1})(\mathbf{z}_k - \hat{\mathbf{z}}_{k|k-1})^T|\mathbf{z}^{k-1}] = \tag{16}$$

$$= E[(\mathbf{h}_k(\mathbf{x}_k) - \hat{\mathbf{z}}_{k|k-1})(\mathbf{h}_k(\mathbf{x}_k) - \hat{\mathbf{z}}_{k|k-1})^T|\mathbf{z}^{k-1}] + \mathbf{R}_k$$

$$\approx \sum_{i=0}^{2n_x} \mathcal{W}_i (\mathcal{Z}_{i,k|k-1} - \hat{\mathbf{z}}_{k|k-1})(\mathcal{Z}_{i,k|k-1} - \hat{\mathbf{z}}_{k|k-1})^T + \mathbf{R}_k,$$

$$\mathbf{P}_{xz,k|k-1} = E[(\mathbf{x}_k - \hat{\mathbf{x}}_{k|k-1})(\mathbf{z}_k - \hat{\mathbf{z}}_{k|k-1})^T|\mathbf{z}^{k-1}]$$

$$\approx \sum_{i=0}^{2n_x} \mathcal{W}_i (\mathcal{X}_{i,k|k-1} - \hat{\mathbf{x}}_{k|k-1})(\mathcal{Z}_{i,k|k-1} - \hat{\mathbf{z}}_{k|k-1})^T, \tag{17}$$

$$\mathcal{Z}_{i,k|k-1} = \mathbf{h}_k(\mathcal{X}_{i,k|k-1}), \forall i, \tag{18}$$

and $[\{\mathcal{X}_{i,k|k-1}\}, \{\mathcal{W}_i\}] = \text{UTpoints}(\hat{\mathbf{x}}_{k|k-1}, \mathbf{P}_{k|k-1}, \kappa)$.

Again, for presentation clarity, the prediction and filtering steps of the UKF algorithm will be denoted as

$$[\hat{\mathbf{x}}_{k|k-1}, \mathbf{P}_{k|k-1}] = \text{UKF}_{\text{pred}}(\hat{\mathbf{x}}_{k-1|k-1}, \mathbf{P}_{k-1|k-1}). \tag{19}$$

for the prediction part and

$$[\hat{\mathbf{x}}_{k|k}, \mathbf{P}_{k|k}] = \text{UKF}_{\text{filt}}(\hat{\mathbf{x}}_{k|k-1}, \mathbf{P}_{k|k-1}, \mathbf{z}_k). \tag{20}$$

A. Truncation of a pdf

Now, suppose, the unconstrained filtering pdf provided by the UKF is given by the first two moments $\hat{\mathbf{x}}_{k|k}$ and $\mathbf{P}_{k|k}$.

The truncated pdf $p_C(\mathbf{x}_k|\mathbf{z}^k)$ can be expressed as

$$p_C(\mathbf{x}_k|\mathbf{z}^k) = \begin{cases} \xi_k^{-1} p(\mathbf{x}_k|\mathbf{z}^k), & \text{if } \mathbf{x}_k \in \mathcal{C}_k, \\ 0, & \text{otherwise}, \end{cases} \tag{21}$$

where ξ_k is a normalizing constant

$$\xi_k = \text{Prob}\{\mathbf{x}_k \in \mathcal{C}_k|\mathbf{z}^k\} = \int_{\mathcal{C}_k} p(\mathbf{x}_k|\mathbf{z}^k)\mathrm{d}\mathbf{x}_k. \tag{22}$$

Relation (21) represents a closed-form description of the random variable $\mathbf{x}_k$ with respect to the constraint $\mathcal{C}_k$ (3) in the form of a pdf. To use this truncated pdf (21) as the IF, an efficient way of sampling from the pdf is required. However, due to nonlinearity of the mapping $\mathbf{C}_k$, a representation of the constrained pdf, which can be simply sampled from, is hard to find. Therefore, its first two moments $\hat{\mathbf{x}}_{k|k}^c$ and $\mathbf{P}_{k|k}^c$ will be calculated and the truncated pdf will be approximated by a Gaussian pdf with the same first two moments.

$$p_C(\mathbf{x}_k|\mathbf{z}^k) \approx \mathcal{N}\{\mathbf{x}_k; \hat{\mathbf{x}}_{k|k}^c, \mathbf{P}_{k|k}^c\}. \tag{23}$$

The moments in (23) are given by

$$\hat{\mathbf{x}}_{k|k}^c = \int \mathbf{x}_k p_C(\mathbf{x}_k|\mathbf{z}^k)\mathrm{d}\mathbf{x}_k, \tag{24}$$

$$\mathbf{P}_{k|k}^c = \int (\mathbf{x}_k - \hat{\mathbf{x}}_{k|k}^c)(\mathbf{x}_k - \hat{\mathbf{x}}_{k|k}^c)^{\mathrm{T}} p_C(\mathbf{x}_k|\mathbf{z}^k)\mathrm{d}\mathbf{x}_k. \tag{25}$$

Analytical calculation of the mean and covariance matrix is impossible except for a few special cases (e.g. linear inequality constraints where the computation is based on evaluation of the error function [5], [30]), and usually an approximate values have to be found. In this paper, the values of the integrals (24) and (25) will be approximated using MC techniques, more specifically the perfect MC technique and importance sampling (IS). The merit of the usage of the MC techniques lies in their relative ease of implementation and acceptable computational demands not significantly depending on the dimension of the state [31].

1) Perfect Monte Carlo: To approximate the integrals in (24) and (25), first suppose S samples $\mathbf{x}_k^{(i)}$, $i = 1, 2, \ldots, S$ are drawn from $p(\mathbf{x}_k|\mathbf{z}^k)$. The samples are divided into two groups; the samples satisfying the constraint $\mathcal{C}_k$ denoted $\mathbf{x}_k^{c,(j)}$, $j = 1, 2, \ldots, S^c$, $S^c \leq S$ and the samples lying outside the constraint. Then, the approximate mean and covariance matrix of the truncated distribution (21) are given by the following relations

$$\hat{\mathbf{x}}^c_{k|k} \approx \frac{1}{S^c}\sum_{j=1}^{S^c}\mathbf{x}_k^{c,(j)}, \tag{26}$$

$$\mathbf{P}^c_{k|k} \approx \frac{1}{S^c}\sum_{j=1}^{S^c}(\mathbf{x}_k^{c,(j)}-\hat{\mathbf{x}}^c_{k|k})(\mathbf{x}_k^{c,(j)}-\hat{\mathbf{x}}^c_{k|k})^T. \tag{27}$$

For the sake of presentation clarity, this truncation will be denoted as

$$[\hat{\mathbf{x}}^c_{k|k},\mathbf{P}^c_{k|k}] = \mathrm{TRUNC_{MC}}(\hat{\mathbf{x}}_{k|k},\mathbf{P}_{k|k},\mathcal{C}_k). \tag{28}$$

The perfect MC technique suffers from one major difficulty appearing in the situation when only a small volume of $p(\mathbf{x}_k|\mathbf{z}^k)$ lies above the constraint area, or S is chosen small. Then, S^c can be zero and the moments $\hat{\mathbf{x}}^c_{k|k}$ and $\mathbf{P}^c_{k|k}$ are undefined. To overcome this difficulty, the IS technique provides an interesting alternative.

2) Importance Sampling: The IS technique considers the samples to be drawn from an importance function (IF) $q(\mathbf{x}_k)$ which may differ from $p(\mathbf{x}_k|\mathbf{z}^k)$. The IF is subject to only one condition which is $q(\mathbf{x}_k)\neq 0$ for any $\mathbf{x}_k\in\mathrm{R}^{n_x}$ for which $p(\mathbf{x}_k|\mathbf{z}^k)\neq 0$. To achieve high efficiency, the volume of the IF within the constrained region should be close to one.

Let S samples $\mathbf{x}_k^{(i)}$, $i=1,2,\ldots,S$ be drawn from the IF $q(\mathbf{x}_k)$. Then, S^c samples $\mathbf{x}_k^{c,(j)}$, $j=1,2,\ldots,S^c$, lying within the region $\mathcal{C}_k$ are selected and the approximate mean and covariance matrix of the truncated pdf are given by

$$\hat{\mathbf{x}}^c_{k|k} \approx \frac{1}{\sum_{j=1}^{S^c}\mathbf{w}_k^{(j)}}\sum_{j=1}^{S^c}\mathbf{x}_k^{c,(j)}\mathbf{w}_k^{(j)}, \tag{29}$$

$$\mathbf{P}^c_{k|k} \approx \frac{1}{\sum_{j=1}^{S^c}\mathbf{w}_k^{(j)}}\sum_{j=1}^{S^c}(\mathbf{x}_k^{c,(j)}-\hat{\mathbf{x}}^c_{k|k})(\mathbf{x}_k^{c(j)}-\hat{\mathbf{x}}^c_{k|k})^T\mathbf{w}_k^{(j)}, \tag{30}$$

where $\mathbf{w}_k^{(j)} = p(\mathbf{x}_k^{c,(j)})/q(\mathbf{x}_k^{c,(j)})$, $\forall j$. The IS based truncation will further be denoted as

$$[\hat{\mathbf{x}}^c_{k|k},\mathbf{P}^c_{k|k}] = \mathrm{TRUNC_{IS}}(\hat{\mathbf{x}}_{k|k},\mathbf{P}_{k|k},\mathcal{C}_k). \tag{31}$$

The techniques (28) and (31) produce the mean $\hat{\mathbf{x}}^c_{k|k}$ and covariance matrix $\mathbf{P}^c_{k|k}$ of a truncated pdf which will be approximated by a Gaussian pdf with the same moments and later used for drawing samples of the PF.

IV. Truncated Unscented Particle Filter

In this section an algorithm of the Truncated Unscented PF (TUPF) is proposed consisting of the PF with the IF generated by the UKF with its consecutive truncating by the techniques proposed in the previous section.

The PF approximates the filtering pdf $p(\mathbf{x}_k|\mathbf{z}^k)$ by an empirical pdf $r_N(\mathbf{x}_k|\mathbf{z}^k)=\sum_{i=1}^N\mathbf{w}_k^{(i)}\delta(\mathbf{x}_k-\mathbf{x}_k^{(i)})$, where $\delta(\cdot)$ is the Dirac function defined as $\delta(\mathbf{x})=0$ for $\mathbf{x}\neq 0$ and $\int\delta(\mathbf{x})\mathrm{d}\mathbf{x}=1$. The algorithm of the TUPF is itemized below. Suppose, the empirical pdf $r_N(\mathbf{x}_{k-1}|\mathbf{z}^{k-1})$ is given by the samples $\{\mathbf{x}_{k-1}^{(i)}\}_{i=1}^N$ and weights $\{\mathbf{w}_{k-1}^{(i)}\}_{i=1}^N$.

Algorithm 1: Truncated Unscented Particle Filter (TUPF)

Step 1: (Gaussian approximation) Calculate mean $\hat{\mathbf{x}}_{k-1|k-1}$ and covariance matrix $\mathbf{P}_{k-1|k-1}$ of the empirical pdf $r_N(\mathbf{x}_{k-1}|\mathbf{z}^{k-1})$ as

$$\hat{\mathbf{x}}_{k-1|k-1} = \frac{1}{N}\sum_{i=1}^N\mathbf{w}_{k-1}^{(i)}\mathbf{x}_{k-1}^{(i)}, \tag{32}$$

$$\mathbf{P}_{k-1|k-1} = \sum_{i=1}^N\left[\mathbf{x}_{k-1}^{(i)}-\hat{\mathbf{x}}_{k-1|k-1}\right]\left[\mathbf{x}_{k-1}^{(i)}-\hat{\mathbf{x}}_{k-1|k-1}\right]^T\mathbf{w}_{k-1}^{(i)}, \tag{33}$$

Step 2: (UKF) Calculate the first two moments of the unconstrained IF by means of the UKF as

$$[\hat{\mathbf{x}}^{\mathrm{IF}}_{k|k-1},\mathbf{P}^{\mathrm{IF}}_{k|k-1}] = \mathrm{UKF_{pred}}(\hat{\mathbf{x}}_{k-1,k-1},\mathbf{P}_{k-1,k-1}), \tag{34}$$

$$[\hat{\mathbf{x}}^{\mathrm{IF}}_{k|k},\mathbf{P}^{\mathrm{IF}}_{k|k}] = \mathrm{UKF_{filt}}(\hat{\mathbf{x}}^{\mathrm{IF}}_{k,k-1},\mathbf{P}^{\mathrm{IF}}_{k,k-1},\mathbf{z}_k). \tag{35}$$

Step 3: (Constraining) From the mean $\hat{\mathbf{x}}^{\mathrm{IF}}_{k|k}$ and covariance matrix $\mathbf{P}^{\mathrm{IF}}_{k|k}$ compute constrained mean $\hat{\mathbf{x}}^{\mathrm{IF},c}_{k|k}$ and covariance matrix $\mathbf{P}^{\mathrm{IF},c}_{k|k}$ of the truncated pdf using either the perfect MC technique

$$[\hat{\mathbf{x}}^{\mathrm{IF},c}_{k|k},\mathbf{P}^{\mathrm{IF},c}_{k|k}] = \mathrm{TRUNC_{MC}}(\hat{\mathbf{x}}^{\mathrm{IF}}_{k|k},\mathbf{P}^{\mathrm{IF}}_{k|k},\mathcal{C}_k) \tag{36}$$

or the IS technique

$$[\hat{\mathbf{x}}^{\mathrm{IF},c}_{k|k},\mathbf{P}^{\mathrm{IF},c}_{k|k}] = \mathrm{TRUNC_{IS}}(\hat{\mathbf{x}}^{\mathrm{IF}}_{k|k},\mathbf{P}^{\mathrm{IF}}_{k|k},\mathcal{C}_k). \tag{37}$$

The mean $\hat{\mathbf{x}}^{\mathrm{IF},c}_{k|k}$ and the covariance matrix $\mathbf{P}^{\mathrm{IF},c}_{k|k}$ are the first two moments of the IF used for drawing samples.
Step 4: (Drawing samples) Draw samples from the IF

$$\pi(\mathbf{x}_k|\mathbf{z}^k,\mathbf{u}_{k-1}) = \mathcal{N}\{\mathbf{x}_k;\hat{\mathbf{x}}^{\mathrm{IF},c}_{k|k},\mathbf{P}^{\mathrm{IF},c}_{k|k}\} \tag{38}$$

until for N samples $\{\mathbf{x}_k^{(i)}\}_{i=1}^N$ the condition $\mathbf{x}_k\in\mathcal{C}_k$ holds.
Step 5: (Computing weights) The weights corresponding to the drawn samples are given by

$$\mathbf{w}_k^{(i)} = \frac{p(\mathbf{z}_k|\mathbf{x}_k^{(i)})\mathcal{N}\{\mathbf{x}_k^{(i)};\hat{\mathbf{x}}^{\mathrm{IF}}_{k|k-1},\mathbf{P}^{\mathrm{IF}}_{k|k-1}\}}{\mathcal{N}\{\mathbf{x}_k^{(i)};\hat{\mathbf{x}}^{\mathrm{IF},c}_{k|k},\mathbf{P}^{\mathrm{IF},c}_{k|k}\}}, \quad i=1,2,\ldots,N. \tag{39}$$

The weights $\{\mathbf{w}_k^{(i)}\}_{i=1}^N$ are then normalized to unity, i.e. $\sum_{i=1}^N\mathbf{w}_k^{(i)}=1$.
The samples $\{\mathbf{x}_k^{(i)}\}_{i=1}^N$ and weights $\{\mathbf{w}_k^{(i)}\}_{i=1}^N$ then constitute the empirical filtering pdf $r_N(\mathbf{x}_k|\mathbf{z}^k)$
Increase k and continue with step 1.

Note that the measurement pdf $p(\mathbf{z}_k|\mathbf{x}_k)$ in (39) is given by $p(\mathbf{z}_k|\mathbf{x}_k)=p_{\mathbf{v}_k}(\mathbf{z}_k-\mathbf{h}_k(\mathbf{x}_k))$. The algorithm starts from the predictive pdf $p(\mathbf{x}_0|\mathbf{z}^{-1})=p(\mathbf{x}_0)$ and therefore in the first time step, only the filtering part of the UKF algorithm given by (37) is utilized.

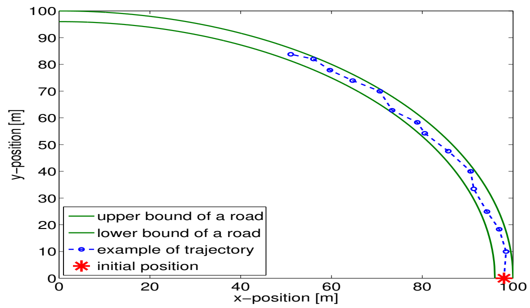

Fig. 1: Outline of the road and example of a trajectory for the tracking example.

V. Numerical Illustration

For illustration of the proposed TUPF, a problem of tracking of a moving vehicle on a circular road is considered. The road is defined by the two arcs with the radii $r_1 = 100$ and $r_2 = 96$ meters [m] with the center at the origin of the Cartesian coordinate system. The vehicle is supposed to keep angular velocity ω within $\omega \in [2.85, 5.7]$ degrees per second [deg/sec]. The equivalent vehicle speed is thus maximally 10 [m/sec]. This situation is depicted in Figure 1 with a sample trajectory. The vehicle starts from the point (initial position of the vehicle) $[98,\ 0]^T$ determining the position in the x and y directions. For the estimation purposes, the vehicle is supposed to follow the continuous white noise acceleration motion model. The state of the vehicle is defined as $\mathbf{x}_k = [x_{1,k},\ x_{2,k},\ x_{3,k},\ x_{4,k}]^T = [x_k,\ \dot{x}_k,\ y_k,\ \dot{y}_k]^T$ (i.e. it consists of the positions and velocities in the x and y directions) which evolves according to

$$\mathbf{x}_{k+1} = \begin{bmatrix} 1 & T & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & T \\ 0 & 0 & 0 & 1 \end{bmatrix} \mathbf{x}_k + \begin{bmatrix} 0.5T^2 & 0 \\ T & 0 \\ 0 & 0.5T^2 \\ 0 & T \end{bmatrix} \mathbf{w}_k, \tag{40}$$

where $T = 1$ [sec] is the sampling period and $\mathbf{w}_k$ is the Gaussian zero-mean state noise with covariance matrix $\mathbf{Q} = \mathbf{I}_2$. The vehicle is supposed to travel for $k = 0, 1, \ldots, K$, $K = 20$.

The vehicle is tracked by a sensor with the sampling interval T measuring the range and the bearings, i.e. the measurement equation is

$$\mathbf{z}_k = \begin{bmatrix} \sqrt{x_{1,k}^2 + x_{3,k}^2} \\ \tan^{-1}\left(\frac{x_{3,k}}{x_{1,k}}\right) \end{bmatrix} + \mathbf{v}_k, \tag{41}$$

where $\mathbf{v}_k$ is a Gaussian zero-mean measurement noise with the covariance matrix $\mathbf{R} = \mathrm{diag}([8, 10^{-3}])$. The function $\mathrm{diag}(\mathbf{x})$ represents a diagonal matrix with elements of the vector $\mathbf{x}$ on its diagonal.

The initial condition is

$$p(\mathbf{x}_0) = \mathcal{N}\left\{\mathbf{x}_0 : \begin{bmatrix} 98 \\ 0 \\ 0 \\ 10 \end{bmatrix} \begin{bmatrix} 10 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 10 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\right\}. \tag{42}$$

TABLE I: Estimation performance of filters for the tracking example.

	PF	UPF	RPF	TUPF
MSE	5.7160	5.5458	4.4249	3.3119
V	13.8265	14.0802	66.88440	9.3582
time [sec]	0.85	0.97	6.92	4.73

The constraint is defined, with respect to (3), as $r_2 \leq \sqrt{x_{1,k}^2 + x_{3,k}^2} \leq r_1$. In the example the performance of two unconstrained PF's

- generic PF with prior IF (*PF*),
- unscented PF (*UPF*),

and two constrained PF's

- the PF rejecting the samples lying outside the constrained area [29] (*RPF*),
- the newly proposed *TUPF*

was analyzed.

The scaling parameter for the UKF within the TUPF and UPF was set to $\kappa = 0$. Within the TUPF, the truncation using the IS-based method (31) was preferred. Sample size is set for all the PF's to $N = 10^3$.

The PF's performance was measured using two criteria based on $M = 10^3$ MC simulations; the mean-square error (MSE)

$$\mathrm{MSE} = \frac{\sum_{m=1}^{M} \sum_{k=0}^{K} \sum_{i=1}^{n_x} (x_{i,k}^{(m)} - \hat{x}_{i,k|k}^{(m)})^2}{M(K+1)n_x}, \tag{43}$$

and the criterion characterizing the average filtering covariance matrix [24] given by

$$V = \frac{\sum_{m=1}^{M} \sum_{k=0}^{K} \mathrm{trace}(\mathbf{P}_{k|k}^{(m)})}{M(K+1)}, \tag{44}$$

where $x_{i,k}^{(m)}$ is the i-th component of the true state in the m-th MC simulation at time k, $\hat{x}_{i,k|k}^{(m)}$ is its filtering estimate, and $\mathrm{trace}(\mathbf{P}_{k|k}^{(m)})$ is the trace of the filtering covariance matrix in the m-th MC simulation. The average computational costs for one MC simulation are measured as well.

An example of the true trajectory and its estimates by means of the PF's is depicted on Figure 2. The results are summarized in Table I where the MSE (43) and criterion V (44) consider only position of the object.

The results indicate superior performance of the proposed TUPF which provides the highest estimation quality with reduction of the computational costs in relation to the RPF. This is caused by the fact the IF produced by the truncated UKF accommodates the constrained region much better than the prior IF used in the RPF.

The PF and UPF provide low quality of estimates as they ignore the information about the state given by the constraint.

VI. Conclusion

The paper dealt with the state estimation of nonlinear non-Gaussian systems with the state subject to a nonlinear

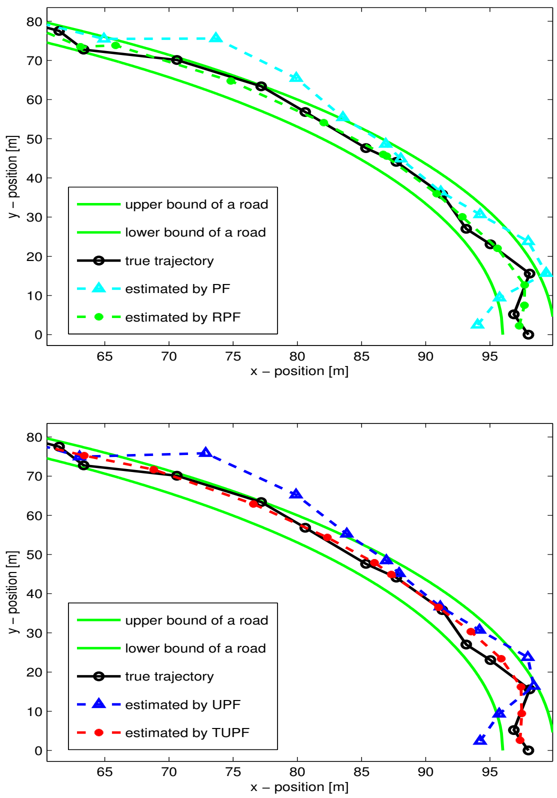

Fig. 2: An example of the true trajectory and its estimates by the PF, UPF, RPF and TUPF.

inequality constraint. The truncated unscented particle filter has been proposed, which utilizes a modified unscented Kalman filter as a generator of the importance function. The modification of the UKF consists in an additional truncation step that accommodates the resultant pdf of the UKF to the constraint. This leads to an efficient IF which exploits the current measurement and simultaneously respects the nonlinear inequality constraint. The proposed TUPF was shown in the numerical example to be superior to the acceptance/rejection based PF in both computational and quality aspects.

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