2011 American Control Conference
on O'Farrell Street, San Francisco, CA, USA
June 29 - July 01, 2011

# Generalized Dynamic Inversion for Multiaxial Nonlinear Flight Control

Ismail Hameduddin and Abdulrahman H. Bajodah

*Abstract*—**The nonlinear problem of aircraft trajectory tracking is tackled in the framework of multiple linear time-varying constrained control using the newly developed paradigm of generalized dynamic inversion. The time differential forms of the multiple constraints encapsulate the control objectives, and are inverted to obtain the reference trajectory-realizing control law. The inversion process utilizes the Moore-Penrose generalized inverse, and it predictably involves the problematic generalized inversion singularity. Thus, a singularity avoidance scheme based on a new type of dynamic scaling factors is introduced that guarantees asymptotically stable tracking and singularity avoidance. The steady state closed loop system allows for two inherently noninterfering control actions working towards a unified goal to exploit the aircraft's control authority over the entire state space. One control action is performed on the range space of the constraint generalized inverse matrix, and it works to impose the prescribed aircraft constrained dynamics. The other control action is performed on the complementary orthogonal nullspace of the constraint matrix, and it provides aircraft's global inner stability using a novel type of dynamically scaled nullprojection control Lyapunov functions. Numerical simulations of a multiaxial aircraft coordinated maneuver verify the efficacy of designing nonlinear flight control systems via this technique.**

## I. INTRODUCTION

Nonlinear flight control has been an active area of research for the past three decades. Early attempts [10], [11] at solving the problem of nonlinear aircraft control involved formulating control laws that would cancel system nonlinearities in closed-loop control via inversion, and hence enable use of already established linear control methods. Another perspective in the theory of control via dynamic inversion is to invert a prescribed dynamic constraint on the original system rather than inverting the system itself. This alleviates the need to make simplifying assumptions regarding the controlled plant that is often required in classical dynamic inversion to make deriving the inverse equations of motion feasible. This paradigm shift in the approach to control via dynamic inversion was first proposed in [13]. A similar approach is adopted within a larger context in the present work.

The Generalized Dynamic Inversion (GDI) control design methodology [6] provides a framework wherein multiple design techniques working towards a unified goal may be consolidated. The methodology works on the principle, as discussed previously, of inverting prescribed dynamic constraints on the original dynamical system. Control by inversion of dynamic constraints is achieved through the Moore-Penrose Generalized Inverse (MPGI)-based Greville formula [7]. The Greville formula provides a single parameterized form to the infinite number of solutions possible for a consistent underdetermined system of equations. In the framework of GDI, this formula allows for essentially two cooperating controllers; one that enforces the original prescribed constraints and another that provides an extra degree of design freedom. Different design methodologies may be incorporated within GDI via this extra degree of freedom. The use of a nullprojection matrix ensures the noninterference of the two controllers and thus both work towards a unified goal. The control action noninterference is a key aspect of GDI since the problem of interference becomes important in aircraft control where there is invariably some augmentation of separate controllers.

The theory behind GDI [6], [1] has been shown effective in the control of spacecraft [3], [4], including underactuated spacecraft [2] in addition to robot manipulators [5]. Of particular interest is the comparative study in [3] between GDI and classical dynamic inversion. The current work extends GDI to control of a more complex and higher order system. In the process several new ideas and techniques are presented. Time-varying constraints are proposed to reduce the large initial control effort usually associated with GDI control solutions. In order to maintain coordinated maneuvering, a multiple constraint version of GDI is introduced with each constraint operating at a different relative degree, in the sense of explicitness in control. Additionally, a new dynamically scaled nullprojection Lyapunov function is constructed to design the stabilizing null-control vector. The problem of the MPGI singularity is tackled using a newly formulated scheme that utilizes a modified generalized dynamic inversion. The new approach, the Extended Dynamically Scaled Generalized Inverse (EDSGI), does not truncate the MPGI near the singularity as in the SR-inverse [12], [15]. This allows asymptotic tracking. Furthermore, unlike previous approaches [1], multiple time scales can be efficiently handled through the introduction of a delay term in the EDSGI scaling term.

## II. AIRCRAFT MATHEMATICAL MODEL

The aircraft model that used for control and simulation is largely based on a nonlinear UAV aircraft model given by Dogan & Venkataramanan [8]. The state vector of the

This work is supported by the King Abdulaziz University Deanship of Scientific Research.

Ismail Hameduddin is a research staff member with the Aeronautical Engineering Department, King Abdulaziz University, Jeddah, Saudi Arabia i.hameduddin@gmail.com

Abdulrahman H. Bajodah is an associate professor with the Aeronautical Engineering Department, King Abdulaziz University, Jeddah, Saudi Arabia abajodah@kau.edu.sa

nonlinear model under consideration is

$$\mathbf{x} = \begin{bmatrix}\phi & \theta & \psi & \alpha & \beta & V_T & p & q & r\end{bmatrix}^T \quad (1)$$

where $\phi$, $\theta$ and $\psi$ are the Euler attitude angles representing roll, pitch and yaw respectively. The angle-of-attack, sideslip and tangential velocity of the aircraft are given by $\alpha$, $\beta$ and $V_T$ respectively. The aircraft body angular rates $p$, $q$ and $r$ describe the roll, pitch and yaw rates. Conventional surfaces and throttle are the only available control authority. Hence elevator deflection $\delta_e$, aileron deflection $\delta_a$, rudder deflection $\delta_r$ and commanded throttle $\xi$ form the vector of control variables

$$\mathbf{u} = \begin{bmatrix}\delta_e & \delta_a & \delta_r & \xi\end{bmatrix}^T. \quad (2)$$

We partition the state vector $\mathbf{x}$ into three separate state vectors

$$\mathbf{x} = \begin{bmatrix}\mathbf{x}_u^T & \mathbf{x}_o^T & \mathbf{x}_i^T\end{bmatrix}^T \quad (3)$$

where $\mathbf{x}_u$ are the unactuated states, namely the states with *relative degree two* (Euler angles in this case), $\mathbf{x}_o$ represents the outer states that have *relative degree one* but whose dynamics are relatively slow ($\alpha$ and $\beta$). Lastly, $\mathbf{x}_i$ is the vector of inner states which includes the fast states, that are directly and significantly affected by the control, i.e., $V_T$, $p$, $q$ and $r$. Hence

$$\mathbf{x}_u = \begin{bmatrix}\phi & \theta & \psi\end{bmatrix}^T \quad (4)$$

$$\mathbf{x}_o = \begin{bmatrix}\alpha & \beta\end{bmatrix}^T \quad (5)$$

$$\mathbf{x}_i = \begin{bmatrix}V_T & p & q & r\end{bmatrix}^T. \quad (6)$$

Accordingly, the rotational equations of motion are

$$\dot{\mathbf{x}}_u = H(\mathbf{x}_u)\mathbf{x}_i = \begin{bmatrix}0 & 1 & \tan\theta\sin\phi & \tan\theta\cos\phi \\ 0 & 0 & \cos\phi & -\sin\phi \\ 0 & 0 & \sin\phi/\cos\theta & \cos\phi/\cos\theta\end{bmatrix}\mathbf{x}_i. \quad (7)$$

For the sake of brevity, only a control-explicit form of the aircraft model is shown here. The reader is referred to the original reference for a more natural and physically insightful form of the model. Define $g_1 : \mathbb{R}^3 \times \mathbb{R}^2 \times \mathbb{R}^4 \to \mathbb{R}$ and $g_2 : \mathbb{R}^3 \times \mathbb{R}^2 \times \mathbb{R}^4 \to \mathbb{R}$ as

$$g_1(\mathbf{x}_u, \mathbf{x}_o, \mathbf{x}_i) = q - (p\cos\alpha + r\sin\alpha)\tan\beta + \frac{g}{V_T\cos\beta}(\cos\alpha\cos\phi\cos\theta + \sin\alpha\sin\theta) \quad (8)$$

and

$$g_2(\mathbf{x}_u, \mathbf{x}_o, \mathbf{x}_i) = -r\cos\alpha + p\sin\alpha + \frac{g}{V_T}(-\cos\phi\cos\theta\sin\alpha\sin\beta + \cos\beta\cos\theta\sin\phi + \cos\alpha\sin\beta\sin\theta) \quad (9)$$

then ontrol-explicit state space model for the outer state dynamics is

$$\dot{\mathbf{x}}_o = \Lambda_o(\mathbf{x}_u, \mathbf{x}_o, \mathbf{x}_i) + \mathbf{B}_o(\mathbf{x}_o, \bar{q})\mathbf{u} \quad (10)$$

where $\Lambda_o : \mathbb{R}^3 \times \mathbb{R}^2 \times \mathbb{R}^4 \to \mathbb{R}^2$ is given by

$$\Lambda_o(\mathbf{x}_u, \mathbf{x}_o, \mathbf{x}_i) = \begin{bmatrix} g_1(\mathbf{x}_u, \mathbf{x}_o, \mathbf{x}_i) - \bar{q}\mathcal{S}(C_{L_0} + C_{L_\alpha}\alpha + C_{L_{\alpha^2}}(\alpha - \alpha_{\text{ref}})^2 + C_{L_q}\frac{\bar{c}}{2V}q)/(mV_T\cos\beta) \\ g_2(\mathbf{x}_u, \mathbf{x}_o, \mathbf{x}_i) - \bar{q}\mathcal{S}(C_{S_0} + C_{S_\beta}\beta)/(mV_T) \end{bmatrix} \quad (11)$$

and $\mathbf{B}_o : \mathbb{R}^2 \times (0, \infty) \to \mathbb{R}^{2\times4}$ is the input matrix function for outer state dynamics, given by

$$\mathbf{B}_o(\mathbf{x}_o, \bar{q}) = \frac{-\bar{q}\mathcal{S}}{mV_T}\begin{bmatrix}\frac{C_{L_{\delta_e}}}{\cos\beta} & 0 & 0 & \frac{T_{\max}\sin(\alpha+\delta)}{\bar{q}\mathcal{S}\cos\beta} \\ 0 & 0 & C_{S_{\delta_r}} & \frac{T_{\max}\cos(\alpha+\delta)\sin\beta}{\bar{q}\mathcal{S}}\end{bmatrix}. \quad (12)$$

In the previous $\mathcal{S}$ is the wing area, $\bar{c}$ is the wing's mean chord length, $m$ is the mass of the aircraft, $T_{\max}$ is the maximum available thrust at altitude/airspeed, $\delta$ is the thrust deflection angle (constant), and the $\bar{q}$ is the dynamic pressure given by

$$\bar{q} = \frac{1}{2}\rho V_T^2 \quad (13)$$

where $\rho$ is the air density.

The control-explicit system equations for the inner dynamics are

$$\dot{\omega} = \mathbf{I}^{-1}\mathbf{S}^{\times}(\omega)\mathbf{I}\omega + \bar{q}\mathcal{S}\mathbf{I}^{-1}\mathbf{f}(\mathbf{x}_o, \mathbf{x}_i) + \bar{q}\mathcal{S}\mathbf{I}^{-1}\mathbf{B}_\omega\mathbf{u} \quad (14)$$

where $\mathbf{f} : \mathbb{R}^2 \times \mathbb{R}^4 \to \mathbb{R}^3$ is given by

$$\mathbf{f}(\mathbf{x}_o, \mathbf{x}_i) = \begin{bmatrix} b\left[C_{\mathcal{L}_0} + C_{\mathcal{L}_\beta}\beta + \frac{b}{2V_T}(C_{\mathcal{L}_p}p + C_{\mathcal{L}_r}r)\right] \\ \bar{c}\left(C_{\mathcal{M}_0} + C_{\mathcal{M}_\alpha}\alpha + \frac{\bar{c}}{2V_T}C_{\mathcal{M}_q}q\right) \\ b\left[C_{\mathcal{N}_0} + C_{\mathcal{N}_\beta}\beta + \frac{b}{2V_T}(C_{\mathcal{N}_p}p + C_{\mathcal{N}_r}r)\right] \end{bmatrix} \quad (15)$$

and $\mathbf{B}_\omega \in \mathbb{R}^{3\times4}$ is the constant input matrix

$$\mathbf{B}_\omega = \begin{bmatrix} 0 & b\,C_{\mathcal{L}_{\delta_a}} & b\,C_{\mathcal{L}_{\delta_r}} & 0 \\ \bar{c}\,C_{\mathcal{M}_{\delta_e}} & 0 & 0 & T_{\max}\Delta z/(\bar{q}\mathcal{S}) \\ 0 & b\,C_{\mathcal{N}_{\delta_a}} & b\,C_{\mathcal{N}_{\delta_r}} & 0 \end{bmatrix}. \quad (16)$$

In the previous $\mathbf{I}$ is the inertia matrix about the center of mass of the aircraft, $\mathbf{S}^{\times}(\omega)$ is the cross-product matrix corresponding to the angular velocity vector $\omega$, $b$ is the wing span and $\Delta_z$ is the moment arm of the thrust about the body y-axis.

The aircraft tangential velocity rate is

$$\dot{V}_T = g_3(\mathbf{x}_u, \mathbf{x}_o) - \bar{q}\mathcal{S}\frac{C_{D_0} + C_{D_{\alpha^2}}\alpha^2}{m} + \frac{\cos(\alpha+\delta)\cos\beta}{m}T_{\max}\,\xi \quad (17)$$

where $g_3 : \mathbb{R}^3 \times \mathbb{R}^2 \to \mathbb{R}$ is given by

$$g_3(\mathbf{x}_u, \mathbf{x}_o) = g\cos\theta\sin\beta\sin\phi + g\cos\beta(\cos\phi\cos\theta\cos\alpha - \cos\alpha\sin\theta). \quad (18)$$

Using (14) and (17), the aircraft inner state dynamics is described by the following control-explicit state space model

$$\dot{\mathbf{x}}_i = \Lambda_i(\mathbf{x}_u, \mathbf{x}_o, \mathbf{x}_i) + \mathbf{B}_i(\mathbf{x}_o, \bar{q})\mathbf{u} \quad (19)$$

where $\Lambda_i : \mathbb{R}^3 \times \mathbb{R}^2 \times \mathbb{R}^4 \to \mathbb{R}^4$ is given by

$$\Lambda_i(\mathbf{x}_u, \mathbf{x}_o, \mathbf{x}_i) = \begin{bmatrix} g_3(\mathbf{x}_u, \mathbf{x}_o) - \bar{q}\mathcal{S}(C_{D_0} + C_{D_{\alpha^2}}\alpha^2)/m \\ \mathbf{I}^{-1}\mathbf{S}^{\times}(\omega)\mathbf{I}\omega + \bar{q}\mathcal{S}\mathbf{I}^{-1}\mathbf{f}(\mathbf{x}_o, \mathbf{x}_i) \end{bmatrix} \quad (20)$$

and $\mathbf{B}_i : \mathbb{R}^2 \times (0, \infty) \to \mathbb{R}^{4\times4}$ is the input matrix function for inner state dynamics, and is given by

$$\mathbf{B}_i(\mathbf{x}_o, \bar{q}) = \begin{bmatrix} \mathbf{0_{1\times3}} & T_{\max}\cos(\alpha+\delta)\cos\beta/m \\ & \bar{q}\mathcal{S}\mathbf{I}^{-1}\mathbf{B}_\omega \end{bmatrix}. \quad (21)$$

## III. Generalized Dynamic Inversion Control Law

The unactuated, outer and inner state error vectors from their desired trajectories (subscripted by $d$) are

$$\mathbf{e}_u = \mathbf{x}_u - \mathbf{x}_{u_d}(t) \quad (22)$$

$$\mathbf{e}_o = \mathbf{x}_o - \mathbf{x}_{o_d}(t) \quad (23)$$

$$\mathbf{e}_i = \mathbf{x}_i - \mathbf{x}_{i_d}(t) \quad (24)$$

where it is assumed that $\mathbf{e}_{u_d} \in \mathcal{C}^2$, $\mathbf{e}_{o_d} \in \mathcal{C}^1$, and $\mathbf{e}_{i_d} \in \mathcal{C}^0$. We define the deviation functions of the unactuated and outer states in terms of weighted error norm vectors as follows

$$\zeta_1 = \|\mathbf{e}_o\|_w^2 \equiv \chi_1(\alpha - \alpha_d(t))^2 + \chi_2(\beta - \beta_d(t))^2 \quad (25)$$

$$= \chi_1 e_\alpha^2 + \chi_2 e_\beta^2 \quad (26)$$

$$\zeta_2 = \|\mathbf{e}_u\|_w^2 \equiv \chi_3(\phi - \phi_d(t))^2 + \chi_4(\theta - \theta_d(t))^2 + \chi_5(\psi - \psi_d(t))^2 \quad (27)$$

$$= \chi_3 e_\phi^2 + \chi_4 e_\theta^2 + \chi_5 e_\psi^2 \quad (28)$$

where $\chi_i \; \forall \; (i = 1, ..., 5)$ are weighting constants. These are also rewritten in terms of the error quantities $e$, subscripted with the appropriate variables. Note that when the states are at their desired values, both $\zeta_1$ and $\zeta_2$ vanish.

We now define two uniformly asymptotically stable linear time-varying constraint dynamics in the deviation functions $\zeta_1$ and $\zeta_2$. Every constraint dynamics has its order equal to the relative degree of the corresponding deviation function. Hence

$$\dot{\zeta}_1 + c_1(t)\zeta_1 = 0 \quad (29)$$

$$\ddot{\zeta}_2 + c_2(t)\dot{\zeta}_2 + c_3(t)\zeta_2 = 0 \quad (30)$$

where the coefficients $c_i(t) \; \forall \; (i = 1, 2, 3)$ are real scalar functions in time. To lower the initial control signal, these coefficients are designed such that $c_i(0) = 0$ and they exponentially reach their limiting values

$$c_i(t) = \lambda_i \left(1 - e^{-t/\sigma_i}\right) \qquad \forall \; i = 1, 2, 3 \quad (31)$$

where $\lambda_i$ and $\sigma_i$ are positive constants representing the limiting value and time constant, respectively. It is sufficient for the uniform asymptotic stability of (29) that $c_1(t) > 0 \; \forall \; t > 0$ [14], hence (29) is uniformly asymptotically stable. For system (30), the following expressions are used to evaluate uniform asymptotic stability

$$|\dot{c}_2(t)| + |c_3(t)| \quad (32)$$

$$\dot{c}_2(t) + 2c_2(t)c_3(t). \quad (33)$$

It is shown in [9] that if (32) is bounded from above and (33) is bounded from below by positive constants then (30) is uniformly asymptotically stable, which is easily verifiable from (31).

Evaluating the time derivatives of $\zeta_1$ and $\zeta_2$ in (29) and (30) along the aircraft's state space submodels (7), (10), and (19) and substituting in the differential form of constraints (29) and (30) yields the following pointwise-linear algebraic system

$$\mathcal{A}(\mathbf{x}_u, \mathbf{x}_o, \bar{q}, t)\mathbf{u} = \mathcal{B}(\mathbf{x}_u, \mathbf{x}_o, \mathbf{x}_i, t) \quad (34)$$

where $\mathcal{A} : \mathbb{R}^3 \times \mathbb{R}^2 \times (0, \infty) \times [0, \infty) \to \mathbb{R}^{2\times4}$ is the controls coefficient matrix function, and $\mathcal{B} : \mathbb{R}^3 \times \mathbb{R}^2 \times \mathbb{R}^4 \times [0, \infty) \to \mathbb{R}^2$ is the controls load matrix. The pointwise-linear system (34) is consistent underdetermined whenever $\mathbf{e}_u \neq \mathbf{0}_3$ and $\mathbf{e}_o \neq \mathbf{0}_2$ (excluding the states at which $\theta = \pm\pi/2$ rad. where some elements of $H(\mathbf{x}_u)$ are infinite, and the states at which $\beta = \pm\pi/2$ rad. where some elements of $\mathcal{B}_o(\mathbf{x}_o, \bar{q})$ are infinite), implying that $\mathbf{u}$ almost-globally realizes the constraint dynamics (29) and (30), and that (34) has an infinite number of solutions. These are parameterized by the arbitrary null-control vector $\mathbf{y} \in \mathbb{R}^4$ in the Greville formula [7]

$$\mathbf{u} = \mathcal{A}^+(\mathbf{x}_u, \mathbf{x}_o, \bar{q}, t)\mathcal{B}(\mathbf{x}_u, \mathbf{x}_o, \mathbf{x}_i, t) + \mathcal{P}(\mathbf{x}_u, \mathbf{x}_o, \bar{q}, t)\mathbf{y} \quad (35)$$

where $\mathcal{A}^+ : \mathbb{R}^3 \times \mathbb{R}^2 \times (0, \infty) \times [0, \infty) \to \mathbb{R}^{4\times2}$ is the MPGI of $\mathcal{A}$, and $\mathcal{P} : \mathbb{R}^3 \times \mathbb{R}^2 \times (0, \infty) \times [0, \infty) \to \mathbb{R}^{4\times4}$ is the nullprojection matrix function given by

$$\mathcal{P}(\mathbf{x}_u, \mathbf{x}_o, \bar{q}, t) = I_{4\times4} - \mathcal{A}^+(\mathbf{x}_u, \mathbf{x}_o, \bar{q}, t)\mathcal{A}(\mathbf{x}_u, \mathbf{x}_o, \bar{q}, t). \quad (36)$$

The closed-loop actuated subsystems are obtained by substituting (35) into (10) and (19), resulting in

$$\dot{\mathbf{x}}_o = \Lambda_o(\mathbf{x}_u, \mathbf{x}_o, \mathbf{x}_i) + \mathbf{B}_o(\mathbf{x}_o, \bar{q})[\mathcal{A}^+(\mathbf{x}_u, \mathbf{x}_o, \bar{q}, t)\mathcal{B}(\mathbf{x}_u, \mathbf{x}_o, \mathbf{x}_i, t) + \mathcal{P}(\mathbf{x}_u, \mathbf{x}_o, \bar{q}, t)\mathbf{y}] \quad (37)$$

$$\dot{\mathbf{x}}_i = \Lambda_i(\mathbf{x}_u, \mathbf{x}_o, \mathbf{x}_i) + \mathbf{B}_i(\mathbf{x}_o, \bar{q})[\mathcal{A}^+(\mathbf{x}_u, \mathbf{x}_o, \bar{q}, t)\mathcal{B}(\mathbf{x}_u, \mathbf{x}_o, \mathbf{x}_i, t) + \mathcal{P}(\mathbf{x}_u, \mathbf{x}_o, \bar{q}, t)\mathbf{y}]. \quad (38)$$

## IV. Singularity Avoidance in Control Law

**Definition 1.** *The extended dynamically scaled generalized inverse $\mathcal{A}^*$ is given by*

$$\mathcal{A}^* \equiv \mathcal{A}^T(\mathbf{x}_u, \mathbf{x}_o, \bar{q}, t)[\mathcal{A}(\mathbf{x}_u, \mathbf{x}_o, \bar{q}, t)\mathcal{A}^T(\mathbf{x}_u, \mathbf{x}_o, \bar{q}, t) + \nu(t)I_{2\times2}]^{-1} \quad (39)$$

*such that $\nu : [0, \infty) \to \mathbb{R}$ satisfies the following asymptotically stable first-order forced dynamics*

$$\dot{\nu}(t) = \frac{\|\mathbf{e}_i\|_p^p - \nu(t)}{\gamma\left(\|\mathbf{e}_u\|_p^p + \|\mathbf{e}_o\|_p^p\right)}, \quad p \in \mathbb{Z}^+, \quad \nu(0) > 0 \quad (40)$$

*where $\|.\|_p^p$ is the vector $p$-norm raised to the $p$-th power, and $\gamma$ is a specified real constant positive scalar.*

The extended DSGI-based control law is obtained by replacing $\mathcal{A}^+$ by $\mathcal{A}^*$ in the GDI control law (35). Hence

$$\mathbf{u}_* = \mathcal{A}^*(\mathbf{x}_u, \mathbf{x}_o, \bar{q}, t)\mathcal{B}(\mathbf{x}_u, \mathbf{x}_o, \mathbf{x}_i, t) + \mathcal{P}_*(\mathbf{x}_u, \mathbf{x}_o, \bar{q}, t)\mathbf{y} \quad (41)$$

where $\mathcal{P}_*$ is the dynamically scaled nullprojection matrix function given by

$$\mathcal{P}_* = I - \mathcal{A}^*(\mathbf{x}_u, \mathbf{x}_o, \bar{q}, t)\mathcal{A}(\mathbf{x}_u, \mathbf{x}_o, \bar{q}, t). \quad (42)$$

Thus, substituting (41) in (37) and (38) yields the following closed-loop actuated subsystems

$$\begin{aligned}\dot{\mathbf{x}}_o =& \Lambda_o(\mathbf{x}_u, \mathbf{x}_o, \mathbf{x}_i) \\ &+ \mathbf{B}_o(\mathbf{x}_o, \bar{q})[\mathcal{A}^*(\mathbf{x}_u, \mathbf{x}_o, \bar{q}, t)\mathcal{B}(\mathbf{x}_u, \mathbf{x}_o, \mathbf{x}_i, t) \\ &+ \mathcal{P}_*(\mathbf{x}_u, \mathbf{x}_o, \bar{q}, t)\mathbf{y}] \quad (43)\\ \dot{\mathbf{x}}_i =& \Lambda_i(\mathbf{x}_u, \mathbf{x}_o, \mathbf{x}_i) \\ &+ \mathbf{B}_i(\mathbf{x}_o, \bar{q})[\mathcal{A}^*(\mathbf{x}_u, \mathbf{x}_o, \bar{q}, t)\mathcal{B}(\mathbf{x}_u, \mathbf{x}_o, \mathbf{x}_i, t) \\ &+ \mathcal{P}_*(\mathbf{x}_u, \mathbf{x}_o, \bar{q}, t)\mathbf{y}]. \quad (44)\end{aligned}$$

**Proposition 1.** *Consider the closed loop system given by (7), (43), and (44). If*

$$\lim_{t\to\infty} \|\mathbf{e}_u\|_p = \lim_{t\to\infty} \|\mathbf{e}_o\|_p = 0 \quad (45)$$

*then the elements of* $\mathcal{A}^*$ *are bounded, and* $\lim_{t\to\infty} \mathcal{A}^*(\mathbf{x}_u, \mathbf{x}_o, \bar{q}, t) = \mathbf{0}_{4\times 2}$.

*Proof:* It follows from the dynamics of $\nu$ given by (40) that

$$\lim_{\|\mathbf{e}_u\|_p^p + \|\mathbf{e}_o\|_p^p \to 0} \nu(t) = \|\mathbf{e}_i\|_p^p. \quad (46)$$

Therefore, satisfying (45) implies that

$$\begin{aligned}&\lim_{t\to\infty} \mathcal{A}^*(\mathbf{x}_u, \mathbf{x}_o, \bar{q}, t) = \\ &\quad \mathcal{A}^T(\mathbf{x}_u, \mathbf{x}_o, \bar{q}, t)[\mathcal{A}(\mathbf{x}_u, \mathbf{x}_o, \bar{q}, t)\mathcal{A}^T(\mathbf{x}_u, \mathbf{x}_o, \bar{q}, t) \\ &\quad + \|\mathbf{e}_i\|_p^p I_{2\times 2}]^{-1} \quad (47)\end{aligned}$$

which has bounded elements for all $\mathbf{e}_i \neq \mathbf{0}_4$. If additionally $\lim_{t\to\infty} \mathbf{e}_i = \mathbf{0}_4$, i.e., all closed loop state vectors $\mathbf{x}_u$, $\mathbf{x}_o$, and $\mathbf{x}_i$ converge to their desired trajectories, then the vector fields given by (7), (43), and (44) must be bounded, implying bounded elements of $\mathcal{A}^*$. Moreover, satisfying the condition (45) implies

$$\begin{aligned}\lim_{t\to\infty} \mathcal{A}(\mathbf{x}_u, \mathbf{x}_o, \bar{q}, t) &= \lim_{\|\mathbf{e}_u\|_p^p + \|\mathbf{e}_o\|_p^p \to 0} \mathcal{A}(\mathbf{x}_u, \mathbf{x}_o, \bar{q}, t) \quad (48)\\ &= \mathbf{0}_{2\times 4}. \quad (49)\end{aligned}$$

Therefore, (47) implies that

$$\lim_{t\to\infty} \mathcal{A}^*(\mathbf{x}_u, \mathbf{x}_o, \bar{q}, t) = \mathbf{0}_{4\times 2}, \quad \mathbf{e}_i \neq \mathbf{0}_4. \quad (50)$$

If additionally $\lim_{t\to\infty} \mathbf{e}_i = \mathbf{0}_4$, then it follows from (47) that

$$\begin{aligned}&\lim_{t\to\infty} \mathcal{A}^*(\mathbf{x}_u, \mathbf{x}_o, \bar{q}, t) = \\ &\lim_{t\to\infty} \mathcal{A}^T(\mathbf{x}_u, \mathbf{x}_o, \bar{q}, t)[\mathcal{A}(\mathbf{x}_u, \mathbf{x}_o, \bar{q}, t)\mathcal{A}^T(\mathbf{x}_u, \mathbf{x}_o, \bar{q}, t)]^{-1} \\ &= \lim_{t\to\infty} \mathcal{A}^+(\mathbf{x}_u, \mathbf{x}_o, \bar{q}, t). \quad (51)\end{aligned}$$

But since $\mathbf{0}_{2\times 4}^+ = \mathbf{0}_{4\times 2}$, then it follows from (49) and (51) that satisfying the condition (45) implies

$$\lim_{t\to\infty} \mathcal{A}^*(\mathbf{x}_u, \mathbf{x}_o, \bar{q}, t) = \mathbf{0}_{4\times 2}. \quad (52)$$

■

## V. Closed Loop Stability of Unactuated-Outer Aircraft Dynamics

The control vector $\mathbf{u}$ given by (35) almost-globally realizes the asymptotically stable unactuated-outer dynamics given by (29) and (30). Therefore, the corresponding partial equilibrium state $\begin{bmatrix}\mathbf{e}_u^T & \mathbf{e}_o^T\end{bmatrix}^T = \mathbf{0}_{5\times 1}$ of the aircraft's MPGI-based closed loop system equations (7), (37), and (38) is almost-globally asymptotically stable. Under additional assumptions on $\mathbf{e}_i$, the same property is inferred on the partial equilibrium state of the aircraft's DSGI-based closed loop equations of motion (7), (43), and (44), as stated by the following theorem.

**Theorem 1.** *If the elements of* $\mathbf{e}_i$ *corresponding to the closed loop subsystems (7), (43), and (44) are bounded* $\forall\ t \in [0, \infty)$, *then the elements of* $\mathbf{e}_u$ *and* $\mathbf{e}_o$ *are also bounded* $\forall\ t \in [0, \infty)$ *wherever* $\theta \neq \pm\pi/2$ *and* $\beta \neq \pm\pi/2$. *If additionally* $\lim_{t\to\infty} \mathbf{e}_i = \mathbf{0}_{4\times 1}\ \forall\ \mathbf{x}_i(0) \in \mathbb{R}^4$, *then the partial equilibrium state* $\begin{bmatrix}\mathbf{e}_u^T & \mathbf{e}_o^T\end{bmatrix}^T = \mathbf{0}_{5\times 1}$ *of the aircraft's closed loop system is almost-globally asymptotically stable.*

*Proof:* Assuming on the contrary that $\|\mathbf{e}_u\|_p^p + \|\mathbf{e}_o\|_p^p$ is unbounded while $\|\mathbf{e}_i\|_p^p$ is bounded $\forall\ t \in [0, \infty)$, then it follows from the expression of $\mathcal{A}$ and the dynamics of $\nu$ given by (40) that the elements of $\mathcal{A}$ go arbitrarily large while $\lim_{t\to\infty} \nu(t) < \infty$. Therefore, $\lim_{t\to\infty}(\mathcal{A}(\mathbf{x}_u, \mathbf{x}_o, \bar{q}, t)\mathcal{A}^T(\mathbf{x}_u, \mathbf{x}_o, \bar{q}, t) + I_{2\times 2}\nu(t)) = \mathcal{A}\mathcal{A}^T$, and it follows from (39) that

$$\lim_{t\to\infty} \mathcal{A}^*(\mathbf{x}_u, \mathbf{x}_o, \bar{q}, t) = \mathcal{A}^+(\mathbf{x}_u, \mathbf{x}_o, \bar{q}, t) \quad (53)$$

implying asymptotic realization of (29) and (30), which is contradictory to the argument of unboundedness of the elements of $\mathbf{e}_u$ and $\mathbf{e}_o$. Therefore, the elements of $\mathbf{e}_u$ and $\mathbf{e}_o$ are bounded for all $t \in [0, \infty)$, wherever $\theta \neq \pm\pi/2$ and $\beta \neq \pm\pi/2$. Moreover, it follows from the asymptotically stable $\nu$ dynamics given by (40) that

$$\lim_{t\to\infty} \mathbf{e}_i = \mathbf{0}_{4\times 1} \Rightarrow \lim_{t\to\infty} \nu(t) = \lim_{t\to\infty} \|\mathbf{e}_i\|_p^p = 0. \quad (54)$$

Therefore, the definition of $\mathcal{A}^*(\mathbf{x}_u, \mathbf{x}_o, \bar{q}, t)$ given by (39) implies that

$$\lim_{t\to\infty} \mathcal{A}^*(\mathbf{x}_u, \mathbf{x}_o, \bar{q}, t) = \mathcal{A}^+(\mathbf{x}_u, \mathbf{x}_o, \bar{q}, t) \quad (55)$$

and that

$$\lim_{t\to\infty} \mathcal{P}_*(\mathbf{x}_u, \mathbf{x}_o, \bar{q}, t) = \mathcal{P}(\mathbf{x}_u, \mathbf{x}_o, \bar{q}, t) \quad (56)$$

and accordingly $\lim_{t\to\infty} \mathbf{u}_* = \mathbf{u}$. Hence, the closed loop functions given by (43) and (44) uniformly converge to the expressions given by (37) and (38). Since the control vector $\mathbf{u}$ almost-globally realizes the constraint dynamics equations (29) and (30), and since the elements of $\mathbf{e}_u$, $\mathbf{e}_o$, and $\mathbf{e}_i$ of the closed loop subsystems (7), (43), and (44) are bounded for all

$t \in [0, \infty)$ except at $\theta = \pm\pi/2$ and at $\beta = \pm\pi/2$, it follows that $\mathbf{u}_*$ almost-globally *asymptotically* realizes the constraint dynamics equations (29) and (30), implying almost-global attractiveness of $\begin{bmatrix}\mathbf{e}_u^T & \mathbf{e}_o^T\end{bmatrix}^T = \mathbf{0}_{5\times1}$. Furthermore, since the control vector $\mathbf{u}$ almost-globally stabilizes aircraft's partial closed loop equilibrium error state $\begin{bmatrix}\mathbf{e}_u^T & \mathbf{e}_o^T\end{bmatrix}^T = \mathbf{0}_{5\times1}$, then there exists by the converse Lyapunov argument [14] a positive-definite Lyapunov function $V(\mathbf{x}_u, \mathbf{x}_o)$ such that $\dot{V}(\mathbf{x}_u, \mathbf{x}_o, \mathbf{x}_i, t)$ is negative-definite along the closed loop trajectories of the unactuated-outer subsystems (7) and (37)

$$V(\mathbf{x}_u, \mathbf{x}_o) > 0 : \ \dot{V}(\mathbf{x}_u, \mathbf{x}_o, \mathbf{x}_i, t) < 0 \quad \forall\ t \in [0, \infty),\ \theta \neq \pm\pi/2,\ \beta \neq \pm\pi/2. \tag{57}$$

Evaluating the time derivative of $V$ along the trajectories of the extended DSGI closed loop unactuated-outer subsystems (7) and (43) yields $\dot{V}_*(\mathbf{x}_u, \mathbf{x}_o, \mathbf{x}_i, \nu(t), t)$, where

$$\lim_{t\to\infty} \dot{V}_*(\mathbf{x}_u, \mathbf{x}_o, \mathbf{x}_i, \nu(t), t) = \lim_{\nu(t)\to 0} \dot{V}_*(\mathbf{x}_u, \mathbf{x}_o, \mathbf{x}_i, \nu(t), t) = \dot{V}(\mathbf{x}_u, \mathbf{x}_o, \mathbf{x}_{i_d}(t), t). \tag{58}$$

Nevertheless, continuity of $\dot{V}_*$ in $\nu(t)$ implies that for sufficiently small $\nu(0)$, $\dot{V}_*$ is negative definite for all $t \in [0, \infty)$, $\theta \neq \pm\pi/2$, $\beta \neq \pm\pi/2$, implying (local) asymptotic stability of the aircraft's partial closed loop equilibrium error state $\begin{bmatrix}\mathbf{e}_u^T & \mathbf{e}_o^T\end{bmatrix}^T = \mathbf{0}_{5\times1}$. Together with semi-global attractiveness, semi-global asymptotic stability follows. ■

## VI. Closed Loop Stability of Inner Aircraft Dynamics

The vector $\mathbf{y}$ is taken in the present work to be linear in the error of the inner state vector $\mathbf{x}_i$ from its desired value $\mathbf{x}_{i_d}$ as

$$\mathbf{y} = K(\mathbf{x}_i - \mathbf{x}_{i_d}) = K\mathbf{e}_i \tag{59}$$

where the matrix gain $K$ is yet to be determined. Substituting the above written choice of $\mathbf{y}$ in the aircraft's outer and inner dynamics given by (43) and (44) yields

$$\begin{aligned}\dot{\mathbf{x}}_o =& \Lambda_o(\mathbf{x}_u, \mathbf{x}_o, \mathbf{x}_i) \\ &+ \mathbf{B}_o(\mathbf{x}_o, \bar{q})[\mathcal{A}^*(\mathbf{x}_u, \mathbf{x}_o, \bar{q}, t)\mathcal{B}(\mathbf{x}_u, \mathbf{x}_o, \mathbf{x}_i, t) \\ &+ \mathcal{P}_*(\mathbf{x}_u, \mathbf{x}_o, \bar{q}, t)K\mathbf{e}_i] \end{aligned} \tag{60}$$

$$\begin{aligned}\dot{\mathbf{x}}_i =& \Lambda_i(\mathbf{x}_u, \mathbf{x}_o, \mathbf{x}_i) \\ &+ \mathbf{B}_i(\mathbf{x}_o, \bar{q})[\mathcal{A}^*(\mathbf{x}_u, \mathbf{x}_o, \bar{q}, t)\mathcal{B}(\mathbf{x}_u, \mathbf{x}_o, \mathbf{x}_i, t) \\ &+ \mathcal{P}_*(\mathbf{x}_u, \mathbf{x}_o, \bar{q}, t)K\mathbf{e}_i]. \end{aligned} \tag{61}$$

The desired aircraft's inner velocity vector $\mathbf{x}_{i_d}(t)$ is given by

$$\mathbf{x}_{i_d}(t) = \begin{bmatrix}V_{T_d}(t) & \omega_d^T(t)\end{bmatrix}^T = \begin{bmatrix}V_{T_d}(t) & \mathbf{0}_{1\times3}\end{bmatrix}^T \tag{62}$$

where $V_{T_d}(t)$ is the desired value of $V_T$, and zero values of desired angular velocity components $p_d(t)$, $q_d(t)$, and $r_d(t)$ are assumed. The aircraft's desired inner dynamics is obtained by replacing $\mathbf{x}_i$ by $\mathbf{x}_{i_d}$ and $\dot{\mathbf{x}}_i$ by $\dot{\mathbf{x}}_{i_d}$ in (61), resulting in

$$\begin{aligned}\dot{\mathbf{x}}_{i_d} =& \Lambda_i(\mathbf{x}_u, \mathbf{x}_o, \mathbf{x}_{i_d}) \\ &+ \mathbf{B}_i(\mathbf{x}_o, \bar{q}_d(t))\mathcal{A}^*(\mathbf{x}_u, \mathbf{x}_o, \bar{q}_d(t), t)\mathcal{B}(\mathbf{x}_u, \mathbf{x}_o, \mathbf{x}_{i_d}, t)\end{aligned} \tag{63}$$

where $\bar{q}_d(t) = \frac{1}{2}\rho V_{T_d}^2(T)$. The inner closed loop error dynamics is obtained by subtracting (63) from (61), resulting in

$$\begin{aligned}\dot{\mathbf{e}}_i =& \dot{\mathbf{x}}_i - \dot{\mathbf{x}}_{i_d} = \Lambda_i(\mathbf{x}_u, \mathbf{x}_o, \mathbf{x}_i) - \Lambda_i(\mathbf{x}_u, \mathbf{x}_o, \mathbf{x}_{i_d}) \\ &+ \mathbf{B}_i(\mathbf{x}_o, \bar{q})[\mathcal{A}^*(\mathbf{x}_u, \mathbf{x}_o, \bar{q}, t)\mathcal{B}(\mathbf{x}_u, \mathbf{x}_o, \mathbf{x}_i, t) \\ &+ \mathcal{P}_*(\mathbf{x}_u, \mathbf{x}_o, \bar{q}, t)K\mathbf{e}_i] \\ &- \mathbf{B}_i(\mathbf{x}_o, \bar{q}_d(t))\mathcal{A}^*(\mathbf{x}_u, \mathbf{x}_o, \bar{q}_d(t), t)\mathcal{B}(\mathbf{x}_u, \mathbf{x}_o, \mathbf{x}_{i_d}, t). \end{aligned} \tag{64}$$

The previous closed loop inner dynamics is written compactly as

$$\begin{aligned}\dot{\mathbf{e}}_i =& \Delta_i(\mathbf{x}_u, \mathbf{x}_o, \mathbf{x}_i, \mathbf{x}_{i_d}, t) \\ &+ \mathbf{B}_i(\mathbf{x}_o, \bar{q}_d(t))\mathcal{P}_*(\mathbf{x}_u, \mathbf{x}_o, \bar{q}_d(t), t)K\mathbf{e}_i \end{aligned} \tag{65}$$

where

$$\begin{aligned}\Delta_i(\mathbf{x}_u, \mathbf{x}_o, \mathbf{x}_i, \mathbf{x}_{i_d}, t) =& \Lambda_i(\mathbf{x}_u, \mathbf{x}_o, \mathbf{x}_i) - \Lambda_i(\mathbf{x}_u, \mathbf{x}_o, \mathbf{x}_{i_d}) \\ &+ \mathbf{B}_i(\mathbf{x}_o, \bar{q})\mathcal{A}^*(\mathbf{x}_u, \mathbf{x}_o, \bar{q}, t)\mathcal{B}(\mathbf{x}_u, \mathbf{x}_o, \mathbf{x}_i, t) \\ &- \mathbf{B}_i(\mathbf{x}_o, \bar{q}_d(t))\mathcal{A}^*(\mathbf{x}_u, \mathbf{x}_o, \bar{q}_d(t), t)\mathcal{B}(\mathbf{x}_u, \mathbf{x}_o, \mathbf{x}_{i_d}, t). \end{aligned} \tag{66}$$

**Theorem 2.** *If the null-control vector* $\mathbf{y}$ *is chosen as*

$$\mathbf{y} = K\mathbf{e}_i = -(\mathcal{P}_*\mathbf{B}_i\mathcal{P}_*)^{-1}(\mathcal{P}_*\Delta_i + \frac{1}{2}\dot{\mathcal{P}}_*\mathbf{e}_i + \frac{1}{2}Q\mathbf{e}_i) \tag{67}$$

*then the equilibrium point* $\mathbf{e}_i = \mathbf{0}_4$ *of the aircraft's closed loop inner error dynamics given by (65) is globally asymptotically stable.*

*Proof:* Consider the following control Lyapunov function

$$V(\mathbf{e}_i, \mathbf{x}_u, \mathbf{x}_o, \bar{q}, t) = \mathbf{e}_i^T\mathcal{P}_*(\mathbf{x}_u, \mathbf{x}_o, \bar{q}, t)\mathbf{e}_i \tag{68}$$

which is positive definite since $\mathcal{P}_*$ is a symmetric positive definite matrix. Evaluating $\dot{V}$ along the closed loop solution trajectories of the inner state error dynamics given by (65) and omitting functions arguments for notational convenience yields

$$\begin{aligned}\dot{V} &= 2\mathbf{e}_i^T\mathcal{P}_*(\Delta + \mathbf{B}_i\mathcal{P}_*K\mathbf{e}_i) + \mathbf{e}_i^T\dot{\mathcal{P}}_*\mathbf{e}_i \\ &= \mathbf{e}_i^T(2\mathcal{P}_*\Delta + 2\mathcal{P}_*\mathbf{B}_i\mathcal{P}_*K\mathbf{e}_i + \dot{\mathcal{P}}_*\mathbf{e}_i) \end{aligned} \tag{69}$$

where $\dot{\mathcal{P}}_*(\mathbf{x}_u, \mathbf{x}_o, \mathbf{x}_i, t)$ is the elementwise-time derivative of the matrix $\mathcal{P}_*$ along the closed loop solution trajectories of the $\mathbf{x}_u$ dynamics given by (7), the $\mathbf{x}_o$ dynamics given by (60), and the $\bar{q}$ dynamics given in the $\mathbf{x}_i$ dynamics given by (61). It is sufficient for global asymptotic stability of inner closed loop error dynamics that $\dot{V} < 0 \quad \forall\ \mathbf{e}_i \neq 0$. This is guaranteed by the existence of a symmetric positive-definite constant matrix $Q \in \mathbb{R}^{4\times4}$ such that

$$\dot{V} = -\mathbf{e}_i^TQ\mathbf{e}_i < 0. \tag{70}$$

Equating (69) and (70) yields

$$\begin{aligned}&\mathbf{e}_i^T(2\mathcal{P}_*\Delta + 2\mathcal{P}_*\mathbf{B}_i\mathcal{P}_*K\mathbf{e}_i + \dot{\mathcal{P}}_*\mathbf{e}_i + Q\mathbf{e}_i) = 0\ \forall\ \mathbf{e}_i \in \mathbb{R}^4 \\ &\Rightarrow 2\mathcal{P}_*\Delta + 2\mathcal{P}_*\mathbf{B}_i\mathcal{P}_*K\mathbf{e}_i + \dot{\mathcal{P}}_*\mathbf{e}_i + Q\mathbf{e}_i = 0. \end{aligned} \tag{71}$$

Solving for $K\mathbf{e}_i$ yields the expression given by (67). ■

Substitution of the null-control vector $y$ given by (67) in (41) yields the extended GDI control law

$$\mathbf{u}_* = \mathcal{A}^* \mathcal{B} + \mathcal{P}_* y \quad (72)$$

$$= \mathcal{A}^* \mathcal{B} - \mathcal{P}_* (\mathcal{P}_* \mathbf{B}_i \mathcal{P}_*)^{-1} (\mathcal{P}_* \Delta + \frac{1}{2} \dot{\mathcal{P}}_* \mathbf{e}_i + \frac{1}{2} Q \mathbf{e}_i). \quad (73)$$

## VII. Numerical Simulations of GDI Maneuvering Control

The validity and effectiveness of the present GDI control design are verified through numerical simulations of a multiaxial maneuver. The aircraft is initially in steady level unaccelerated flight (SLUF) at an altitude of 3000 m and a tangential velocity of 150 ms$^{-1}$. The desired maneuver is composed of a positive 180° directional angle ($\psi$) change and a simultaneous increase of tangential velocity $V_T$ to the maximum possible with available thrust ($V_T \approx 230$ ms$^{-1}$ at $\xi \approx 1.0$), together with a continuous tracking of a sinusoidal reference banking angle $\phi$ signal of a 30° amplitude, while maintaining coordinated flight by avoiding sideslipping. The angle $\alpha$ is left uncontrolled by setting $\chi_1 = 0$. Instead, the desired value of $\theta$ is specified such that the flight-path angle $\gamma = 0$, which holds the altitude of the aircraft unchanged. The value of $\theta$ for this desired flight condition is approximately $-4.0°$.

Fig. (1) shows the time histories of Euler's angles, which exhibit good closed loop transient response and excellent steady-state tracking. The commanded control surface deflections and throttle are shown in Fig. (2). The control response remains within reasonable bounds and avoids saturation in both limits as well as rates. The new trim condition of the elevator is evident from elevator deflection plot, chosen to enforce $\theta = -4.0°$ and thus $\gamma = 0$.

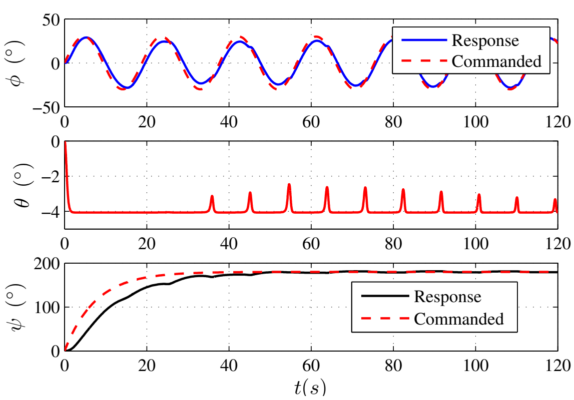


Fig. 1. Time history of the Euler angles.

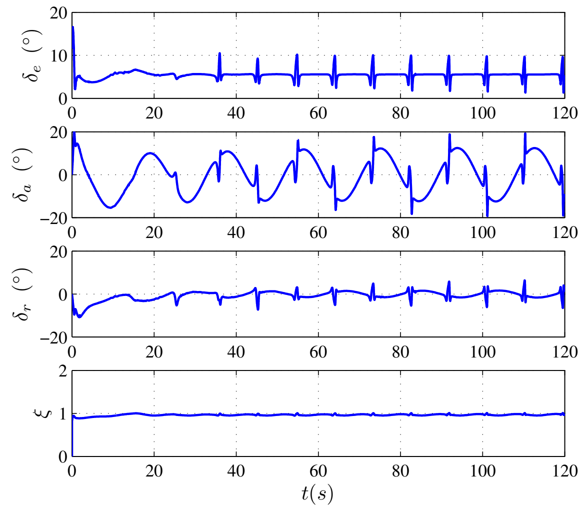


Fig. 2. Time history of control surface deflections and commanded throttle.

## References


[1] A. H. Bajodah, "Singularly perturbed feedback linearization with linear attitude deviation dynamics realization," *Nonlinear Dynamics*, vol. 53, no. 4, pp. 321–343, 2008.

[2] ——, "Asymptotic perturbed feedback linearisation of underactuated euler's dynamics," *International Journal of Control*, vol. 82, no. 10, pp. 1856–1869, 2009.

[3] ——, "Generalised dynamic inversion spacecraft control design methodologies," *IET Control Theory and Applications*, vol. 3, no. 2, pp. 239–251, 2009.

[4] ——, "Asymptotic generalised dynamic inversion attitude control," *IET Control Theory and Applications*, vol. 4, no. 5, pp. 827–840, 2010.

[5] ——, "Servo-constraint generalized inverse dynamics for robot manipulator control design," *International Journal of Robotics and Automation*, vol. 25, no. 1, pp. 38–47, 2010.

[6] A. Bajodah, D. Hodges, and Y.-H. Chen, "Inverse dynamics of servo-constraints based on the generalized inverse," *Nonlinear Dynamics*, vol. 39, no. 1-2, pp. 179–196, 2005, cited By (since 1996) 15.

[7] A. Ben-Israel and T. Greville, *Generalized inverses: Theory and applications*. Springer-Verlag, 2003.

[8] A. Dogan and S. Venkataramanan, "Nonlinear control for reconfiguration of unmanned-aerial-vehicle formation," *Journal of Guidance, Control, and Dynamics*, vol. 28, no. 4, pp. 667–678, 2005.

[9] A. Ignatyev, "Stability of a linear oscillator with variable parameters," *Electronic Journal of Differential Equations*, vol. 1997, pp. 1–6, 1997.

[10] O. Kato and I. Sugiura, "An interpretation of airplane general motion and control as inverse problem," *Journal of Guidance, Control, and Dynamics*, vol. 9, no. 2, pp. 198–204, 1986.

[11] S. Lane and R. Stengel, "Flight control design using non-linear inverse dynamics," *Automatica*, vol. 24, no. 4, pp. 471–483, 1988.

[12] Y. Nakamura and H. Hanafusa, "Inverse kinematic solutions with singularity robustness for robot manipulator control," *Journal of Dynamic Systems, Measurement and Control, Transactions of the ASME*, vol. 108, no. 3, pp. 163–171, 1986.

[13] R. A. Paielli and R. E. Bach, "Attitude control with realization of linear error dynamics," *Journal of Guidance, Control, and Dynamics*, vol. 16, no. 1, pp. 182–189, 1993.

[14] J. Slotine and W. Li, *Applied nonlinear control*. Prentice Hall, 1991.

[15] C. Wampler II, "Manipulator inverse kinematic solutions based on vector formulations and damped least-squares methods," *IEEE Transactions on Systems, Man and Cybernetics*, vol. 16, no. 1, pp. 93–101, 1986.