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Computation of limit cycles in Lur'e systems

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***Abstract*— Computation of limit cycles in autonomous piecewise linear feedback systems in Lur'e form is considered. It is shown how the complementarity representation of the feedback characteristic allows to represent the discretized closed loop system as a linear complementarity system. A static linear complementarity problem, whose solutions correspond to periodic solutions of the discrete–time system, is formulated. The proposed technique is able to compute steady state oscillations with known period for continuous–time systems, so as demonstrated by simulation results on the Chua electrical circuit and on other Lur'e systems which exhibit asymmetric unstable and sliding limit cycles.**

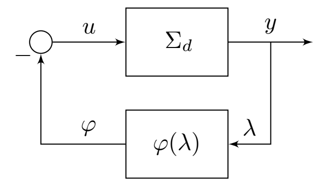

Fig. 1. Block diagram of a Lur'e system.

I. INTRODUCTION

Piecewise linear feedback systems in Lur'e form can be represented as the feedback interconnection of a linear time invariant dynamical system Σ_d with a piecewise linear (PWL) static characteristic $\varphi(\lambda)$, as reported in Fig. 1. The static characteristic $\varphi(\lambda)$ is a piecewise linear multi–input multi–output multi–valued mapping, which includes piecewise linear functions, set–valued functions and unbounded characteristics [1]. This type of Lur'e systems may exhibit several interesting behaviors, for instance such systems tend to oscillate also without external excitation. Various techniques have been proposed to investigate such self-oscillations which are also called limit cycles. Time domain approaches are based mainly on the so-called shooting method [2], [3], which determines the initial condition for the periodic solution by solving a set of initial value nonlinear equations, or a sensitivity matrix equation, with the Newton-Raphson method. The main drawback of this method is the evaluation of the sensitivity matrix, which is often computationally expensive. Other time domain techniques are based on cascaded solutions, but for the model construction an a priori sequence of modes must be assumed [4], [5]. A few of the frequency domain methods include the Tsypkin method [6], [7] and the describing function, [8], [9], [10]. The describing function analysis occasionally fails to predict the limit cycles, particularly when the system under consideration does not satisfy the assumption of filtering out the higher-order harmonics. The Tsypkin method provides useful analytical results but it becomes difficult to be applied when the periodic oscillation is not unimodal, such as in the case of two switchings per period or sliding solution and, above all, when the PWL characteristic is different from relay. The recent literature has shown that the complementarity framework can be useful for investigating PWL systems [11], [12]. In [13] the complementarity approach has been proposed for the computation of limit cycles in autonomous systems in Lur'e form. In this paper we extend those results, showing that the complementarity technique can be used also to compute steady-state oscillations with sliding so as unstable asymmetric limit cycles. The paper is organized as follows. In Section II the construction procedure of a linear complementarity representation for PWL feedback systems is shown. In Section III the proposed complementarity approach for the computation of limit cycles is presented. Section IV includes some numerical tests. The paper is concluded in Section V.

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II. COMPLEMENTARITY REPRESENTATION

The autonomous system in Fig. 1 can be represented in the following state–space form

$$\dot{x} = A_d x + B_d u \tag{1a}$$
$$y = \lambda = C_d x + D_d u \tag{1b}$$
$$u = -\varphi(\lambda), \tag{1c}$$

where (A_d, B_d, C_d, D_d) is a minimal state space realization with $A_d \in \mathbb{R}^{n\times n}$, $B_d \in \mathbb{R}^{n\times m}$, $C_d \in \mathbb{R}^{m\times n}$ and $D_d \in \mathbb{R}^{m\times m}$. With some abuse of notation, $\varphi(\lambda)$ is used for indicating the characteristic even though it can be set–valued. In this paper we consider $\varphi(\lambda)$ passing through the origin. In order to analyze the computation of limit cycles for piecewise linear Lur'e systems we recall some useful definitions.

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Definition 1: A solution of the system (1) is any absolutely continuous function $x(t) : [0, \infty) \mapsto \mathbb{R}^n$ that satisfies equations (1) for almost every $t \geqslant 0$, given an initial condition $x(0)$.

We assume that, for every initial condition $x(0)$, (1a) has at least one solution which satisfies (1b) and (1c). Clearly the origin is a solution (an equilibrium) of (1).

Definition 2: A trajectory, also called orbit, is the locus in the state space of the solution $x(t)$ of the system (1) for all $t \geqslant 0$, [10].

Definition 3: A solution x^* of the system (1) is *periodic* if there exists $T > 0$ such that $x^*(t + T) = x^*(t)$ for all $t \geqslant 0$.

In order to introduce the limit cycle definition, we need to recall the definition of distance of a point from a set.

Definition 4: The distance $d_{\mathfrak{L}}(x)$ between a point x and a set $\mathfrak{L}$ is defined as

$$d_{\mathfrak{L}}(x) = \inf_y \{d(x, y) : y \in \mathfrak{L}\},$$

where $d(x, y)$ is a distance function, e.g. the Euclidean distance.

Definition 5: A limit cycle $\mathfrak{L} \subset \mathbb{R}^n$ is an isolated closed trajectory.

A limit cycle $\mathfrak{L}$ is *symmetric* if for every $x \in \mathfrak{L}$ it is also $-x \in \mathfrak{L}$. A limit cycle is

- *stable* if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that $d_{\mathfrak{L}}(x(0)) < \delta$ implies that $d_{\mathfrak{L}}(x(t)) < \varepsilon$ for all $t > 0$;
- *unstable* if it is not stable;
- *asymptotically stable* if it is stable and δ can be chosen such that

$$d_{\mathfrak{L}}(x(0)) < \delta \Rightarrow \lim_{t \to \infty} d_{\mathfrak{L}}(x(t)) = 0.$$

With the terminology *sliding limit cycle* we intend a limit cycle in which part of the trajectory is characterized by sliding motion [14], [15]. The proposed technique for the computation of limit cycles is based on complementarity representations. Given a vector $q \in \mathbb{R}^p$ and a real matrix $M \in \mathbb{R}^{p \times p}$, the linear complementarity problem, say LCP(q,M), consists of finding, if it exists, a vector $z \in \mathbb{R}^p$ such that

$$z \geqslant 0 \tag{2a}$$
$$q + Mz \geqslant 0 \tag{2b}$$
$$z^T(q + Mz) = 0. \tag{2c}$$

In the sequel conditions (2) will be more compactly indicated as follows

$$0 \leqslant w = (q + Mz) \perp z \geqslant 0. \tag{3}$$

The LCP might have no solution, one solution or multiple solutions [16]. A quite general description of a set–valued PWL mapping $\varphi(\lambda)$ is represented by the following complementarity form [16]

$$\varphi = A_s\lambda + B_s z + g_s \tag{4a}$$
$$w = C_s\lambda + D_s z + h_s \tag{4b}$$
$$0 \leqslant w \perp z \geqslant 0 \tag{4c}$$

with $\varphi \in \mathbb{R}^m$, $\lambda \in \mathbb{R}^m$ and $z, w \in \mathbb{R}^p_+$ with $p \geqslant m$. For a given characteristic $\varphi(\lambda)$, the representation (4) is not unique. Putting together (1) and (4), after some algebraic manipulations the system (1) can be rewritten as

$$\dot{x} = A_c x + B_c z + g_c \tag{5a}$$
$$w = C_c x + D_c z + h_c \tag{5b}$$
$$0 \leqslant w \perp z \geqslant 0, \tag{5c}$$

with

$$\Theta \triangleq I + D_d A_s \tag{6a}$$
$$A_c = A_d - B_d A_s \Theta^{-1} C_d \tag{6b}$$
$$B_c = B_d A_s \Theta^{-1} D_d B_s - B_d B_s \tag{6c}$$
$$C_c = C_s \Theta^{-1} C_d \tag{6d}$$
$$D_c = D_s - C_s \Theta^{-1} D_d B_s \tag{6e}$$
$$g_c = B_d [A_s \Theta^{-1} D_d - I] g_s \tag{6f}$$
$$h_c = h_s - C_s \Theta^{-1} D_d g_s. \tag{6g}$$

A linear system (5a)–(5b) subject to the complementarity constraints (5c) on z and w is called a continuous–time linear complementarity system (LCS), [17].

In order to get (5) we need Θ to be nonsingular. Of course if $D_d = 0$ clearly $\Theta = I$ and then nonsingular, regardless of the complementarity representation chosen for the piecewise linear characteristic. Apart quite general conditions discussed in [18], it is not so obvious to understand which matrix properties make Θ nonsingular. On the other hand, the invertibility of Θ can be assumed without loss of generality. In fact if Θ is singular, one can use an alternative complementarity representation in the form (4) with $A_s = 0$ which implies Θ being invertible. The existence of a complementarity model with $A_s = 0$ for any PWL characteristic can be simply shown. Consider a line $\varphi = \alpha\lambda$ with α positive (analogous arguments can be done for negative α and for multi–input multi–output characteristics). Such a line can be represented by using the following complementarity form

$$\varphi = -z_1 + z_2 \tag{7a}$$
$$w_1 = \alpha\lambda + z_1 \tag{7b}$$
$$w_2 = -\alpha\lambda + z_2 \tag{7c}$$
$$0 \leqslant w \perp z \geqslant 0, \tag{7d}$$

which has $A_s = 0$. More in general one can eliminate a nonzero term $A_s\lambda$ in (4a) by adding a suitable number of pairs of complementarity variables. Furthermore, it is possible to use the same procedure to obtain a nonzero A_s.

III. Limit cycles computation

So far we have shown that it is possible to rewrite the system (1) as the LCS (5). In the following we assume that the system (5) has a periodic absolutely continuous solution, $x(t) = x(t+T)$ $\forall t \geqslant 0$ and that $z(t) = z(t+T)$ for almost all $t \geqslant 0$ with T being the known period of the solution. In order to compute periodic trajectories for (5) we first make a discretization with sampling period $T_N = T/N$, N being an integer. By discretizing (5) by using the backward Zero-Order-Hold technique the following discrete–time LCS is obtained

$$x_k - Ax_{k-1} = Bz_k + g \tag{8a}$$
$$w_k = C_c x_k + D_c z_k + h_c \tag{8b}$$
$$0 \leqslant w_k \perp z_k \geqslant 0 \tag{8c}$$

with $k = 1, \ldots, N$ and matrices

$$A := e^{A_c T_N} \tag{9a}$$
$$B := \int_0^{T_N} e^{A_c \tau} d\tau \; B_c \tag{9b}$$
$$g := \int_{(0}^{T_N} e^{A_c \tau} d\tau \; g_c. \tag{9c}$$

System (8) is a discretized version of (5). One should expect that a periodic solution of period T of (5) corresponds to a periodic solution of period N of (8), and vice–versa. Indeed, in [19] it is proved that the convergence property of the continuous piecewise linear interpolant of the samples sequence x_k given by (8) to a continuous time solution $x(t)$ of (5) as T_N goes to zero, both for initial–value LCSs and for boundary–value LCSs. However, the boundary condition $x(0) = x(T)$ corresponding to a limit cycle is excluded by the assumptions in [19]. Proving the convergence property under the limit cycle boundary condition is a non trivial task and it is out of the scope of this paper. For simplicity, in the following we assume that (8) has a periodic solution of period N, approximating the periodic solution of (5).

Define

$$\bar{x} = \mathrm{col}(x_1,\ x_2, \ldots,\ x_N) \tag{10a}$$
$$\bar{z} = \mathrm{col}(z_1,\ z_2, \ldots,\ z_N) \tag{10b}$$

with $\overline{x} \in \mathbb{R}^{N_x \cdot N}$, $\overline{z} \in \mathbb{R}^{N_z \cdot N}$ and "col" indicating vector obtained by stacking in unique column the column vectors in its argument. By using the periodicity condition $x_0 = x_N$ we can write simultaneously equations (8a) along the period N:

$$\overline{A}\overline{x} = \overline{B}\overline{z} + \overline{g} \tag{11}$$

where

$$\overline{A} = \begin{bmatrix} I_{N_x} & 0_{N_x} & \cdots & 0_{N_x} & -A \\ -A & I_{N_x} & \cdots & 0_{N_x} & 0_{N_x} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0_{N_x} & 0_{N_x} & \cdots & -A & I_{N_x} \end{bmatrix} \tag{12a}$$
$$\overline{B} = I_N \otimes B \tag{12b}$$
$$\overline{g} = 1_N \otimes g \tag{12c}$$

where $\otimes$ indicates the Kronecker product, I_N denotes the $N \times N$ identity matrix, 0_N denotes the $N \times N$ matrix with zero entries and 1_N is the N-th dimensional vector of ones. From (12a) and (9a) we get

$$\det(\overline{A}) = \det(I_{N_x} - A^N) = \det(I_{N_x} - e^{A_c T_N N}). \tag{13}$$

If $\overline{A}$ is invertible or equivalently A_c has no eigenvalues in the origin, from (11) we obtain

$$\overline{x} = \overline{A}^{-1}[\overline{B}\overline{z} + \overline{g}], \tag{14}$$

which is the unique solution of (11). Note that given $\overline{g}$, having a unique solution $\overline{x}$ does not require a unique $\overline{z}$. Since the Lur'e system is autonomous, the limit cycle is defined modulus time translation, [20]. If $(\widetilde{x}(t), \widetilde{z}(t))$ is a periodic solution of (5) each time–shifted version $(\widetilde{x}(t-\tau), \widetilde{z}(t-\tau))$ for any $\tau \in (0, T)$ is also a periodic solution. This is not in contrast with the invertibility of $\overline{A}$ because for any shifted $\overline{z}$ the corresponding shifted version of $\overline{x}$ will be given by (14) (note that the initial value x_0 of the periodic discrete-time solution is not fixed a priori). From (8b) we get

$$\overline{w} = \overline{C}\overline{x} + \overline{D}\overline{z} + \overline{h} \tag{15}$$

where $\overline{w} \in \mathbb{R}^{N_z \cdot N}$ and

$$\overline{C} = I_N \otimes C_c,\ \overline{D} = I_N \otimes D_c,\ \overline{h} = 1_N \otimes h_c. \tag{16}$$

Then by substituting (14) in (15) we obtain

$$\overline{w} = [\overline{C}\overline{A}^{-1}\overline{B} + \overline{D}]\overline{z} + [\overline{C}\overline{A}^{-1}\overline{g} + \overline{h}] = M\overline{z} + q. \tag{17}$$

Through algebraic calculations and by defining $\Pi \triangleq (I_{N_x} - A^N)^{-1}$ it is simple to show that

$$M = \begin{bmatrix} C_c\Pi B + D_c & C_c\Pi A^{N-1}B & \cdots & C_c\Pi AB \\ C_c\Pi AB & C_c\Pi B + D_c & \cdots & C_c\Pi A^2 B \\ \vdots & \vdots & \ddots & \vdots \\ C_c\Pi A^{N-1}B & C_c\Pi A^{N-2}B & \cdots & C_c\Pi B + D_c \end{bmatrix} \tag{18}$$

and

$$q = 1_N \otimes (C_c(I_{N_x} - A)^{-1}g + h_c). \tag{19}$$

If A_c has a zero eigenvalue then the matrix $\overline{A}$ in (11) is singular. In this case a solution $\overline{x}$ for (11) can be obtained by using the Gauss-Jordan elimination technique [21]. That algorithm, through algebraic transformations, allows to re-formulate (11) in the form

$$\overline{\overline{A}}\overline{x} = \overline{\overline{B}}\overline{z} + \overline{\overline{g}} + \overline{\gamma} \tag{20}$$

with the constraint

$$\overline{\overline{B}}_{(N\cdot N_x,\bullet)}\overline{z} + \overline{\overline{g}}_{N\cdot N_x} = 0 \tag{21}$$

where

$$\overline{\overline{A}} = \begin{bmatrix} 1 & 0 & \cdots & 0 & -\overline{a}_{(1,N\cdot N_x)} \\ 0 & 1 & \cdots & 0 & -\overline{a}_{(2,N\cdot N_x)} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & -\overline{a}_{((N-1)\cdot N_x, N\cdot N_x)} \\ 0 & 0 & \cdots & 0 & 1 \end{bmatrix}, \tag{22}$$

$\overline{\overline{B}}_{(N\cdot N_x,\bullet)}$ is the last row of the matrix $\overline{\overline{B}}$, $\overline{\overline{g}}_{N\cdot N_x}$ is the last element of the vector $\overline{\overline{g}}$ and $\overline{\gamma}$ is a column vector with zero entries except the last one that is equal to a generic parameter γ. The Gauss-Jordan transformation is such that if (21) is satisfied, than the last component of $\overline{x}$ is equal to γ, which is the free parameter to be chosen. Now we can calculate $\overline{x}$ from (20) and substitute this value in (15). So we obtain

$$\begin{aligned} \overline{w} &= [\overline{C}\overline{\overline{A}}^{-1}\overline{\overline{B}} + \overline{D}]\overline{z} + [\overline{C}\overline{\overline{A}}^{-1}\overline{\overline{g}} + \overline{h}] + \overline{C}\overline{\overline{A}}^{-1}\overline{\gamma} \\ &= \overline{M}\overline{z} + \overline{q} + \overline{C}\overline{\overline{A}}^{-1}\overline{\gamma}. \end{aligned} \tag{23}$$

The new complementarity problem LCP($\overline{q} + \overline{C}\overline{\overline{A}}^{-1}\overline{\gamma}, \overline{M}$) is dependent on the parameter γ.

Then we have transformed the problem of finding a periodic solution of (8) into the problem of finding a solution of the following LCP(q, M)

$$0 \leqslant q + M\overline{z} \perp \overline{z} \geqslant 0 \tag{24}$$

with M and q given by (18) and (19), respectively. Once calculated the value of $\overline{z}$ we can get the state evolution through the equation (14). Any solution of LCP(q, M) will correspond to a periodic oscillation of the system (8). Therefore finding different solutions of the LCP(q, M) is of primary importance. Several algorithms for the detection of all solutions of a LCP have been proposed, [22], [23]. The numerical results presented in next section are obtained by implementing a combination between the iterative algorithm described in [16] and the well–known PATH algorithm [24].

IV. Simulation results

In this section we show the usefulness of the proposed technique for computing limit cycles in autonomous relay feedback systems in the case that for asymmetric unstable limit cycles, for periodic oscillations with sliding motion and for the steady state oscillations of a practical electrical circuit.

A. An example with A_d having zero eigenvalue

Consider the continuous–time system with the following matrices

$$A_d = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & -2 \end{pmatrix} \quad B_d = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \tag{25}$$

with the output being the first state variable and the PWL characteristic a relay. By using the describing function approach or the procedure proposed in [25], it is simple to verify that the system exhibits a limit cycle. A possible complementarity model of the relay in the form (4) is given by the matrices:

$$A_s = \begin{pmatrix} 0 \end{pmatrix} \qquad B_s = \begin{pmatrix} -2 & 0 \end{pmatrix} \qquad g_s = \begin{pmatrix} 1 \end{pmatrix} \tag{26a}$$

$$C_s = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad D_s = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad h_s = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{26b}$$

Since the matrix A_d has a null eigenvalue, as we discussed above, we cannot use the relay complementarity model (26), because being $A_s = 0$ would result in a singular $\bar{A}$. Instead, it is possible to construct the closed–loop model (6) by using the alternative relay characteristic model obtained from (26) with the procedure presented in Section II. By choosing $T = 6.424s$ and by discretizing the system with $N = 350$ the results shown in Fig. 2 have been obtained.

B. Asymmetric unstable limit cycle

Consider the third–order system described by the matrices

$$A_d = \begin{pmatrix} 3 & 1 & 0 \\ -3 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \quad B_d = \begin{pmatrix} 1 \\ 1.2 \\ 0.36 \end{pmatrix}, \tag{27a}$$

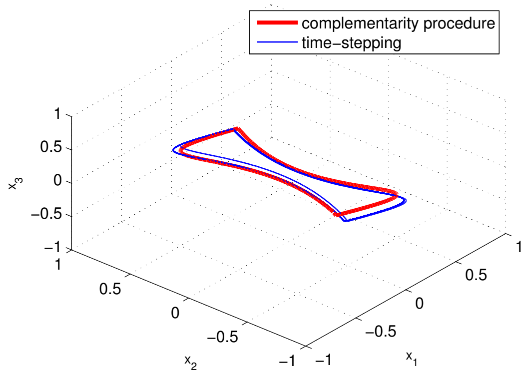

Fig. 2. Limit cycle of the relay feedback system computed by using the LCP procedure and a time–stepping transient evolution.

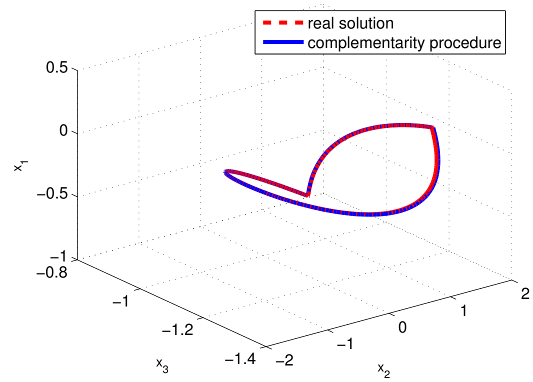

Fig. 3. Steady state solution computed by using the complementarity procedure with iterative method and real solution.

the output is the first state variable. In [15] it is shown that such system with relay feedback can exhibit asymmetric periodic solutions. Consider the relay model (26). By selecting $T = 3.723s$ and $N = 200$ and by solving the LCP (24) the asymmetric unstable limit cycle depicted in Fig. 3 has been obtained. Note that the LCP solution is obtained without knowing a priori the sequence of modes nor applying the Poincaré map. Furthermore, in general the unstable solution is hard to find with methods based on time-stepping simulations.

C. Limit cycles with sliding

Under certain conditions, it has been shown that relay feedback systems with $D_d = 0$ can exhibit periodic oscillations partially lying on the switching hyperplane $S = \{x \in \mathbb{R}^n : C_d x = 0\}$. This peculiar type of oscillation is also called sliding limit cycle [15]. With the following example we show that the proposed approach can be used for finding this type of oscillation. Consider the third order relay feedback system described by the matrices

$$A_d = \begin{pmatrix} -3 & 1 & 0 \\ -3 & 0 & 1 \\ -1 & 0 & 0 \end{pmatrix} \quad B_d = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}, \tag{28}$$

the output is the first state variable. By using the relay complementarity representation (26), by fixing $T = 8.642s$ and by choosing $N = 46$ samples per period, the matrices (18) and (19) can be simply obtained. In Fig. 4 it is shown the solution obtained through the LCP (24) together with the numerical results of a time stepping simulation of the system. These results confirm the good accuracy obtained by using the proposed approach for computing the sliding limit cycle.

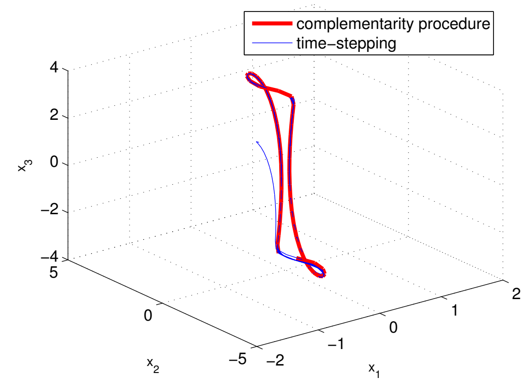

Fig. 4. Sliding limit cycle of a relay feedback system.

D. A practical electrical system

Fig. 5 shows the so called Chua's circuit, [26]. By applying the Kirchhoff laws we obtain

$$A_d = \begin{pmatrix} -\frac{1}{RC_1} & \frac{1}{RC_1} & 0 \\ \frac{1}{RC_2} & -\frac{1}{RC_2} & -\frac{1}{C_2} \\ 0 & \frac{1}{L} & 0 \end{pmatrix} \quad B_d = \begin{pmatrix} \frac{1}{C_1} \\ 0 \\ 0 \end{pmatrix}. \tag{29}$$

Consider as output of the linear part the first state variable. The characteristic of the PWL element $\varphi(\lambda)$ is the saturation function, which has the following complementarity representation:

$$A_s = 0 \qquad B_s = \begin{pmatrix} 1 & -1 \end{pmatrix} \qquad g_s = -1 \tag{30a}$$

$$C_s = \begin{pmatrix} -1 \\ -1 \end{pmatrix} \qquad D_s = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad h_s = \begin{pmatrix} -1 \\ 1 \end{pmatrix}. \tag{30b}$$

The circuit parameters are $C_1 = 0.05\,\mathrm{F}$, $C_2 = 0.4\,\mathrm{F}$, $L = 0.5\,\mathrm{H}$ and $R = 0.3\,\Omega$. By fixing $T = 4.2776s$ and by

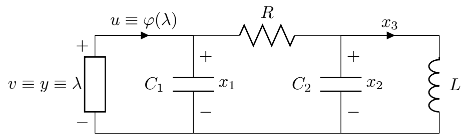

Fig. 5. Chua's circuit with piecewise linear element.

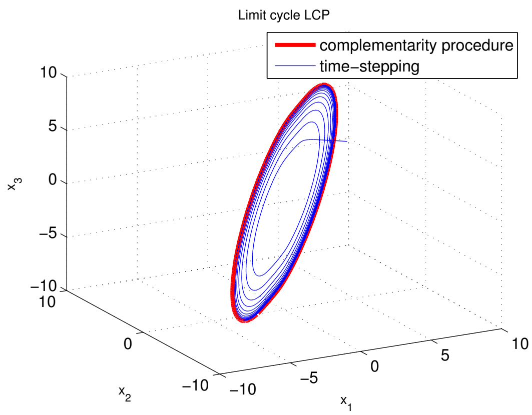

Fig. 6. Limit cycle exhibited by the Chua's circuit.

choosing $N = 200$ samples per period results are reported in Fig. 6.

V. CONCLUSION

The complementarity formalism has been shown to be useful for computing periodic oscillations in Lur'e systems. Limit cycles are computed by solving a suitable static linear complementarity problem. The proposed approach is able to compute classical limit cycles as well as sliding limit cycles and asymmetric unstable oscillations of the continuous–time system. Future work will investigate the use of the proposed tool for the calculation of limit cycles bifurcation diagrams and a choosing strategy for the PWL characteristic complementarity representation.

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