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Dynamic Disturbance Attenuation and Approximate Optimal Control for Fully Actuated Mechanical Systems

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Abstract— **The standard solutions of the $\mathcal{L}_2$-disturbance attenuation and optimal control problems hinge upon the computation of the solution of a Hamilton-Jacobi (HJ), Hamilton-Jacobi-Bellman (HJB) respectively, partial differential equation or inequality, which may be difficult or impossible to obtain in closed-form. Herein we focus on the matched disturbance attenuation and on the optimal control problems for fully actuated mechanical systems. We propose a methodology to avoid the solution of the resulting HJ (HJB, respectively) partial differential inequality by means of a *dynamic* state feedback. It is shown that for planar mechanical systems the solution of the matched disturbance attenuation and the optimal control problems can be given in closed-form.**

I. Introduction

In recent years the problem of controlling mechanical systems has become crucial in several applications, $e.g.$ robotics and home-automation, flight and aircraft control or surgery applications, to name just a few. The main task consists in determining a control action (open-loop) or a control law (closed-loop) such that either the mechanical system evolves according to a desired reference trajectory, regardless of disturbances acting on the system, or such that the trajectories of the mechanical system and the control input minimize a criterion of optimality. It is well-known that the standard solutions of the control problems informally defined above are related to a Hamilton-Jacobi, Hamilton-Jacobi-Bellman, respectively, partial differential equation or inequality, the solution of which may be, however, difficult or impossible to compute in practical cases [1], [7], [20], [21].

Therefore several approaches to approximate, in a neighborhood of the origin, with a desired degree of accuracy, the solution of the HJ and of the HJB partial differential equation or inequality have been proposed, see [4], [6], [10], [11], [13], [14], [19], [22]. Most of these results rely either on the iterative computation of the coefficients of a local expansion of the solution, provided that all the functions of the nonlinear system are analytic or on the solution of the HJ (HJB, resp.) inequality or equation along the trajectories of the system. Finally, a large effort has been devoted to avoid the hypothesis of differentiability of the storage function, interpreting the HJ (HJB, resp.) inequality or equation in the viscosity sense, see [3], [5], [16].

In [2] the problem of disturbance attenuation and set point regulation for a rigid robot is approached. It is shown that a PD type controller is sufficient to render the closed-loop system dissipative with respect to the *supply rate* associated to the notion of $\mathcal{L}_2$-gain. In [9] the H_∞ control of a rigid spacecraft is considered and a H_∞ suboptimal feedback is proposed. In particular, it is shown that if the torque does not appear in the penalty variable then almost disturbance decoupling can be obtained.

The optimal control problem for mechanical systems has been addressed in [8]. In particular the same assumptions are considered herein and in [8], namely the equations of the rigid body are perfectly known and the positions and the velocities are measurable. Therein, however, a preliminary feedforward term is designed to compensate for the effect of gravity, hence only the components of the control forces that affect the kinetic energy are considered in the optimization, neglecting the gravitation-dependent torques. Moreover, the control law resulting from the optimization is a linear static state feedback and the value function that solves the Hamilton-Jacobi equation resembles the Hamiltonian function of the mechanical system with the potential energy replaced by a quadratic function.

The main contribution of this article is a method to construct *dynamically*, $i.e.$ by means of a dynamic extension, an exact solution of a (modified) HJ inequality or a HJB equation for fully actuated mechanical systems without actually solving any partial differential equation. The methodology yields a dynamic state feedback control law achieving a desired level of disturbance attenuation or a criterion of optimality while guaranteeing at the same time asymptotic stability of the zero equilibrium of the closed-loop system.

The rest of the article is organized as follows. In Section II the basic definitions and results of the methodology proposed in [17], [18] are summarized. In Section III the definition of the problems, $i.e.$ the matched disturbance attenuation and optimal control problems for mechanical systems, is given and discussed. The main result is presented in Section IV, where it is also shown that for planar mechanical systems a closed-form solution can be obtained. Finally, the application of the method to a two-degree-of-freedom (2DOF) planar robot is discussed and conclusions are drawn in Sections V and VI, respectively.

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II. Preliminaries [17], [18]

Consider the first-order quadratic partial differential inequality defined as[1]

$$V_x f(x) + \frac{1}{2} V_x \mathcal{G}(x) V_x^T + \frac{1}{2}\mathcal{H}(x) \leq 0\,, \tag{1}$$

with $x \in \mathbb{R}^n$, where $f : \mathbb{R}^n \to \mathbb{R}^n$ is a smooth vector field, $\mathcal{G} = \mathcal{G}^T : \mathbb{R}^n \to \mathbb{R}^{n\times n}$, and $\mathcal{H} : \mathbb{R}^n \to \mathbb{R}_+$ is a positive semidefinite smooth mapping with $\mathcal{H}(0) = 0$, $\mathcal{H}_x(0) = 0$. Assume that $f(0) = 0$, hence $f(x) = F(x)x$, for some continuous mapping $F : \mathbb{R}^n \to \mathbb{R}^{n\times n}$, possibly not unique. Note that the partial differential equations or inequalities arising in the $\mathcal{L}_2$-disturbance attenuation and in the optimal control problems have the *form* of the *pde* in (1).

Note that the quadratic approximation of the inequality (1) is solved by the quadratic function $V = \frac{1}{2}x^T \bar{P} x$ with $\bar{P} = \bar{P}^T > 0$ given by the algebraic Riccati equation

$$\bar{P}A + A^T\bar{P} + \bar{P}G\bar{P} + H = 0\,, \tag{2}$$

with

$$A \triangleq \left.\frac{\partial f}{\partial x}\right|_{x=0},\;\; H \triangleq \left.\frac{\partial^2 \mathcal{H}}{\partial x^2}\right|_{x=0},\;\; G \triangleq \mathcal{G}(0)\,. \tag{3}$$

We now define the following notion of solution of the inequality (1).

Definition 1: A $\mathcal{C}^1$ mapping $P : \mathbb{R}^n \to \mathbb{R}^{1\times n}$, zero at zero, is said to be an *algebraic $\bar{P}$ solution* of (1) if there exists a mapping $\Sigma : \mathbb{R}^n \to \mathbb{R}^{n\times n}$, with $x^T\,\Sigma(x)\,x > 0$, for all $x \in \mathbb{R}^n \setminus \{0\}$, such that

$$P(x)f(x) + \frac{1}{2}P(x)\mathcal{G}(x)P(x)^T + \frac{1}{2}\mathcal{H}(x) + x^T\Sigma(x)x \leq 0\,, \tag{4}$$

and $P(x)$ is *tangent* at $x = 0$ to the symmetric positive definite solution of (2), *i.e.* $\left.\frac{\partial P(x)^T}{\partial x}\right|_{x=0} = \bar{P}$. ◇

Note that $P(x)$ is not assumed to be a gradient vector. A similar approach is proposed in [12] where the solutions of the Hamilton-Jacobi partial differential inequality are characterized in terms of nonlinear matrix inequalities (NLMI). Therein, however, it is assumed that the solution of the NLMI is a gradient vector. Using the *algebraic $\bar{P}$ solution* $P(x)$, define the function

$$V(x,\xi) = P(\xi)x + \frac{1}{2}(x-\xi)^T R(x-\xi)\,, \tag{5}$$

with $\xi \in \mathbb{R}^n$ and $R = R^T \in \mathbb{R}^{n\times n}$ positive definite.

Remark 1: Consider V as in (5) and note that there exist a non-empty compact set $\Omega_1 \subseteq \mathbb{R}^{2n}$ containing the origin and a positive definite matrix $\bar{R}$ such that for all $R \geq \bar{R}$ the function $V(x,\xi)$ in (5) is positive definite for all $(x,\xi) \in \Omega_1 \subseteq \mathbb{R}^{2n}$. In fact, since $P(x)$ is *tangent* at $x = 0$ to the solution of the algebraic Riccati equation (2), the function $P(x)x : \mathbb{R}^n \to \mathbb{R}$ is, locally around the origin, quadratic and moreover has a local minimum for $x = 0$[2]. Hence, the existence of $\bar{R}$ can be proved noting that the function $P(\xi)x$ is (locally) quadratic in (x,ξ) and, restricted to the manifold $\mathcal{M} = \{\xi \in \mathbb{R}^n : \xi = x\}$, is positive definite for all $x \neq 0$ in Ω_1. ▲

To streamline the presentation define

$$\Delta(x,\xi) = (R - \Phi(x,\xi))\Lambda(\xi)^T\,, \tag{6}$$

with $\Lambda(\xi) = \Psi(\xi)R^{-1}$, where $\Phi(x,\xi) \in \mathbb{R}^{n\times n}$ is a continuous mapping such that $P(x) - P(\xi) = (x-\xi)^T\Phi(x,\xi)^T$, $\Psi(\xi) \in \mathbb{R}^{n\times n}$ is the Jacobian matrix of the mapping $P(\xi)$ and $A_{cl}(x) = F(x) + \mathcal{G}(x)N(x)$, with $N(x)$ such that $P(x) = x^T N(x)^T$.

Theorem 1: [17], [18] Let $P(x)$ be an *algebraic $\bar{P}$ solution* of (1). Let the matrix $R > 0$ be such that $V(x,\xi)$ is positive definite in a set $\Omega \subseteq \mathbb{R}^{2n}$ containing the origin and such that

$$\frac{1}{2}A_{cl}(x)^T\Delta + \frac{1}{2}\Delta^T A_{cl}(x) + \frac{1}{2}\Delta^T\mathcal{G}(x)\Delta < \Sigma(x)\,, \tag{7}$$

for all $(x,\xi) \in \Omega \setminus \{0\}$. Then there exists $\bar{k}$ such that for all $k \geq \bar{k}$ the function $V(x,\xi) > 0$ satisfies

$$\begin{aligned}\mathcal{HJ}(x,\xi) \triangleq & V_x(x,\xi)f(x) + V_\xi(x,\xi)\alpha(x,\xi) + \frac{1}{2}\mathcal{H}(x)\\ &+ \frac{1}{2}V_x(x,\xi)\mathcal{G}(x)V_x(x,\xi)^T \leq 0\,,\end{aligned} \tag{8}$$

for all $(x,\xi) \in \Omega$, with $\alpha(x,\xi) = -kV_\xi^T = -k(\Psi(\xi)^T x - R(x-\xi))$.

Theorem 2: [17], [18] Let $P(x)$ be an *algebraic $\bar{P}$ solution* of (1) with $\Sigma(0) > 0$. Then, there exist a matrix $R > 0$, a neighborhood of the origin $\Omega \subseteq \mathbb{R}^{2n}$ and $\bar{k}$ such that for all $k \geq \bar{k}$ the function $V(x,\xi) > 0$ satisfies the partial differential inequality (8) for all $(x,\xi) \in \Omega$.

III. Problem Definition

Consider fully actuated mechanical systems described by the Euler-Lagrange equation [15], namely

$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} + G(q) = \tilde{\tau}\,, \tag{9}$$

where $G(q)$ describes the potential forces and the matrix $C(q,\dot{q})$, which is linear in the second argument, describes the Coriolis and centripetal forces.

Defining the variables $x_1 = q$ and $x_2 = \dot{q}$, with $x_1(t) \in \mathbb{R}^n$, $x_2(t) \in \mathbb{R}^n$ and $x(t) = (x_1(t), x_2(t))$, equation (9) is equivalent to a system of first-order ordinary differential equations, namely

$$\begin{aligned}\dot{x}_1 &= x_2\,,\\ \dot{x}_2 &= M(x_1)^{-1}\left[\tilde{\tau} - C(x_1,x_2)x_2 - G(x_1)\right]\,.\end{aligned} \tag{10}$$

Assume that the preliminary feedback $\tilde{\tau} = \hat{\tau} + G(0)$, $\hat{\tau} \in \mathbb{R}^n$, is applied to compensate for the effect of gravity

[1] In what follows we use the notation V_x to denote the gradient of the scalar function V with respect to the vector x.

[2] This can be easily proved considering that the first-order derivative of the function is zero in $x = 0$ and $(P(x)x)_{xx} = 2\bar{P} > 0$, where V_{xx} denotes the Hessian matrix of the scalar function $V(x)$ with respect to the vector x.

at the origin. Under this assumption, the gravitational term becomes $\tilde{G}(x_1) = G(x_1) - G(0)$ which is zero at the origin. Therefore, there exists a continuous mapping $\bar{G} : \mathbb{R}^n \to \mathbb{R}^{n\times n}$ such that $\tilde{G}(x_1) = \bar{G}(x_1)x_1$, for all $x_1 \in \mathbb{R}^n$.

Finally, suppose that the variables of interest - to optimize or for which a desired attenuation level needs to be guaranteed in the optimal control or the disturbance attenuation problems, respectively - are the positions x_1 and the velocities x_2 of the joints.

Problem 1: ($\mathcal{L}_2$-disturbance attenuation). Consider system (10) and let $\gamma > 1$ and $\hat{\tau} = u + d$, where $u \in \mathbb{R}^n$ is the control input whereas $d \in \mathbb{R}^n$ is an unknown disturbance signal. The regional dynamic state feedback $\mathcal{L}_2$-disturbance attenuation problem with stability consists in determining an integer $\tilde{n} \geq 0$, a dynamic control law of the form

$$\begin{aligned} \dot{\xi} &= \alpha(x, \xi)\,, \\ u &= \beta(x, \xi)\,, \end{aligned} \tag{11}$$

with $\xi(t) \in \mathbb{R}^{\tilde{n}}$, $\alpha : \mathbb{R}^{2n} \times \mathbb{R}^{\tilde{n}} \to \mathbb{R}^{\tilde{n}}$, $\beta : \mathbb{R}^{2n} \times \mathbb{R}^{\tilde{n}} \to \mathbb{R}^n$ and a set $\bar{\Omega} \subset \mathbb{R}^{2n} \times \mathbb{R}^{\tilde{n}}$, containing the origin of $\mathbb{R}^{2n} \times \mathbb{R}^{\tilde{n}}$, such that the closed-loop system (10)-(11) has the following properties.

(i) The zero equilibrium of the system (10)-(11) with $d(t) = 0$, for all $t \geq 0$, is asymptotically stable with region of attraction containing $\bar{\Omega}$.

(ii) For every $d \in \mathcal{L}_2(0, T)$, such that the trajectories of the system remain in $\bar{\Omega}$, the $\mathcal{L}_2$-gain of (10)-(11) from d to $z = [x^T \; u^T]^T$ is less than or equal to γ, *i.e.*

$$\int_0^T \|x(t)\|^2 dt + \int_0^T \|u(t)\|^2 dt \leq \gamma^2 \int_0^T \|d(t)\|^2 dt\,,$$

for all $T \geq 0$. $\diamond$

Problem 2: (Optimal Control). Consider system (10) with $\hat{\tau} = u$. The approximate regional dynamic optimal control problem consists in determining an integer $\tilde{n} \geq 0$, a dynamic control law (11) and a set $\bar{\Omega} \subset \mathbb{R}^{2n} \times \mathbb{R}^{\tilde{n}}$ containing the origin of $\mathbb{R}^{2n} \times \mathbb{R}^{\tilde{n}}$ such that the closed-loop system (10)-(11) has the following properties.

(i) The zero equilibrium of the system (10)-(11) is asymptotically stable with region of attraction containing $\bar{\Omega}$.

(ii) For any $\bar{u}$ and any (x_0, ξ_0) such that the trajectories of the system (10)-(11) remain in $\bar{\Omega}$

$$J((x_0, \xi_0), \beta) \leq J((x_0, \xi_0), \bar{u})\,,$$

where

$$J((x_0, \xi_0), u) = \frac{1}{2}\int_0^\infty \left(L(x, \xi) + u^T u\right) dt\,,$$

and $L(\cdot, \cdot) : \mathbb{R}^{4n} \to \mathbb{R}_+$ is a positive semidefinite function. $\diamond$

IV. Disturbance attenuation and approximate optimal control

In order to approach the matched $\mathcal{L}_2$-disturbance attenuation and the optimal control problems within the same framework, define $\epsilon^2 = -(1/\gamma^2 - 1)$, where $\gamma \in (1, \infty]$ is the desired disturbance attenuation level. Note that $\epsilon \in (0, 1]$, where the case $\epsilon = 1$, that is $\gamma = \infty$, represents the optimal control problem since no restriction on the attenuation level is imposed. The choice of $\gamma \in (1, \infty)$ yields a compromise between the optimal strategy and the maximum achievable robustness with respect to $\mathcal{L}_2$ exogenous inputs.

Remark 2: Let $\mathcal{H}(x) = \mu_1^2 x_1^T x_1 + \mu_2^2 x_2^T x_2$,

$$f(x) \triangleq \begin{bmatrix} x_2 \\ -M(x_1)^{-1}\left(C(x_1, x_2)x_2 + \tilde{G}(x_1)\right) \end{bmatrix}, \tag{12}$$

and

$$\mathcal{G}(x) \triangleq -\epsilon^2 \begin{bmatrix} 0 & 0 \\ 0 & M(x_1)^{-2} \end{bmatrix}. \tag{13}$$

When $\gamma = \infty$ the instantaneous cost minimized by the optimal control law is given by $L(x, \xi) = (\mathcal{H}(x) + \rho(x, \xi))$, where $\rho(x, \xi)$ is defined as $\rho(x, \xi) = -\mathcal{HJ}(x, \xi) \geq 0$, with $\mathcal{HJ}(x, \xi)$, $f(x)$ and $\mathcal{G}(x)$ as in (8), (12) and (13), respectively. Moreover, the actual cost *paid* is $V(x(0), \xi(0))$, with $V(x, \xi)$ defined in (5), hence the cost can be minimized with a proper initialization of the dynamic controller, *i.e.* for a given initial condition x_0, it is possible to select the initial condition of the dynamic extension $\xi(0)$ such that

$$\xi(0) = \arg\min_{\xi} V(x_0, \xi)\,. \tag{14}$$

$\blacktriangle$

A. Fully actuated mechanical systems

In this section the solutions to Problems 1 and 2 are given for the fully actuated mechanical systems described by the equations (10).

To begin with the definition of an *algebraic $\bar{P}$ solution* for the system (10) requires the computation of the positive definite matrix $\bar{P}$ that solves the linearized problem. To this end consider the first-order approximation of the nonlinear system (10), namely

$$\dot{x} = \begin{bmatrix} 0 & I_n \\ -M(0)^{-1}D & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ M(0)^{-1} \end{bmatrix} u\,.$$

where $D = D^T \in \mathbb{R}^{n\times n}$ is a constant matrix defined as $D = \partial G/\partial x_1(0)$. Note that D is the Hessian matrix of the potential energy $\mathcal{U}(x_1)$ evaluated in $x_1 = 0$, therefore if the potential energy has a strict local minimum at the origin, then D is positive definite. The corresponding algebraic Riccati equation is given by

$$\bar{P}A + A^T\bar{P} - \epsilon^2\bar{P}BB^T\bar{P} + H^T H = 0\,, \tag{15}$$

where $H = diag\{\mu_1 I_n,\, \mu_2 I_n\}$. Partitioning the matrix $\bar{P}$ as

$$\bar{P} = \begin{bmatrix} \bar{P}_1 & \bar{P}_2 \\ \bar{P}_2^T & \bar{P}_3 \end{bmatrix},$$

let the matrices $\bar{P}_1, \bar{P}_2$ and $\bar{P}_3 \in \mathbb{R}^{n\times n}$ be defined as the solutions of the system of quadratic matrix equations

$$\begin{aligned} \mu_1^2 I_n \quad &= \bar{P}_2 M(0)^{-1} D + D^T M(0)^{-1} \bar{P}_2^T + \epsilon^2 \bar{P}_2 M(0)^{-2} \bar{P}_2^T \\ \bar{P}_1 \quad &= D^T M(0)^{-1} \bar{P}_3 + \epsilon^2 \bar{P}_2 M(0)^{-2} \bar{P}_3 \,, \\ \bar{P}_3 \quad &= \frac{1}{\epsilon} M(0) [\bar{P}_2 + \bar{P}_2^T + \mu_2^2 I_n]^{1/2} \,. \end{aligned} \tag{16}$$

Note that the solution of the matrix equations (16) exists and is unique.

Proposition 1: Consider the mechanical system (10). Suppose that $\Sigma_i(0) > 0$, $i = 1, 2$, let $\gamma \in (1, \infty]$. Let $x_i^T \Upsilon_i^T(x) \Upsilon_i(x) x_i = x_i^T(\mu_i^2 I_n + \Sigma_i(x)) x_i > 0$, $i = 1, 2$, let $W_1(x_1, x_2)$ be such that

$$\Upsilon_1^T(x)\Upsilon_1(x) = W_1 M(x_1)^{-1}\bar{G}(x_1) + \bar{G}(x_1)^T M(x_1)^{-1} W_1 + \epsilon^2 W_1 M(x_1)^{-2} W_1$$

and let

$$\begin{aligned} V_1(x) &= W_1 M(x_1)^{-1}\left[C(x) + \epsilon^2 M(x_1)^{-1} W_2\right] \\ &\quad + \bar{G}(x_1)^T M(x_1)^{-1} W_2 \,, \\ V_2(x) &= W_2 M(x_1)^{-1}\left[C(x) + \frac{\epsilon^2}{2} M(x_1)^{-1} W_2\right] \\ &\quad - \frac{1}{2}\Upsilon_2(x)^T \Upsilon_2(x) \,, \end{aligned}$$

with $W_2(x_1, x_2)$ such that $W_2(0,0) = \bar{P}_3$. Then there exist a matrix $R > 0$, a neighborhood of the origin $\Omega \subseteq \mathbb{R}^{4n}$ and $\bar{k}$ such that for all $k \geq \bar{k}$ the function $V(x, \xi)$ as in (5), with

$$P(x) = [x_1^T V_1 + x_2^T V_2, x_1^T W_1 + x_2^T W_2] \,, \tag{17}$$

is positive definite and satisfies for all $(x, \xi) \in \Omega$ the Hamilton-Jacobi partial differential inequality (8), hence the dynamic control law

$$\begin{aligned} \dot{\xi} \quad &= -k(\Psi(\xi)^T x - R(x - \xi)) \,, \\ u \quad &= -g(x)^T \left[P(x)^T + (R - \Phi(x, \xi))(x - \xi)\right] \,, \end{aligned} \tag{18}$$

solves Problem 1 if $\epsilon \in (0, 1)$ and Problem 2 if $\epsilon = 1$.

Proof: Let $\Sigma_i(0) > 0$, $i = 1, 2$, $f(x)$ and $\mathcal{G}(x)$ as in (12) and (13), respectively, $\epsilon \in (0, 1)$, and note that $P(x)$ as in (17) is an *algebraic $\bar{P}$ solution* of the equation (4), *i.e.*

$$\begin{aligned} &2x_1^T V_1 x_2 + 2x_2^T V_2 x_2 - 2x_1^T W_1 M^{-1}\tilde{G} - 2x_2^T W_2 M^{-1}\tilde{G} \\ &- \epsilon^2 \left(x_1^T W_1 + x_2^T W_2\right) M^{-2} \left(W_1 x_1 + W_2 x_2\right) \\ &- 2x_1^T W_1 M^{-1} C x_2 - 2x_2^T W_2 M^{-1} C x_2 + x_1^T \Upsilon_1^T \Upsilon_1 x_1 \\ &+ x_2^T \Upsilon_2^T \Upsilon_2 x_2 = 0. \end{aligned} \tag{19}$$

Then, by Theorems 1 and 2 there exist k, R and a set $\Omega \subseteq \mathbb{R}^{4n}$ such that the dynamic control law (18) solves the regional dynamic state feedback $\mathcal{L}_2$-disturbance attenuation problem with stability. If $\epsilon = 1$ in (13) then the dynamic control law (18) solves Problem 2. □

Remark 3: Let $\mu_1 \neq 0$. The mechanical system (10) is *zero-state detectable* with respect to the output $h(x) = [\mu_1 x_1,\ \mu_2 x_2]^T$. Therefore, considering the condition (8), by LaSalle's invariance principle and zero-state detectability, the feedback (18) asymptotically stabilizes the zero equilibrium of the closed-loop system. ▲

B. Planar Mechanical Systems

In the case of planar mechanical systems the solutions of the linearized disturbance attenuation problem and of the equation (19) can be given in closed-form. Consider planar fully actuated mechanical systems. Defining the variables $x_1 = q$ and $x_2 = \dot{q}$, with $x_1(t) \in \mathbb{R}^n$, $x_2(t) \in \mathbb{R}^n$, equation (9) is equivalent to a system of first-order ordinary differential equations, namely

$$\begin{aligned} \dot{x}_1 \quad &= x_2 \,, \\ \dot{x}_2 \quad &= M(x_1)^{-1}\left[\hat{\tau} - C(x_1, x_2) x_2\right] \,. \end{aligned} \tag{20}$$

The solution of the algebraic Riccati equation (15) for the system (20) is

$$\begin{aligned} \bar{P}_1 \quad &= \mu_1 \left[\frac{2\mu_1}{\epsilon} M(0) + \mu_2^2 I_n\right]^{1/2} \,, \\ \bar{P}_2 \quad &= \frac{\mu_1}{\epsilon} M(0) \,,\ \bar{P}_3 = \frac{1}{2}[S + S^T] \,, \end{aligned} \tag{21}$$

with $S = \epsilon^{-1} M(0)\left[2\mu_1 \epsilon^{-1} M(0) + \mu_2^2 I_n\right]^{1/2}$.

The following result provides solutions to Problems 1 and 2 for fully actuated planar mechanical systems (20).

Corollary 1: Consider the mechanical system (20). Suppose that $\Sigma_i(0) > 0$, $i = 1, 2$, let $\gamma \in (1, \infty]$. Let $W_1(x_1, x_2) = \dfrac{1}{\epsilon}\Upsilon(x)^T M(x_1)$ and

$$V_1(x_1, x_2) = \Upsilon(x)\left[\frac{1}{\epsilon} C(x_1, x_2) + \epsilon M(x_1)^{-1} W_2\right] \,,$$

$$\begin{aligned} V_2(x_1, x_2) &= W_2 M(x_1)^{-1}\left[C(x_1, x_2) + \frac{\epsilon^2}{2} M(x_1)^{-1} W_2\right] \\ &\quad - \frac{1}{2}\Upsilon_2(x)^T \Upsilon_2(x) \,, \end{aligned}$$

with $W_2(x_1, x_2)$ such that $W_2(0,0) = \bar{P}_3$. Then there exist a matrix $R > 0$, a neighborhood $\Omega \subset \mathbb{R}^{4n}$ containing the origin and $\bar{k}$ such that for all $k \geq \bar{k}$ the dynamic control law defined in (18), with $P(x)$ as in (17), solves Problem 1 if $\epsilon \in (0, 1)$ and Problem 2 if $\epsilon = 1$.

V. 2 DOF PLANAR ROBOT

Consider a planar fully actuated robot with two rotational joints and let $x_1 = [\chi_1, \chi_2] \in \mathbb{R}^2$ be the relative positions of the joints and $x_2 = [\chi_3, \chi_4] \in \mathbb{R}^2$ be the corresponding velocities. The dynamics of the mechanical system can be described by equations of the form (20) with

$$M(\chi_1, \chi_2) = \begin{bmatrix} a_1 + 2a_2\cos(\chi_2) & a_2\cos(\chi_2) + a_3 \\ a_2\cos(\chi_2) + a_3 & a_3 \end{bmatrix} \,,$$

$$C(\chi) = \left[\begin{array}{cc} 0 & -a_2 \sin(\chi_2)(\chi_4 + 2\chi_3) \\ a_2 \sin(\chi_2)\chi_3 & 0 \end{array} \right]$$

where $a_1 = I_1 + m_1 d_1^2 + I_2 + m_2 d_2^2 + m_2 l_1^2$, $a_2 = m_2 l_1 d_2$ and $a_3 = I_2 + m_2 d_2^2$, with I_i, m_i, l_i and d_i the moment of inertia, the mass, the length and the distance between the center of mass and the tip of the i-th joint, $i = 1, 2$, respectively. In the simulations we let $m_1 = m_2 = 0.5Kg$, $l_1 = l_2 = 0.3m$, $d_1 = d_2 = 0.15m$ and $I_1 = I_2 = 0.0037Kgm^2$. To begin with suppose that the action of the actuators is corrupted by white noise and consider a desired attenuation level on the position of the joints close to $\gamma = 1$, $e.g.$ define $\epsilon = 0.1$. Let $\Sigma(\chi_1, \chi_2) = diag\{10^{-3}(1 + \chi_1^2), 10^{-3}(1 + \chi_2^2)\}$ and determine the *algebraic $\bar{P}$ solution* defined in Proposition 1. Since the nonlinear Hamilton-Jacobi partial differential equation or inequality that yields the solution of the matched disturbance attenuation problem for the considered planar robot does not admit a closed-form solution, the performance of the dynamic control law defined in (18) is compared with the solution of the linearized problem, $i.e.$ the static state feedback given by $u_o = -B^T \bar{P} x$. In the dynamic control law, the matrix R is selected as $R = \alpha\Phi(0, 0)$, with $\alpha \in \mathbb{R}_+$. Let the initial condition of the planar robot be $[\chi_1(0), \chi_2(0), \chi_3(0), \chi_4(0)] = [\pi/2, \pi/2, 0, 0]$. Figure 1 displays the time histories of the angular positions of the joints for different values of the parameter α when the linearized control law u_o and the dynamic control law (18) are applied, dashed and solid lines, respectively. In all the plots the same disturbance affects the actuators. The behavior of the joints with $\alpha = 0.8$ is displayed in the top graph and it can be noted that the dynamic control law guarantees worse rejection of the matched disturbances than the control law u_o. Increasing the value of the parameter α improves the performance of the dynamic control law. In particular, the choice $R = \Phi(0, 0)$ (middle graph) yields a solution almost identical to u_o whereas selecting $\alpha = 5$ the disturbance attenuation is significantly improved (bottom graph).

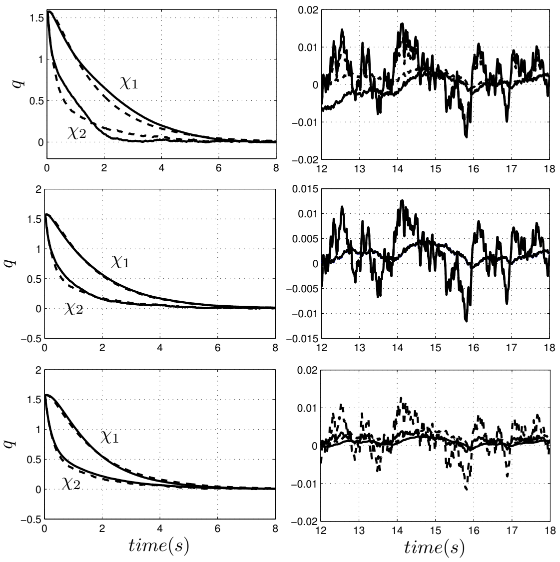

Fig. 1. Time histories of the angular positions of the joints for different values of the parameter α (Top graph: $\alpha = 0.8$. Middle graph: $\alpha = 1$. Bottom graph: $\alpha = 5$) when the linear control law u_o and the dynamic control law are applied, dashed and solid lines, respectively.

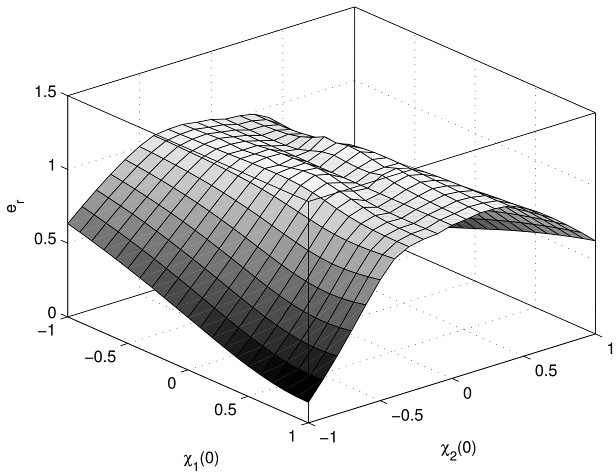

Fig. 2. Ratio between the costs yielded by the dynamic control law - considering the optimal value of $[\xi_1(0), \xi_2(0)]$ for each $\chi(0)$, letting $[\xi_3(0), \xi_4(0)] = [0, 0]$ - and by the optimal static state feedback for the linearized system.

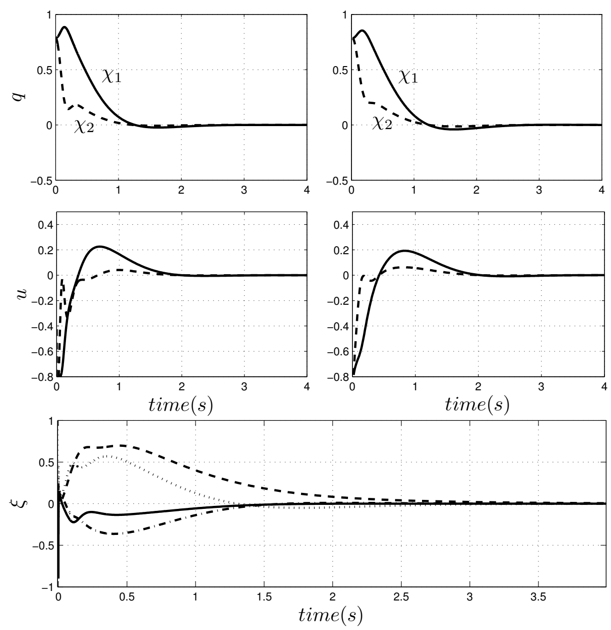

Fig. 3. Top graph: time histories of the angular positions of the two joints when the dynamic control law and the optimal state feedback u_o are applied, left and right graph, respectively. Middle graph: time histories of the dynamic control law and u_o, left and right graph, respectively. Bottom graph: time histories of the dynamic extension ξ.

Consider the ideal case of absence of disturbances and let $\epsilon = 1$, $i.e.$ $\gamma = \infty$. Figure 2 displays, for different initial conditions, the ratio between the costs yielded by the dynamic control law - considering the optimal value of $[\xi_1(0), \xi_2(0)]$ for each $\chi(0)$, letting $[\xi_3(0), \xi_4(0)] = [0, 0]$ - and by the optimal static state feedback for the linearized system, namely $\rho = V_d(\chi(0), \xi(0))/V_o(\chi(0))$. Obviously,

$\rho < 1$ implies that the cost paid by the dynamic control law is smaller than the cost of the optimal static state feedback for the linearized system.

Let $\alpha = 1.2$ and $\chi(0) = [\pi/4, \pi/4, 0, 0]$. As above, the dynamic control law is compared with the optimal solution of the linearized problem obtained with $\epsilon = 1$, whose optimal cost is $V(\chi(0)) = \frac{1}{2}\chi(0)^T \bar{P} \chi(0) = 0.2606$. Set $[\xi_3(0), \xi_4(0)] = [0, 0]$. The optimization of the value function $V(\chi(0), \xi(0))$, which gives the optimal cost *paid* by the solution, with respect to $\xi_1(0)$ and $\xi_2(0)$, yields the values $\xi_1^o(0) = 0.3$, $\xi_2^o(0) = -0.9$ and the corresponding value attained by the function is $V(\chi(0), \xi_1^o(0), \xi_2^o(0), 0, 0) = 0.2081$. The top graph of Figure 3 shows the time histories of the angular positions of the two joints when the dynamic control law, with $\alpha = 1.2$, and the optimal state feedback u_o are applied, left and right graph, respectively. The time histories of the dynamic control law and the optimal local state feedback are displayed in the middle graph of Figure 3, left and right graph respectively, whereas the bottom graph shows the time histories of the state of the dynamic extension, ξ.

In the last simulation a comparison between the dynamic control law (18) and the linear control law proposed in [8] is performed. In the following the variables to minimize are the positions together with the velocities of the joints, since the conditions of existence for the control law in [8] can not be satisfied considering only the minimization of the positions of the joints, as in the previous numerical example.

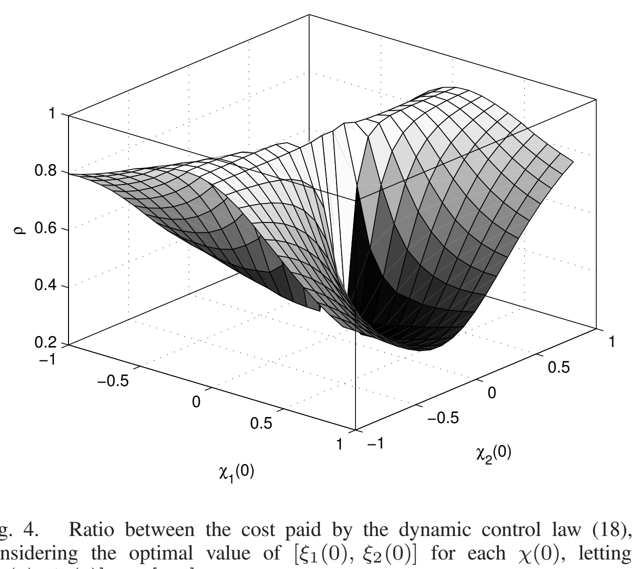

Fig. 4. Ratio between the cost paid by the dynamic control law (18), considering the optimal value of $[\xi_1(0), \xi_2(0)]$ for each $\chi(0)$, letting $[\xi_3(0), \xi_4(0)] = [0, 0]$, and the control law proposed in [8].

Figure 4 displays the ratio between the cost paid by the dynamic control law (18), considering the optimal value of $[\xi_1(0), \xi_2(0)]$ for each $\chi(0)$, letting $[\xi_3(0), \xi_4(0)] = [0, 0]$, and the linear control law proposed in [8]. It can be noted that the cost paid by the dynamic solution is lower than the cost of the linear control law of [8].

VI. Conclusions

The matched $\mathcal{L}_2$-disturbance attenuation problem and the optimal control of fully actuated mechanical systems are studied. It is shown that a dynamic control law can be designed by means of a *dynamic extension*. The methodology hinges upon the solution of an algebraic Riccati equation without involving any partial differential equation. Moreover, for planar mechanical systems the solution can be given in closed-form. The article is concluded with an example of application of the proposed methodology to a 2 DOF planar robot.

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